Enriched Categories

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Proseminar “Category Theory”, Summer 2020

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Example: **Vec**

$$\mathbb{R}^5 \xrightarrow{f} \mathbb{R}^3$$

- The morphisms between two objects are a *Set*
  - $\text{Hom}(\mathbb{R}^5, \_) : \textbf{Vec} \to \textbf{Set}$
- **Set** is special. Can we try to get away from that?

Special structure of morphisms

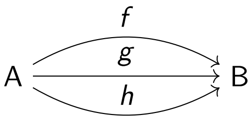
- We often have special structure in morphisms
- A notion of “combining two morphisms” into one
  - Concatenation: $\circ : \mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$
  - Tensoring $\otimes : \mathrm{Hom}(A, B) \otimes \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$
- Can we generalize the notion of *morphisms* to reflect that?

Motivation for Enriched Categories

Two central questions:

1. Can we generalize over **Set**?
2. Can we employ special structure of morphisms ($\Rightarrow$ *relationships*) between objects

Answer: Instead of sets of morphisms (*hom-sets*)



we use *hom-objects from another category* $\mathcal{V}$ to represent the relationship between A and B!

$$\mathrm{A} = \mathcal{C}(A, B) \Rightarrow \mathrm{B}$$

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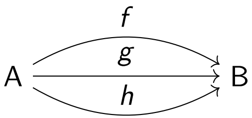

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Classical categories can be seen as enriched over **Set**.

An example: $\mathbb{R}$-**Vec** as a **Set**-Category

$\mathrm{Hom}(\mathbb{R}^3, \mathbb{C})$

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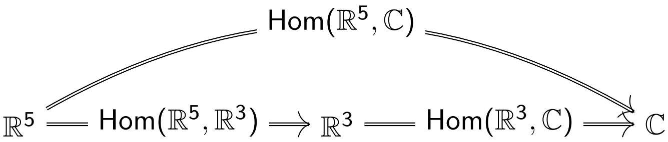
$\mathrm{Hom}(\mathbb{R}^5, \mathbb{R}^3)$



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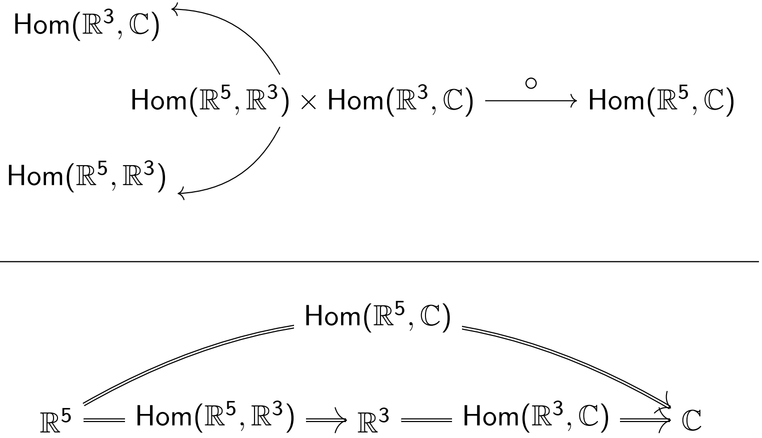
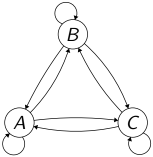


Another point of view

- Essentially, enriched categories can be thought of as *directed graphs with edge labels*
- The edge labels come from another (regular, non-enriched) category $\mathcal{V}$
- We have a hom-object between any two objects $\implies$ the graph is *complete*



Important! We do not have individual morphisms anymore!

Step 1: Monoidal Categories

- What structure should our category of hom-objects $\mathcal{V}$ have?
- We saw that something like a cartesian product $\times$ could be useful

Definition (Monoidal Category)

A monoidal category consists of

- A category $\mathcal{V}$
- A functor $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ ("tensor product")
- An element $\mathbb{1} \in Ob(\mathcal{V})$ ("unit")

such that

- $\otimes$ is associative (up to isomorphism)
- $\mathbb{1}$ is the left and right unit w.r.t $\otimes$ (up to isomorphism)
- certain coherence diagrams commute (see next slide)

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- certain coherence diagrams commute (see next slide)

$\Longrightarrow$ essentially a categorical version of a monoid

Some coherence Axioms

$$
\begin{array}{ccccc}
((W \otimes X) \otimes Y) \otimes Z & \longrightarrow & (W \otimes X) \otimes (Y \otimes Z) & \longrightarrow & W \otimes (X \otimes (Y \otimes Z)) \\
\downarrow & & & & \uparrow \\
(W \otimes (X \otimes Y)) \otimes Z & & \longrightarrow & & W \otimes ((X \otimes Y) \otimes Z)
\end{array}
$$

$$
\begin{array}{ccc}
(X \otimes \mathbb{1}) \otimes Y & \longrightarrow & X \otimes (\mathbb{1} \otimes Y) \\
& \searrow \quad \swarrow & \\
& X \otimes Y &
\end{array}
$$

Examples

- **Set** with $\times$ and $\{\cdot\}$
- k-**Vec** with $\times$ $(=\oplus)$ and $\{0\}$
- k-**Vec** with $\otimes$ and k
- **Grp** with $\times$ and $\{e\}$
- **Graph** with the tensor product of graphs and 
- **Cat** with $\times$ and 
- $\overline{\mathbb{R}}_+$ (with $\geq$ for morphisms), $+$ and 0.

