Monoidal Categories

Main Definitions

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Def 1 (Monoid) *Let C be a set with $\cdot : C \times C \to C$ and an element $1 \in C$, such that*

$$\forall a, b, c \in C.\ (a \cdot b) \cdot c = a \cdot (b \cdot c) \tag{1}$$
$$\forall a \in C.\ 1 \cdot a = a \cdot 1 = a \tag{2}$$

then $\langle C, \cdot, 1 \rangle$ is a monoid*.*

Def 2 (Monoidal Category) *Let C be a category with a bifunctor $\otimes : C \times C \to C$, an element $1 \in C$ and*

$$a : \forall X, Y, Z \in C.\ (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \tag{3}$$
$$\iota : 1 \otimes 1 \xrightarrow{\sim} 1 \tag{4}$$

satisfying the Pentagon Axiom and the Unit Axiom, then $\langle C, \otimes, a, 1, \iota \rangle$ is a monoidal category*.*

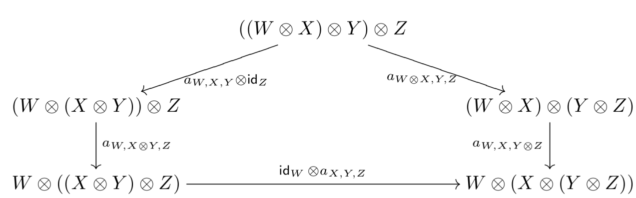
Def 3 (Strict Monoidal Category) *Let C be a monoidal category, such that*

$$\forall X, Y, Z \in C.\ (X \otimes Y) \otimes Z = X \otimes (Y \otimes Z) \tag{5}$$
$$\forall Z.\ 1 \otimes X = X \otimes 1 = X \tag{6}$$

and the associativity and the Unit Constraints are identity maps, then C is a strict monoidal category*.*

Def 4 (Rigid Monoidal Category) *Let C be a monoidal category. If every object has left and right duals, then C is a* rigid category*.*

Def 5 (Pentagon Axiom) *For all $X, Y, Z, W \in C$ the diagram*



commutes

Def 6 (Unit Axiom) *The functors*

$$L_1 : X \mapsto 1 \otimes X \tag{7}$$
$$R_1 : X \mapsto X \otimes 1 \tag{8}$$

are autoequivalences of C.

Def 7 (Unit Object) *The pair $\langle 1, \iota \rangle$ is the* unit object *of C.*

Def 8 (Unit Constraints) *Define natural isomorphisms*

$$l_X : 1 \otimes X \to X \tag{9}$$
$$r_x : X \otimes 1 \to X \tag{10}$$

such that

$$L_1(l_x) = 1 \otimes (1 \otimes X) \xrightarrow{a^{-1}_{1,1,X}} (1 \otimes 1) \otimes X \xrightarrow{\iota \otimes id_X} 1 \otimes X \tag{11}$$
$$R_1(r_X) = (X \otimes 1) \otimes 1 \xrightarrow{a_{X,1,1}} X \otimes (1 \otimes 1) \xrightarrow{id_X \otimes \iota} X \otimes 1 \tag{12}$$

then l_X is the left and r_X is the right unit constraint

Def 9 (Triangle axiom) *The diagram*

$$\begin{array}{ccc} (X \otimes \mathbf{1}) \otimes Y & \xrightarrow{a_{X,\mathbf{1},Y}} & X \otimes (\mathbf{1} \otimes Y) \\ & {\scriptstyle r_X \otimes \mathsf{id}_Y} \searrow \quad \swarrow {\scriptstyle \mathsf{id}_X \otimes l_Y} & \\ & X \otimes Y & \end{array}$$

commutes.

