From Posets to Categories

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Seminar on Category Theory

Posets

A partially-ordered set is a pair $(X, \leq)$, where X is a set and $\leq$ is a binary relation on X, s.t.

1. $\leq$ is reflexive; $x \leq x$
2. $\leq$ is transitive; $x \leq y \wedge y \leq z \implies x \leq z$
3. $\leq$ is anti-symmetric; $x \leq y \wedge y \leq x \implies x = y$

Dictionary

Posets	**Cat theory**
elements of the poset	objects
order structure $x \leq y$	morphisms $x \to y$

Top & Bottom objects

The top object of a poset, if it exists, is an element $\top$ such that $\forall x. x \leq \top$.

The bottom object of a poset, if it exists, is an element $\bot$ such that $\forall x. \bot \leq x$.

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Binary Meets & Joins

The meet $a \sqcap b$ of two elements a, b is their greatest lower bound, if it exists.

$a \sqcap b$ satisfies $a \sqcap b \leq a \wedge a \sqcap b \leq b$, and if also $c \leq a$ and $c \leq b$, then $c \leq a \sqcap b$.

The join $a \sqcup b$ is the least upper bound, if it exists.

$a \sqcup b$ satisfies $a \leq a \sqcup b \wedge b \leq a \sqcup b$, and if also $a \leq c$ and $b \leq c$, then $a \sqcup b \leq c$.

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top object $\top$	terminal object $\mathbf{1}$
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binary meet $x \sqcap y$	product $x \times y$
binary join $x \sqcup y$	co-product $x \amalg y$

Arbitrary Meets & Joins

Given a set $S \subseteq X$, the meet $\sqcap S$ of S (if it exists) is the greatest lower bound of S.

$\sqcap S \leq s \ \forall s \in S$ and if $c \leq s \ \forall s \in S$, then $c \leq \sqcap S$.

Similarly, $\sqcup S$ is the least upper bound of S.

$s \leq \sqcup S \ \forall s \in S$ and if $s \leq c \ \forall s \in S$, then $\sqcup S \leq c$.

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arbitrary meet $\sqcap S$	limit
arbitrary join $\sqcup S$	co-limit

Order-preserving Functions

Let $P = (X, \leq_X)$ and $Q = (Y, \leq_Y)$ be posets.
An order-preserving function between P and Q is a function

$$f : X \to Y$$

s.t.

$$x \leq_X x' \Rightarrow f(x) \leq_Y f(x').$$

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order-preserving functions	functors between poset categories

(Monotone) Galois Connections

A (monotone) Galois connection between two posets $P = (X, \leq_X)$ and $Q = (Y, \leq_Y)$ is a pair of order-preserving functions

$$\mathrm{P} \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \mathrm{Q}$$

such that $\alpha(p) \leq_Y q \iff p \leq_X \gamma(q)$.

γ is an *upper adjoint*, α a *lower adjoint*.

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arbitrary meet $\sqcap S$ arbitrary join $\sqcup S$	limit co-limit
order-preserving functions	functors between poset categories
Galois connections	adjoint functors