# Adjunctions

Category Theory Seminar SS20 at University of Saarland

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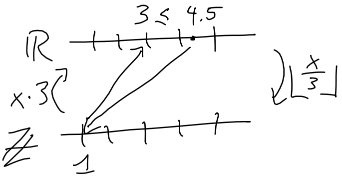
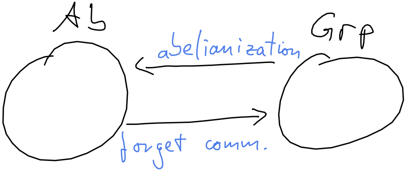
# Motivation

- *"Adjoint functors arise everywhere"* – Saunders MacLane
- Two functors that are not inverse, but almost
  - category of integers, dividing und multiplying by a constant
  - category of groups and abelian groups. Abelization goes one way and inclusion the other but you don't end up in the same spot.

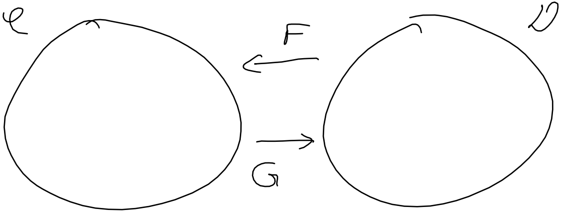


# Definitions

## Definition

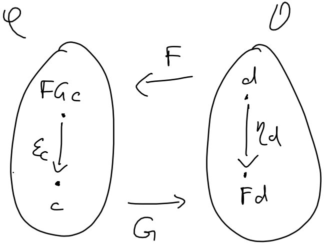
- Categories $\mathcal{C}$, $\mathcal{D}$
- left-adjoint functor $\mathcal{D} \xrightarrow{F} \mathcal{C}$
- right-adjoint functor $\mathcal{C} \xrightarrow{G} \mathcal{D}$
- $F \dashv G$
- $F, G$ are unique up to unique isomorphism

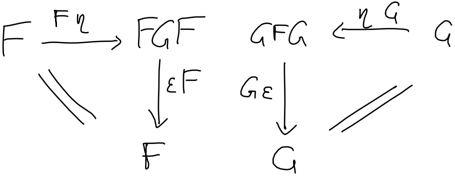


# #1 Definition using unit/counit

## Definition

- left-adjoint $\mathcal{D} \xrightarrow{F} \mathcal{C}$, right-adjoint $\mathcal{C} \xrightarrow{G} \mathcal{D}$
- Natural transformations $1_{\mathcal{D}} \xrightarrow{\eta} GF$, $FG \xrightarrow{\epsilon} 1_{\mathcal{C}}$
- Satisfying the **triangle equalities**

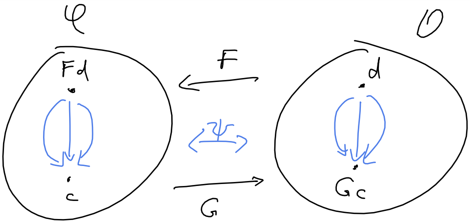


# #2 Definition using hom-functors

## Definition

- left-adjoint $\mathcal{D} \xrightarrow{F} \mathcal{C}$, right-adjoint $\mathcal{C} \xrightarrow{G} \mathcal{D}$
- Natural isomorphism of hom-functors
  $Hom_{\mathcal{C}}(Fd, c) \;\simeq\; Hom_{\mathcal{D}}(d, Gc)$
- $\Psi_{xy} : Hom_{\mathcal{C}}(Fd, c) \to Hom_{\mathcal{D}}(d, Gc)$
- $\bar{f}$ notation for $\Psi$, $\bar{\bar{f}} = f$



