From Posets to Categories

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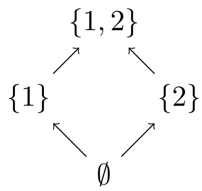
Main premise/motivation: we've seen some basic definitions and concepts, now apply it all to one example – Posets

Definition 1. *A partially-ordered set $(X, \leq)$ is a set X equipped with $\leq \subseteq X \times X$, s.t. $\leq$ is*

1. *reflexive, $x \leq x$*
2. *transitive, $x \leq y \Rightarrow y \leq z \Rightarrow x \leq z$*
3. *anti-symmetric, $x \leq y \Rightarrow y \leq x \Rightarrow x = y$*

For instance:

- powerset ordered by inclusion, e.g. $(\mathcal{P}\{1, 2\}, \subseteq)$



- natural numbers ordered by the divisibility relation $(\mathbb{N}, |)$
- acyclic directed graphs (V, E): the set is the set of vertices V and $v_1 \leq v_2$ iff v_2 is reachable from v_1, i.e. $(v_1, v_2) \in E^*$, the reflexive-transitive closure of the set of edges E

  anti-symmetry is implied by acyclicity

We can view this as a (concrete) category: we have objects X, morphisms $x \to y$ iff $x \leq y$.

Composition of arrows corresponds to transitivity of the order. The identity morphisms correspond to reflexivity.

This results in the following correspondences: ("dictionary")

elements of the poset	objects
order structure	morphisms

There are some specialties:

thin: a poset category is thin, meaning $|\mathsf{Hom}(x, y)| \leq 1$

skeletal: every two isomorphic objects are equal (we even have that every preorder has a poset as its skeleton)

small: the collection of objects is a set, not a (proper) class

Thin, skeletal, and small categories are exactly posets.

Let's continue with posets. Maximum object $\top$: $\forall x, x \leq \top$; dually, minimum object $\bot$: $\forall x, \bot \leq x$.

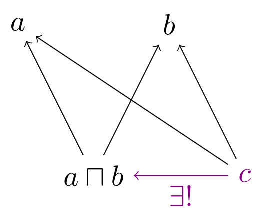
- in the above powerset poset, this is clear
- $(\mathbb{N}, |)$ does have the top object 0 (everything divides 0) and a bottom object (1)
- in DAGs: sources and sinks

These correspond to initial and terminal objects in category theory!

We can define (partial, in general) binary operations **meet** $\sqcap$ and **join** $\sqcup$ in a poset. $a \sqcap b$ is the maximal object satisfying $a \sqcap b \leq a \wedge a \sqcap b \leq b$. Imagine meets as infima and joins as suprema.

- the powerset lattice: meet is intersection and join is union
- DAGs: lowest common ancestor and highest common descendant, but these might not be unique and thus meet/join are partial
- natural numbers: gcd, lcm

As a diagram/universal property:



similarly for join.

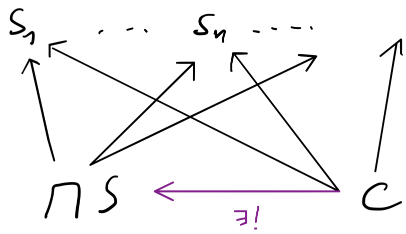
That diagram looks the same as the diagram for products!

(binary) meet	product
(binary) join	co-product

A poset with all binary joins and meets is called a *lattice*.

- the powerset poset is a lattice
- $(\mathbb{N}, |)$ is a lattice
- DAGs aren't in general

We can extend that to joins and meets of arbitrary sets $\subseteq X$.



- for natural numbers, the gcd of arbitrary sets exists, but only finite joins (lcm)
- the powerset lattice: join is union, meet is intersection

For instance: the empty meet is the *unique* maximum object $\top$ (if it exists).

In general: arbitary meets/joins are limits and co-limits.

arbitray meet	limit
arbitrary join	co-limit

Why? the commutativity condition of cones is trivial: every diagram in thin categories does automatically commute!

Lemma 1. *We have arbitrary joins iff we have arbitrary meets.*

Proof. For deriving arbitrary meets from joins, just take the join of all smaller-or-equal elements. (greatest lower bound = least upper bound of smaller elements)

Define $S' := \{s' \in X | \forall s \in S. s' \leq s\}$ and $\sqcap S := \sqcup S'$. First we show that $\sqcap S \leq s \forall s \in S$, or equivalently, that $\sqcup S' \leq s$.

Note that, if $s' \in S'$, then in particular $s' \leq s$. Thus, by the universal property of $\sqcup$, we have $\sqcup S' \leq s$.

