Monoidal 🐱

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Monoid

Definition

Definition (Monoid)
Let C be a set with $1 \in C$ and $\cdot : C \times C \to C$, such that

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$
$$a \cdot 1 = 1 \cdot a = a$$

then $\langle C, \cdot, 1 \rangle$ is a *monoid*.

Examples

- Strings with concatenation and the empty string

$$(\text{“}\mathit{foo}\text{”} +\!\!+ \text{“}\mathit{bar}\text{”}) +\!\!+ \text{“}\mathit{baz}\text{”} = \text{“}\mathit{foo}\text{”} +\!\!+ (\text{“}\mathit{bar}\text{”} +\!\!+ \text{“}\mathit{baz}\text{”})$$

$$\text{“}\mathit{foo}\text{”} +\!\!+ \text{“”} = \text{“}\mathit{foo}\text{”} = \text{“”} +\!\!+ \text{“}\mathit{foo}\text{”}$$

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- Integers with addition and 0

$$(123 + 456) + 789 = 123 + (456 + 789)$$
$$42 + 0 = 42 = 0 + 42$$

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- Integers with addition and 0

$$(123 + 456) + 789 = 123 + (456 + 789)$$
$$42 + 0 = 42 = 0 + 42$$

- Sets with union and the empty set

$$(\{A, B\} \cup \{B, C\}) \cup \{A, D\} = \{A, B\} \cup (\{B, C\} \cup \{A, D\})$$
$$\{A\} \cup \emptyset = \{A\} = \emptyset \cup \{A\}$$

Strict Monoidal Categories

Definition

Let C be a category with $1 \in C$ and a bifunctor $\otimes : C \times C \to C$, such that

$$(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$$
$$1 \otimes X = X \otimes 1 = X$$

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and

$$a : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) = id$$
$$\iota : 1 \otimes 1 \xrightarrow{\sim} 1 = id$$

then $\langle C, \otimes, a, 1, \iota \rangle$ is a *strict monoidal category*.

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Let C be a category, then $End(C)$ is a strict monoidal category.

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  by composition laws.
- $1 \otimes 1 = 1$ by identity laws.

$\square$

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Let $\mathbb{F}$ be a field, then $Mat(\mathbb{F})$ is a strict monoidal category

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- 1 by single 1-matrix

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Proof.

- $n \otimes m = n \cdot m$, $f \otimes g$ by Kronecker product
- 1 by single 1-matrix
- $\forall X, Y, Z \in C.\ (X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$ by properties of Kronecker product
- $1 \otimes 1 = 1$ obvious

$\square$

Non-Strict Monoidal Categories

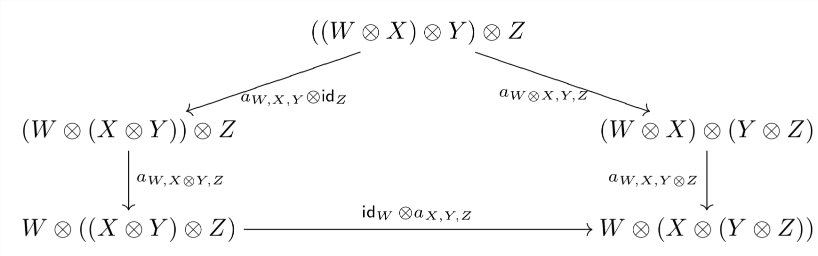
Definition

Definition (Monoidal Category)
Let C be a category with $1 \in C$, a bifunctor $\otimes : C \times C \to C$, and

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$$
$$\iota : 1 \otimes 1 \xrightarrow{\sim} 1$$

satisfying the *pentagon* and the *unit axiom*, then $\langle C, \otimes, a, 1, \iota \rangle$ is a *monoidal category*

The diagram



commutes

Unit Axiom

The functors

$$L_1 : X \mapsto 1 \otimes X$$
$$R_1 : X \mapsto X \otimes 1$$

are autoequivalences of C.