Remark 10 (More triangles) *The diagrams*

$$\begin{array}{ccc} (\mathbf{1} \otimes X) \otimes Y & \xrightarrow{a_{\mathbf{1},X,Y}} & \mathbf{1} \otimes (X \otimes Y) \\ & {\scriptstyle l_X \otimes \mathsf{id}_Y} \searrow \quad \swarrow {\scriptstyle l_{X \otimes Y}} & \\ & X \otimes Y & \end{array}$$

$$\begin{array}{ccc} (X \otimes Y) \otimes \mathbf{1} & \xrightarrow{a_{X,Y,\mathbf{1}}} & X \otimes (Y \otimes \mathbf{1}) \\ & {\scriptstyle r_{X \otimes Y}} \searrow \quad \swarrow {\scriptstyle \mathsf{id}_X \otimes r_Y} & \\ & X \otimes Y & \end{array}$$

commute.

Def 11 (Dual Objects) *Let C be a monoidal category and $X \in C$. If there exist $ev_X : X^* \otimes X \to 1$ and $coev_X : 1 \to X \otimes X^*$ such that*

$$X \xrightarrow{coev_X \otimes id_X} (X \otimes X^*) \otimes X \xrightarrow{a_{X,X^*,X}} X \otimes (X^* \otimes X) \xrightarrow{id_X \otimes ev_X} X \tag{13}$$

$$X^* \xrightarrow{id_{X^*} \otimes coev_X} X^* \otimes (X \otimes X^*) \xrightarrow{a^{-1}_{X^*,X,X^*}} (X^* \otimes X) \otimes X^* \xrightarrow{ev_X \otimes id_{X^*}} X^* \tag{14}$$

then X^* is the left dual *of X.*
If there exist $ev'_X : X \otimes {}^*X \to 1$ and $coev'_X : 1 \to {}^*X \otimes X$ such that

$$X \xrightarrow{id_X \otimes coev'_X} X \otimes ({}^*X \otimes X) \xrightarrow{a^{-1}_{X,{}^*X,X}} (X \otimes {}^*X) \otimes X \xrightarrow{ev'_X \otimes id_X} X \tag{15}$$

$${}^*X \xrightarrow{coev'_X \otimes id_{{}^*X}} ({}^*X \otimes X) \otimes {}^*X \xrightarrow{a_{{}^*X,x,{}^*X}} {}^*X \otimes (X \otimes {}^*X) \xrightarrow{id_{{}^*X} \otimes ev'_X} X \tag{16}$$

then ${}^*X$ is the right dual *of X.*

Def 12 (Monoidal Functor) *Let $\langle C, \otimes, 1, a, \iota \rangle$ and $\langle C', \otimes', 1', a', \iota' \rangle$ be monoidal categories, $F : C \to C'$ be a functor with a natural isomorphism*

$$J_{X,Y} : F(X) \otimes' F(Y) \xrightarrow{\sim} F(X \otimes' Y) \tag{17}$$

such that $F(1)$ is isomorphic to $1'$ and the diagram

$$\begin{array}{ccc}
(F(X) \otimes^{\wr} F(Y)) \otimes^{\wr} F(Z) & \xrightarrow{a^{\wr}_{F(X),F(Y),F(Z)}} & F(X) \otimes^{\wr} (F(Y) \otimes^{\wr} F(Z)) \\
\downarrow {\scriptstyle J_{X,Y} \otimes^{\wr} \mathsf{id}_{F(Z)}} & & \downarrow {\scriptstyle \mathsf{id}_{F(X)} \otimes^{\wr} J_{Y,Z}} \\
F(X \otimes Y) \otimes^{\wr} F(Z) & & F(X) \otimes^{\wr} F(Y \otimes Z) \\
\downarrow {\scriptstyle J_{X \otimes Y,Z}} & & \downarrow {\scriptstyle J_{X,Y \otimes Z}} \\
F((X \otimes Y) \otimes Z) & \xrightarrow{F(a_{X,Y,Z})} & F(X \otimes (Y \otimes Z))
\end{array}$$

commutes, then $\langle F, H \rangle$ is a monoidal functor.