Definitions for Adjunctions

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Two functors $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ are called *adjoint*

1. if there are two *natural transformations*

$$\eta : 1_{\mathcal{D}} \to GF \qquad \varepsilon : FG \to 1_{\mathcal{C}}$$

satisfying the *triangle equalities*

$$\varepsilon_{Fd} \circ F\eta_d = id_{Fd} \qquad G\varepsilon_c \circ \eta_{Gc} = id_{Gc}$$

2. if there is a *natural isomorphism* of homfunctors

$$Hom_{\mathcal{C}}(Fd, c) \simeq Hom_{\mathcal{D}}(d, Gc)$$

that is natural in c & d

F is *left-adjoint* and G is *right-adjoint*

F & G are unique up to unique isomorphism

$$F \dashv G \,,\; F \dashv G' \Rightarrow G \simeq G'$$

Diagonal functor $\mathcal{C} \xrightarrow{\Delta} \mathcal{C} \times \mathcal{C}$, $\Delta c = \langle c, c \rangle$

Product functor $\mathcal{C} \times \mathcal{C} \xrightarrow{Prod} \mathcal{C}$, $Prod\langle a, b \rangle = a \times b$

a single object

A *free functor* is defined to be left-adjoint to a *forgetful functor*

Right Adjoints Preserve Limits

For any diagram $J \xrightarrow{D} \mathcal{C}$ we have

$$\text{RAPL}: \quad G(Lim\, D) = Lim\,(G \circ D)$$

dually *left adjoints preserve colimits*.

A *contravariant functor* turns a *colimit* into a *limit*.

An *Adjunct Functor Theorem* allows you to derive an adjoint functor F from a given functor G

A *Monad* is a functor $T : \mathcal{D} \to \mathcal{D}$ with natural transformations $\eta : 1_{\mathcal{D}} \to T$ and $\mu : T^2 \to T$

and some coherence laws

$$\begin{tikzcd}
T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\
T^2 \arrow[r, "\mu"'] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r, "T\eta"] \arrow[dr, equal] & T^2 \arrow[d, "\mu"] & T \arrow[l, "\eta T"'] \arrow[dl, equal] \\
& T &
\end{tikzcd}$$

Every adjunction gives rise to a monad *$G \circ F$*