Unit constraints

Define natural isomorphisms

$$l_X : 1 \otimes X \to X$$
$$r_x : X \otimes 1 \to X$$

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$$r_x : X \otimes 1 \to X$$

such that

$$L_1(l_x) = 1 \otimes (1 \otimes X) \xrightarrow{a^{-1}_{1,1,X}} (1 \otimes 1) \otimes X \xrightarrow{\iota \otimes \mathrm{id}_X} 1 \otimes X$$
$$R_1(r_X) = (X \otimes 1) \otimes 1 \xrightarrow{a_{X,1,1}} X \otimes (1 \otimes 1) \xrightarrow{\mathrm{id}_X \otimes \iota} X \otimes 1$$

Unit constraints

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$$R_1(r_X) = (X \otimes 1) \otimes 1 \xrightarrow{a_{X,1,1}} X \otimes (1 \otimes 1) \xrightarrow{\mathrm{id}_X \otimes \iota} X \otimes 1$$

then l_X is the left and r_X is the right *unit constraint*

Unit constraints

For all $X \in C$

$$l_{1 \otimes x} = id_1 \otimes l_X$$
$$r_{X \otimes 1} = r_X \otimes id_1$$

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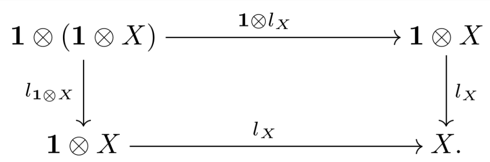
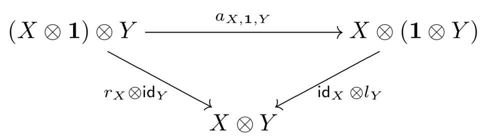
Proof.



□

Triangle

The diagram



commutes

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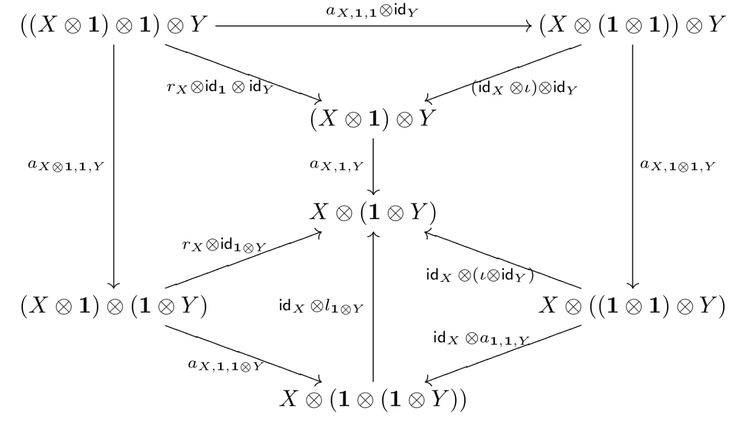
$((X \otimes \mathbf{1}) \otimes \mathbf{1}) \otimes Y$
$a_{X,\mathbf{1},\mathbf{1}} \otimes \mathrm{id}_Y$
$(X \otimes (\mathbf{1} \otimes \mathbf{1})) \otimes Y$
$r_X \otimes \mathrm{id}_{\mathbf{1}} \otimes \mathrm{id}_Y$
$(\mathrm{id}_X \otimes \iota) \otimes \mathrm{id}_Y$
$(X \otimes \mathbf{1}) \otimes Y$
$a_{X \otimes \mathbf{1},\mathbf{1},Y}$
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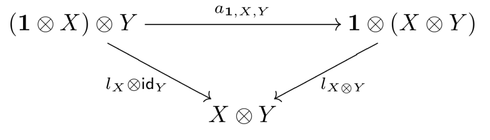
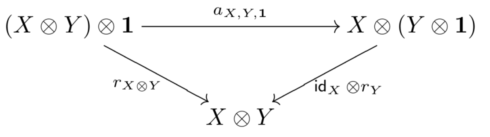
$X \otimes (\mathbf{1} \otimes Y)$
$r_X \otimes \mathrm{id}_{\mathbf{1} \otimes Y}$
$\mathrm{id}_X \otimes (\iota \otimes \mathrm{id}_Y)$
$(X \otimes \mathbf{1}) \otimes (\mathbf{1} \otimes Y)$
$\mathrm{id}_X \otimes l_{\mathbf{1} \otimes Y}$
$X \otimes ((\mathbf{1} \otimes \mathbf{1}) \otimes Y)$
$\mathrm{id}_X \otimes a_{\mathbf{1},\mathbf{1},Y}$
$a_{X,\mathbf{1},\mathbf{1} \otimes Y}$
$X \otimes (\mathbf{1} \otimes (\mathbf{1} \otimes Y))$