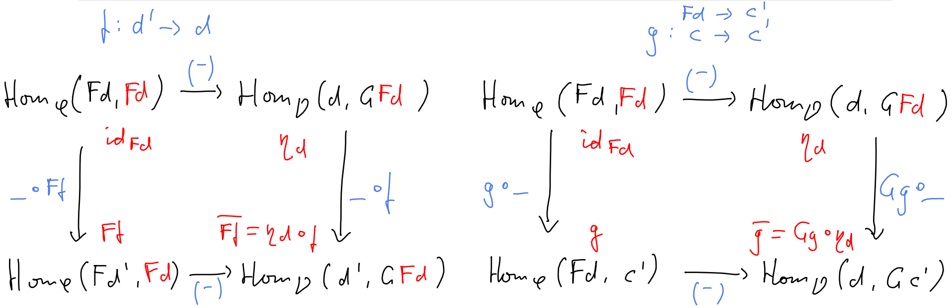
# homsets → (co)unit

**Proof.**

Given $Hom_{\mathcal{C}}(Fd, c) \simeq Hom_{\mathcal{D}}(d, Gc)$, define $\eta_d = \overline{id_{Fd}}$, $\epsilon_c = \overline{id_{Gc}}$

$$(1) \quad \overline{Ff} = \eta_d \circ f \qquad (2) \quad \overline{f} = \epsilon_c \circ Ff$$

$$(3) \quad \overline{g} = Gg \circ \eta_d \qquad (4) \quad \overline{Gg} = g \circ \epsilon_c$$

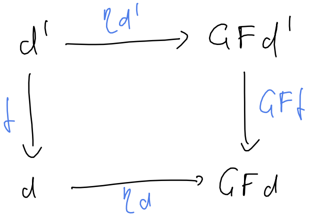


# homsets → (co)unit

**Proof.**

$$(1)\quad \overline{Ff} = \eta_d \circ f \qquad (2)\quad \overline{f} = \epsilon_c \circ Ff$$
$$(3)\quad \overline{g} = Gg \circ \eta_d \qquad (4)\quad \overline{Gg} = g \circ \epsilon_c$$

- $\eta$ is a natural transformation, $GFf \circ \eta_{d'} = \eta_d \circ f$ using (1), (3)
- triangle equalities using (2), (3) and naturality of $\eta, \epsilon$




$$GFf \circ \eta_{d'} = \eta_d \circ f$$
$$\overline{Ff} = \overline{Ff}$$

$$\varepsilon_{Fd} \circ F\eta_d = id_{Fd}$$
$$\overline{\eta_d} = id_{Fd}$$
$$\eta_d = \overline{id_{Fd}}$$

# (co)unit $\to$ homsets

**Proof sketch.**

Given $1_{\mathcal{D}} \xrightarrow{\eta} GF$, $FG \xrightarrow{\epsilon} 1_{\mathcal{C}}$ define

- $\Psi_{dc}\left(g : Fd \to c\right) = Gg \circ \eta_d$
- $\Psi_{dc}^{-1}\left(f : d \to Gc\right) = \epsilon_c \circ Ff$

Proof needs triangle equality and naturality of $\eta, \epsilon$

Or use the Yoneda Lemma (Brandenburg p. 193) □

# Galois Connections

- In a poset category $(A, \leq)$ there is at most one morphism between objects
- $Hom_{\mathcal{C}}(Fd, c) \;\simeq\; Hom_{\mathcal{D}}(d, Gc)$
- $Fd \leq c \Leftrightarrow d \leq Gc$
- So a (monotone) galois connection is a special adjunction

# Coproduct $\dashv \Delta \dashv$ Product

## Definition

diagonal functor $\Delta : \mathcal{C} \xrightarrow{\Delta} \mathcal{C} \times \mathcal{C}$ with $\Delta c = \langle c, c \rangle$
product functor $\mathbf{Prod} : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ with $\mathbf{Prod}\langle a, b \rangle = a \times b$

Per universal construction for any $\langle c, c \rangle \xrightarrow{\langle p,q \rangle} \langle a, b \rangle$ there exists a unique $c \xrightarrow{h} a \times b$ with $\pi_1 \circ h = p, \pi_2 \circ h = q$.
For any $c \xrightarrow{h} a \times b$ there exist $c \xrightarrow{\pi_1 \circ h} a, c \xrightarrow{\pi_2 \circ h} b$.
So we can define $\langle p, q \rangle \mapsto h,\ h \mapsto \langle \pi_1 \circ h, \pi_2 \circ h \rangle$
$Hom_{\mathcal{C} \times \mathcal{C}}(\langle c, c \rangle, \langle a, b \rangle) \ \simeq \ Hom_{\mathcal{C}}(c, a \times b)$