For universality, assume that $c \leq s \forall s \in S$. We show $c \leq \sqcap S$. As $c \leq s \forall s \in S$, in particular $c \in S'$ by definition. So $c \leq \sqcup S' = \sqcap S$. □

Remark 1. *The correponding theorem for categories does not hold, i.e. a complete category (a category having all limits indexed by small categories) is not necessarily cocomplete, and vice versa. As an example, take the partially-ordered class of ordinal numbers as a category. It is cocomplete (for every set of ordinals S, $\bigcup S$ is again an ordinal), but it is not complete as it does not have a terminal object.*

A poset with arbitrary meets/joins is called a *complete lattice*. Example: the powerset lattice is complete.

Now let's look at two posets $(X, \leq_X)$ and $(Y, \leq_Y)$. Can we establish any relationship between them?

Maybe there's an order-preserving function between them: $\alpha : (X, \leq_X) \to (Y, \leq_Y)$ such that

$$\forall x\ y, x \leq y \to \alpha\ x \leq \alpha\ y.$$

Example: $(P, \leq_P) = (\mathcal{P}(\mathbb{Z}), \subseteq)$ and

$$(Q, \leq_Q) = (\{[l,h] | l \leq h \in \mathbb{Z} \cup \{-\infty, \infty\}\} \cup \{\bot\}, \leq)$$

($\bot$ is the minimum in $(Q, \leq_Y)$, $[-\infty, \infty]$ the maximum).

$$\alpha\ S := \begin{cases} \bot & S = \emptyset \\ [\inf\ S, \sup\ S] & \text{othw} \end{cases}$$

In the category-theoretic view, this is just a functor between the two poset categories. If $a : x \to y$, then $\alpha(a) : \alpha x \to \alpha y$, since α is order-preserving.

Galois Connections We'll consider *monotone* Galois connections here.

Consider again the order-preserving function from before. We can also map back.

$$\gamma(I) = \begin{cases} \emptyset & I = \bot \\ \{z \in \mathbb{Z} | l \leq z \leq h\} & I = [l, h] \end{cases}$$

$$\text{P} \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \text{Q}$$

In this particular example, we can view α as an "abstraction" function (from integer sets to integer intervals) and γ as a concretisation function.

We can show that: $\alpha(p) \leq q \iff p \leq \gamma(q)$.

This is a particular instance of a Galois connection.

Definition 2. *A (*monotone*!) Galois connection between two posets* $(P, \leq_P)$ *and* $(Q, \leq_Q$ *) are two monotone maps*

$$P \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} Q$$

such that $\alpha(p) \leq q \iff p \leq \gamma(q)$.

α *is also called a lower/left adjoint and* γ *an upper/right adjoint.*

This is actually equivalent to requiring

$$\alpha(\gamma q) \leq q$$
$$p \leq \gamma(\alpha p).$$

Essentially, this states that α is the best possible abstraction (the smallest interval containing everything) and γ is the largest possible concretisation.

We show one direction of this, using the equivalence from the definition:

$$\alpha(\gamma q) \leq q \iff \gamma q \leq \gamma q$$
$$p \leq \gamma(\alpha p) \iff \alpha p \leq \alpha p$$

Other quick examples:

- For a ($\mathbb{R}/\mathbb{C}$-) vector space V, consider the posets $(\mathcal{P}V, \subseteq)$ and $(\mathsf{Subspace}(V), \subseteq)$. We can define $\alpha(S)$ to be the span $\langle S \rangle$ generated by S and $\gamma(V')$ to be the underlying set. Then (α, γ) form a monotone Galois connection.
- monotone Galois connections naturally appear in computer science in abstract interpretation (in verification/invariant generation)

Lemma 2 (An instance of RAPL)**.** *The upper adjoint* γ *of a Galois connection* $(\alpha, \gamma) : (P, \leq_P) \leftrightarrows (Q, \leq_Q)$ *preserves meets.*

Proof. Let $\sqcap S$ in Q be given. We show that $\gamma(\sqcap S) = \sqcap S'$, where $S' := \{\gamma s | s \in S\}$ (and that the latter does actually exist).

- We have $\sqcap S \leq s \forall s \in S$. By monotonicity, $\gamma(\sqcap S) \leq \gamma s \forall s \in S$. Thus $\gamma(\sqcap S) \leq s' \forall s' \in S'$.
- We have to show: if $c \leq \gamma s \forall s \in S$, then $c \leq \gamma(\sqcap S)$, or equivalently $\alpha(c) \leq \sqcap S$.

  By the property of Galois connections, we have $c \leq \gamma s \iff \alpha c \leq s$ for all $s \in S$, which is equivalent to $\alpha c \leq \sqcap S$. This completes the proof.

□

Lemma 3. *The lower adjoint* α *of a Galois connection* (α, γ) *preserves joins.*

Galois connections | Adjoint functors