□

Triangles

The diagrams





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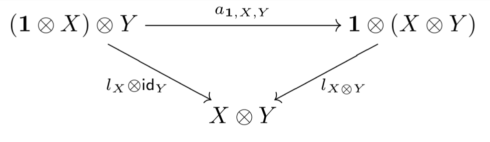

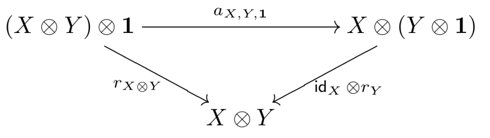

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Proof.
similar □

Proposition

Corollary
In any monoidal category $l_1 = r_1 = \iota$

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Proof.
Set $X = Y = 1$ *in the second triangle. We have*

$$l_1 \otimes id_1 = l_{1 \otimes 1} \circ a_{1,1,1} = (id_1 \otimes l_1) \circ a_{1,1,1}$$

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By unit constraint

$$(id_1 \otimes l_1) \circ a_{1,1,1} = \iota \otimes id_1$$

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Hence

$$r_1 \otimes id_1 = l_1 \otimes id_1 = \iota \otimes id_1$$

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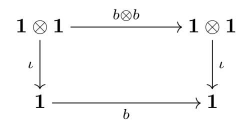

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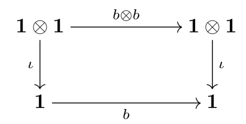
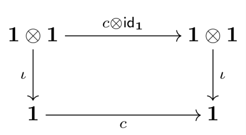


Note that



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Sets with the Cartesian product and the singleton set are a monoidal category.

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Example
$\mathbb{K}$-**Vec** with $\otimes = \otimes_{\mathbb{K}}$ and $1 = \mathbb{K}$ is a monoidal category.

Mac Lane's strictness theorem

Theorem (Mac Lane)
Any monoidal category is monoidally equivalent to a strict monoidal category

Mac Lane's strictness theorem

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Proof.
Tensor Categories - P. 37 □

Rigid Monoidal Categories

Duals

Definition (Left Dual)
Let C be a monoidal category and $X \in C$. If there exist $ev_X : X^* \otimes X \to 1$ and $coev_X : 1 \to X \otimes X^*$ such that

$$X \xrightarrow{coev_X \otimes id_X} (X \otimes X^*) \otimes X \xrightarrow{a_{X,X^*,X}} X \otimes (X^* \otimes X) \xrightarrow{id_X \otimes ev_X} X$$

$$X^* \xrightarrow{id_{X^*} \otimes coev_X} X^* \otimes (X \otimes X^*) \xrightarrow{a^{-1}_{X^*,X,X^*}} (X^* \otimes X) \otimes X^* \xrightarrow{ev_X \otimes id_{X^*}} X^*$$

then X^* is the *left dual* of X.

Definition (Right Dual)
Let C be a monoidal category and $X \in C$. If there exist $ev'_X : X \otimes {}^*X \to 1$ and $coev'_X : 1 \to {}^*X \otimes X$ such that

$$X \xrightarrow{id_X \otimes coev'_X} X \otimes ({}^*X \otimes X) \xrightarrow{a^{-1}_{X,{}^*X,X}} (X \otimes {}^*X) \otimes X \xrightarrow{ev'_X \otimes id_X} X$$

$${}^*X \xrightarrow{coev'_X \otimes id_{{}^*X}} ({}^*X \otimes X) \otimes {}^*X \xrightarrow{a_{{}^*X,x,{}^*X}} {}^*X \otimes (X \otimes {}^*X) \xrightarrow{id_{{}^*X} \otimes ev'_X} X$$

then ${}^*X$ is the *right dual* of X.