# Coproduct $\dashv \Delta \dashv$ Product

- $Hom_{\mathcal{C}\times\mathcal{C}}(\langle c, c\rangle, \langle a, b\rangle) \ \simeq \ Hom_{\mathcal{C}}(c, a \times b)$
- restatement of the universal construction of a product
- any right adjoint of the diagonal functor is the product
- coproduct analogous but on the left of $\Delta$

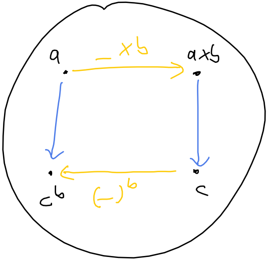
# Tensor-Hom Adjunction

Works for any category with tensor product and internal hom (object that corresponds to the homset). Here product and function type.

- Endofunctors $(- \times b)$ and $(b \Rightarrow -)$
- $Hom_{\mathcal{C}}(a \times b, c) \simeq Hom_{\mathcal{C}}(a, b \Rightarrow c)$
- Known as **currying** in programming.

```
curry :: ((a, b) → c) → (a → (b → c))
curry f = λa ⇒ λb ⇒ f (a, b)

uncurry :: (a → (b → c)) → ((a, b) → c)
uncurry g = λ(a, b) ⇒ g a b
```



# Free $\dashv$ Forgetful

## Definition

For a construct $(A, U)$ if free objects exist we can define a free functor $F \dashv U$
E.g. free group, free monoid, free category

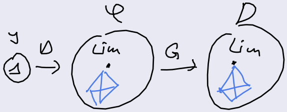
## Example

In **Mon** the free monoid $\Sigma^*$ for a set of generators $\Sigma$ is the set of all finite sequences of elements of $\Sigma$ with concatenation as the operation.
$\{a, b\} \mapsto (\{\epsilon, a, b, aa, ab, ba, bb, \ldots\}, \circ)$
homset isomorphism: any mapping of the generators to another set defines a monoid homomorphism and vice-versa

# Right Adjoints Preserve Limits

## Theorem (RAPL)

- *functors* $\mathcal{D} \xrightarrow{F} \mathcal{C}$, $\mathcal{C} \xrightarrow{G} \mathcal{D}$
- *index category* $\mathcal{I}$, *diagram* $\mathcal{I} \xrightarrow{D} \mathcal{C}$




$$G(Lim\ D) \simeq Lim\ (G \circ D)$$

## Proof using homset isomorphism $\overline{g}$.

$(Lim\ D, \lambda_i)$ is a cone for $D$, therefore $(G(Lim\ D), G(\lambda_i))$ is a cone for $GD$. Take any cone $(X, \mu_i)$ for $GD$. We find a unique factorizing $X \xrightarrow{h} G(Lim\ D)$ so that $G\lambda_i \circ h = \mu_i$.
$(FX, \overline{\mu_i})$ is a cone for $D$, therefore there exists a unique factorizing morphism $FX \xrightarrow{\overline{\phi}} Lim\ D$. Now put $h := \phi$. Naturality tells us $G\lambda_i \circ \phi = \mu_i$. □

# $G(Lim\ D) \simeq Lim(GD)$

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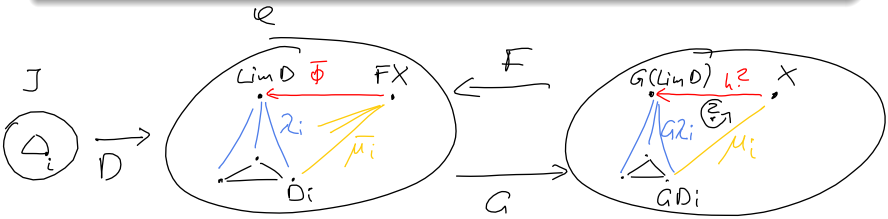
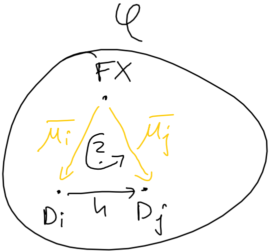
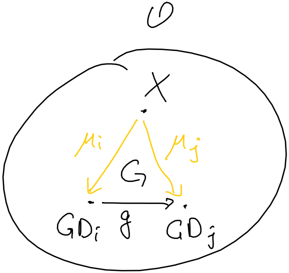


