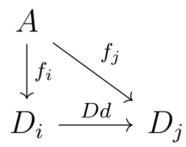
Limits, colimits, sources, sinks, pullbacks, pushouts

Definition 1: Let I be a class. A *source* is a pair $(A, (f_i)_{i \in I})$ consisting of an object A and a family of morphisms $f_i : A \longrightarrow A_i$. The object A is called the *domain of the source*. The family of objects $(A_i)_{i \in I}$ is called the *codomain of the source*.

Definition 2: Let $\mathbf{I}$ and $\mathbf{A}$ be categories. A functor $D : \mathbf{I} \longrightarrow \mathbf{A}$ is called a *diagram*. The domain $\mathbf{I}$ is called *scheme*. The diagram is called *small* (or *finite*), if $\mathbf{I}$ is small (or finite).

Definition 3: Let $D : \mathbf{I} \longrightarrow \mathbf{A}$ be a diagram. An $\mathbf{A}$-source $(A, (f_i)_{i \in \mathrm{Ob}(\mathbf{I})})$ with codomain $(D_i)_{i \in \mathrm{Ob}(\mathbf{I})}$ is called *natural for D* or *cone*, if for each $\mathbf{I}$-morphism $d : i \longrightarrow j$ the triangle



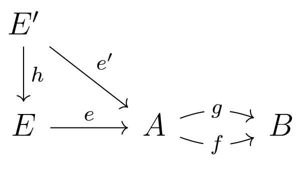
commutes.

Let $(L, (l_i)_{i \in \mathrm{Ob}(\mathbf{I})})$ be a natural source for D with codomain $(D_i)_{i \in \mathrm{Ob}(\mathbf{I})}$. $(L, (l_i)_{i \in \mathrm{Ob}(\mathbf{I})})$ is called a *limit*, if it fulfils the following universal property: For each natural source $(A, (f_i)_{i \in \mathrm{Ob}(\mathbf{I})})$ with codomain $(D_i)_{i \in \mathrm{Ob}(\mathbf{I})}$ there exists a unique morphism $f : A \longrightarrow L$ with $f_i = l_i \circ f$ for each $i \in \mathrm{Ob}(\mathbf{I})$.

Definition 4: A source $(P, (p_i)_{i \in I})$ with codomain $(A_i)_{i \in I}$ that has the property that for each source $(A, (f_i)_{i \in I})$ with codomain $(A_i)_{i \in I}$ there exists a unique morphism $f : A \longrightarrow P$ such that $f_i = p_i \circ f$ for each $i \in I$ is called *product*.

A product with codomain $(A_i)_{i \in I}$ is called a *product of the family* $(A_i)_{i \in I}$.

Definition 5: Let $f, g : A \longrightarrow B$ be two morphisms. A morphism $e : E \longrightarrow A$ with the properties that $f \circ e = g \circ e$ and that for any morphism $e' : E' \longrightarrow A$ with $f \circ e' = g \circ e'$ there exists a unique morphism h such that the triangle



commutes, is called an *equalizer of f and g*.

Definition 6: A commuting square

$$\begin{array}{ccc} P & \xrightarrow{\tilde{f}} & B \\ \Big\downarrow{\scriptstyle \tilde{g}} & & \Big\downarrow{\scriptstyle g} \\ A & \xrightarrow{f} & C \end{array}$$

is called a *pullback square* provided that for any commuting square

$$\begin{array}{ccc} \hat{P} & \xrightarrow{\hat{f}} & B \\ \Big\downarrow{\scriptstyle \hat{g}} & & \Big\downarrow{\scriptstyle g} \\ A & \xrightarrow{f} & C \end{array}$$

there exists a unique morphism $k : \hat{P} \longrightarrow P$ for which the following diagram commutes

$$\begin{array}{cccc} \hat{P} & & \overset{\hat{f}}{\frown\!\!\!\longrightarrow} & \\ & \searrow{\scriptstyle k} & & \\ \hat{g} & & P & \xrightarrow{\tilde{f}} & B \\ & & \Big\downarrow{\scriptstyle \tilde{g}} & & \Big\downarrow{\scriptstyle g} \\ & \searrow & A & \xrightarrow{f} & C. \end{array}$$

In this case the source $(P, (\tilde{g}, \tilde{f}))$ is called a *pullback* of the sink $(C, (f, g))$ and $\tilde{f}$ is called *pullback of* f *along* g.

Theorem 7: *Let* $f : A \longrightarrow C$ *and* $g : B \longrightarrow C$ *be morphisms.*

If a source $(A \times B, (\pi_A, \pi_B))$ *with codomain* (A, B) *is a product of* A *and* B, *and* $e : E \longrightarrow A \times B$ *is an equalizer of* $f \circ \pi_A$ *and* $g \circ \pi_B$, *then*

$$\begin{array}{ccc} E & \xrightarrow{\pi_A \circ e} & A \\ \Big\downarrow{\scriptstyle \pi_B \circ e} & & \Big\downarrow{\scriptstyle f} \\ B & \xrightarrow{g} & C \end{array}$$

is a pullback square.

Definition 8: Let **A** be a category.

If for each finite diagram **A** there exists a limit, then **A** is said to be *finitely complete*.

If for each small diagram **A** there exists a limit, then **A** is said to be *complete*.