Examples

Example

$\mathbb{K}$-Vec is rigid.

- Dual of V is the dual vector space V^*
- $ev_V : V^* \otimes V \to \mathbb{K}$ by contraction
- $coev_V : \mathbb{K} \to V^* \otimes V$ by embedding

Unique by duals

If $X \in C$ has a left / right dual, then it is unique up to a unique isomorphism.

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If $X \in C$ has a left / right dual, then it is unique up to a unique isomorphism. Define $\alpha : X_1^* \to X_2^*$ by

$$X_1^* \xrightarrow{id_{X_1^*} \otimes c_2} X_1^* \otimes (X \otimes X_2^*) \xrightarrow{a^{-1}_{X_1^*, X, X_2^*}} (X_1^* \otimes X) \otimes X_2^* \xrightarrow{e_1 \otimes id_{X_2^*}} X_2^*$$

and $\beta : X_2^* \to X_1^*$ similar.

If $X \in C$ has a left / right dual, then it is unique up to a unique isomorphism. Define $\alpha : X_1^* \to X_2^*$ by

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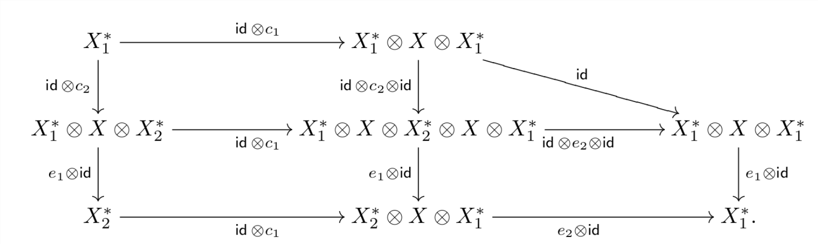
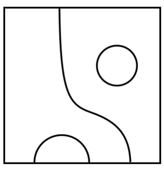
and $\beta : X_2^* \to X_1^*$ similar.
Show $\beta \circ \alpha$ and $\alpha \circ \beta$ are *id*.



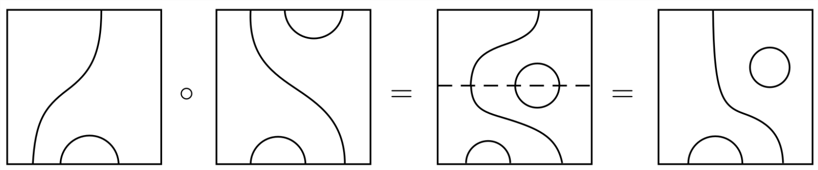
Temperley-Lieb categories

Diagrams

Definition (simple Temperley-Lieb diagram)
Let m, n be non-negative. Consider unit square with m and n points on top and bottom. A *simple Temperley-Lieb diagram* consists of smooth, non-crossing arcs between top and bottom.



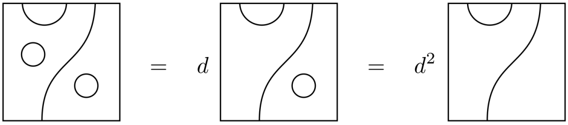
Composition



Category

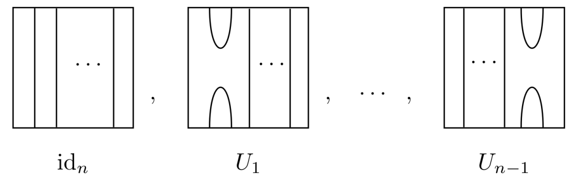
The *Temperley-Lieb category* is a category where

- Objects are non-negative $\mathbb{N}$
- *Hom*(m, n) is the $\mathbb{F}$-linear span of diagrams from m to n, modulo d-equivalence.