# $(FX, \overline{\mu_i})$ is a cone for $D$

**Proof.**

Need to show for any $D_i \xrightarrow{h} D_j$ that $h \circ \overline{\mu_i} = \overline{\mu_j}$. We know for any $GD_i \xrightarrow{g} GD_j$ that $g \circ \mu_i = \mu_j$, therefore $Gh \circ \mu_i = \mu_j$.
Then using the homset isomorphism $\overline{\mu_j} = \overline{Gh \circ \mu_i} = h \circ \overline{\mu_i}$ where the last equality follows from naturality. □



# RAPL & LAPC

You see this pattern in a lot of different fields of math

- Products/coproducts/exponentials are also limits. So you get some algebraic laws.
  - $U \otimes (V \oplus W) \ \simeq \ (U \otimes V) \oplus (U \otimes W)$
  - $c^{a+b} \ \simeq \ c^a \times c^b$
- Free group on disjoint union is free product of free groups
  - $F(A \sqcup B) \ \simeq \ F(A) * F(B)$
- For a function $f : A \to B$ the function $f^{-1} : \mathcal{P}(B) \to \mathcal{P}(A)$ is is left adjoint to $f_* : \mathcal{P}(A) \to \mathcal{P}(B)$
  - $f^{-1}(\bigcup_i B_i) = \bigcup_i f^{-1}(B_i)$

# Adjunct Functor Theorem(s)

## Definition

Right adjoint functors preserve all limits that exist in their domain.
An adjoint functor theorem is a statement that (under certain conditions) the converse holds: a functor $\mathcal{C} \xrightarrow{G} \mathcal{D}$ which preserves limits is a right adjoint.
In the general theorem the conditions are that $\mathcal{C}$ has small limits and is small and that some morphisms constituting the **solution set criterion** for $G$ exist.

# Restricts to Isomorphism of Subcategories

## Definition

A fixpoint of $\eta$ is a $d \in \mathcal{D}$ so that $\eta_d : d \to G(F(d))$ is an isomorphism. $Fix(\eta)$ is all such fixpoints. Analogous for $\epsilon$.

## Theorem

*$Fix(\eta), Fix(\epsilon)$ are subcategories of $\mathcal{D}, \mathcal{C}$.*
*And there exists an equivalence of categories $Fix(\eta) \simeq Fix(\epsilon)$.*

## Example

The functor that maps a vector space to its dual
$D : \mathbf{Vect}_K^{op} \to \mathbf{Vect}_K$ is left-adjoint to $D^{op} : \mathbf{Vect}_K \to \mathbf{Vect}_K^{op}$.
The unit is the embedding of a space $V$ into its bidual space $V^{**}$
The fixpoints are the finite-dimensional vector spaces. So we get that $\mathbf{FinVect}_K \simeq \mathbf{FinVect}_K^{op}$

# Monads

**Definition**

A Monad is an endofunctor $\mathcal{D} \xrightarrow{T} \mathcal{D}$ with two natural transformations $1_{\mathcal{D}} \xrightarrow{\eta} T$ and $T^2 \xrightarrow{\mu} T$ and some coherence laws.

**Theorem**

*Any pair of adjoint functors $F, G$ gives rise to a monad, namely $G \circ F : \mathcal{D} \to \mathcal{D}$.*
*$\eta$ stays the same. $\mu$ can be defined by $T^2 = GFGF \xrightarrow{G\epsilon F} GF = T$.*
*Cohence laws follow from triangle equalities.*

**Theorem**

*For any monad $T$ we can find multiple adjunctions that give give to it. A whole category even!*

# Summary



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