From Posets to Categories — Definitions

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Definition 1 (Posets)**.** *A partially-ordered set is a pair $(X, \leq)$, where X is a set and $\leq$ is a binary relation on X, s.t.*

1. *$\leq$ is reflexive; for all x, $x \leq x$*
2. *$\leq$ is transitive; for all x, y, z, $x \leq y$ and $y \leq z$ implies that $x \leq z$*
3. *$\leq$ is anti-symmetric; for all x, y, $x \leq y$ and $y \leq x$ implies $x = y$*

Definition 2 (Binary Meets & Joins)**.** *The meet $a \sqcap b$ of two elements a, b is their greatest lower bound, if it exists. The join $a \sqcup b$ is the least upper bound, if it exists.*

Formally, $a \sqcap b$ satisfies $a \sqcap b \leq a$ and $a \sqcap b \leq b$, and if also $c \leq a$ and $c \leq b$, then $c \leq a \sqcap b$.

Definition 3 (Lattices)**.** *A poset with all binary meets and joins is called a lattice.*

Definition 4 (Top & Bottom objects)**.** *The top object of a poset, if it exists, is an element $\top$ such that for all x, $x \leq \top$. The bottom object of a poset, if it exists, is an element $\bot$ such that for all x, $\bot \leq x$. These objects are automatically determined uniquely.*

Definition 5 (Arbitrary Meets & Joins)**.** *Given a set $S \subseteq X$, the meet $\sqcap S$ of S (if it exists) is the greatest lower bound of S. Similarly, $\sqcup S$ is the least upper bound of S.*

Formally, $\sqcap S \leq s \forall s \in S$ and if $c \leq s \forall s \in S$, then $c \leq \sqcap S$.

Definition 6 (Complete Lattices)**.** *A lattice with all arbitrary meets and joins is called complete.*

Definition 7 (Order-preserving Functions)**.** *Let two posets $P = (X, \leq_X)$ and $Q = (Y, \leq_Y)$ be given. An order-preserving function between P and Q is a function $f : X \to Y$ s.t. $x \leq_X x' \Rightarrow f(x) \leq_Y f(x')$.*

Definition 8 ((Monotone) Galois Connections)**.** *A (monotone) Galois connection between two posets $P = (X, \leq_X)$ and $Q = (Y, \leq_Y)$ is a pair of order-preserving functions*

$$P \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} Q$$

such that $\alpha(p) \leq_Y q \iff p \leq_X \gamma(q)$.

Dictionary

Posets	**Cat theory**
elements of the poset	objects
order structure $x \leq y$	morphisms $x \to y$
maximum object $\top$	terminal object $\mathbf{1}$
minimum object $\bot$	initial object $\mathbf{0}$
binary meet $x \sqcap y$	product $x \times y$
binary join $x \sqcup y$	co-product $x \amalg y$
arbitrary meet $\sqcap S$	limit
arbitrary join $\sqcup S$	co-limit
order-preserving functions	functors between poset categories
Galois connections	adjoint functors