Generators

All morphisms $f \in Hom(m, n)$ can be written canonically



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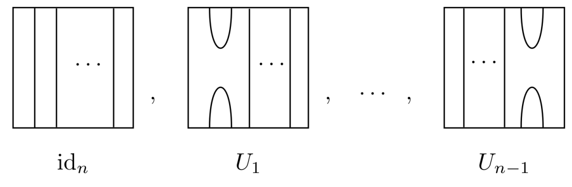
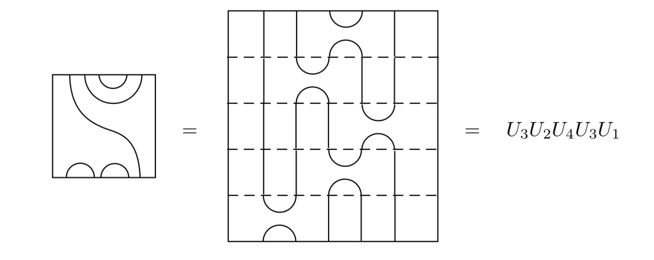
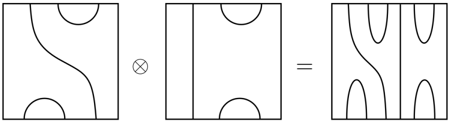




Strict Monoidal

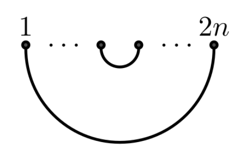
The Temperley-Lieb category is a strict monoidal category where

- 1 is the 0-diagram
- for all objects $a, b : \mathbb{N}$, $a \otimes b$ is $a + b$
- for all diagrams $f : m_1 \to n_1, g : m_2 \to n_2$,
  $f \otimes g : m_1 \otimes m_2 \to n_1 \otimes n_2$ is juxtaposition



Rigid

Definition (unit)
For all n, the arrow $\eta_n : 0 \to n + n$



is the *unit* (*counit* $\epsilon_n : n + n \to 0$ similar).

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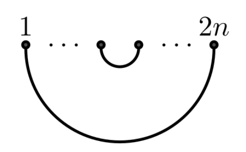

is the *unit* (*counit* $\epsilon_n : n + n \to 0$ similar).

Using this unit, all objects are self-dual.

Monoidal Functors

Definition

Let $\langle C_1, \otimes_1, 1_1, a_1, \iota_1 \rangle$ and $\langle C_2, \otimes_2, 1_2, a_2, \iota_2 \rangle$ be monoidal categories, $F : C_1 \to C_2$ be a functor with a natural isomorphism

$$J_{X,Y} : F(X) \otimes_2 F(Y) \xrightarrow{\sim} F(X \otimes_1 Y)$$

such that $F(1_1)$ is isomorphic to 1_2 and the diagram

$$\begin{array}{ccc}
(F(X) \otimes^\wr F(Y)) \otimes^\wr F(Z) & \xrightarrow{a^\wr_{F(X),F(Y),F(Z)}} & F(X) \otimes^\wr (F(Y) \otimes^\wr F(Z)) \\
\Big\downarrow{\scriptstyle J_{X,Y} \otimes^\wr \mathsf{id}_{F(Z)}} & & \Big\downarrow{\scriptstyle \mathsf{id}_{F(X)} \otimes^\wr J_{Y,Z}} \\
F(X \otimes Y) \otimes^\wr F(Z) & & F(X) \otimes^\wr F(Y \otimes Z) \\
\Big\downarrow{\scriptstyle J_{X \otimes Y, Z}} & & \Big\downarrow{\scriptstyle J_{X, Y \otimes Z}} \\
F((X \otimes Y) \otimes Z) & \xrightarrow{F(a_{X,Y,Z})} & F(X \otimes (Y \otimes Z))
\end{array}$$

commutes, then $\langle F, H \rangle$ is a *monoidal functor*.

Example

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Forgetful functors are monoidal.