Special cases and variations

A monoidal category is called...

- *symmetric* if $\otimes$ is commutative up to isomorphism (and some additional coherence diagramms commute)

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- *closed* if for every X, the functor $\_ \otimes X$ has a right adjoint
  - We call this adjoint functor the *internal hom*

Step 2: Enriched Categories

Remember that we are replacing hom-sets with hom-objects.
We need to have a *composition morphism* $\circ := \circ_{ABC}$

$$\mathcal{C}(A, B) \otimes \mathcal{C}(B, C) \xrightarrow{\circ} \mathcal{C}(A, C)$$

In $\mathcal{V} = \mathbf{Set}$, this is just the usual composition of functions.

- But what about the identity morphism id_A?
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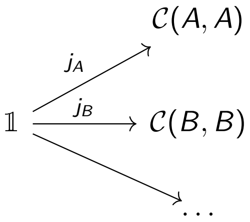


in $\mathcal{V}$

Definition (Enriched Category)

Let $(\mathcal{V}, \otimes, \mathbb{1})$ be a monoidal category.
A $\mathcal{V}$-Category $\mathcal{C}$ consists of

- A collection of objects $Ob(\mathcal{C})$
- For $A, B \in Ob(\mathcal{C})$ a *hom-object* $\mathcal{C}(A, B) \in \mathcal{V}$
- For $A, B, C \in Ob(\mathcal{C})$ a *composition morphism*

$$\circ_{ABC} : \mathcal{C}(A, B) \otimes \mathcal{C}(B, C) \to \mathcal{C}(A, C)$$

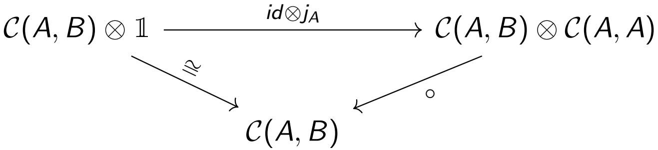
- For $A \in Ob(\mathcal{C})$ an *identity selector* $j_A : \mathbb{1} \to \mathcal{C}(A, A)$

such that the associativity and unit laws (next slides) hold.

Associativity of $\circ$

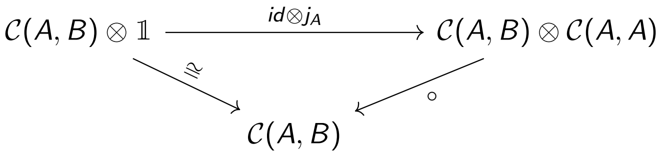
$$
\begin{array}{ccc}
(\mathcal{C}(A,B)\otimes\mathcal{C}(B,C))\otimes\mathcal{C}(C,D) & \leftrightarrow & \mathcal{C}(A,B)\otimes(\mathcal{C}(B,C)\otimes\mathcal{C}(C,D)) \\
\downarrow{\scriptstyle \circ\otimes id} & & \downarrow{\scriptstyle id\otimes\circ} \\
\mathcal{C}(A,C)\otimes\mathcal{C}(C,D) & & \mathcal{C}(A,B)\otimes\mathcal{C}(B,D) \\
\downarrow{\scriptstyle \circ} & \swarrow{\scriptstyle \circ} & \\
\mathcal{C}(A,D) & &
\end{array}
$$

Unit law for j_A



(symmetric law omitted)

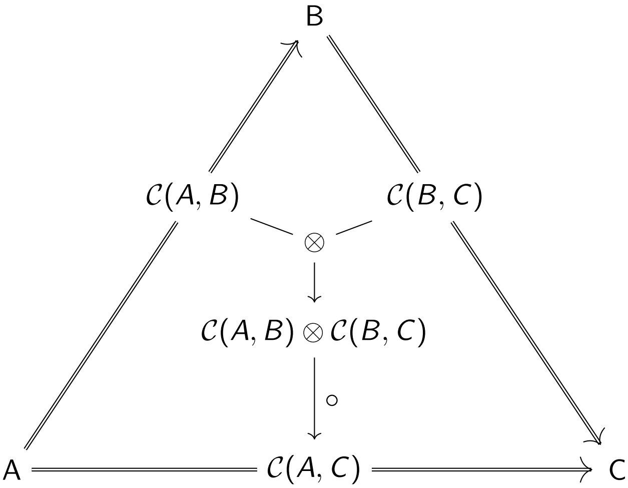
Unit law for j_A



(symmetric law omitted)

"If we pick the right identity, it will act as a unit w.r.t $\circ$!"

The Triangle of Truth

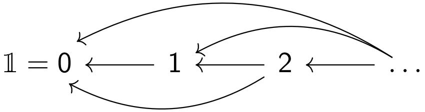


Examples of Enriched Categories

- Any category $\mathcal{C}$ can be seen as enriched over **Set**
- $k -$ **Vec** as a $(k -$ **Vec**$)$-Category with $\otimes$ and k
- $k -$ **Vec** as a $(k -$ **Vec**$)$-Category with $\oplus$ and $\{0\}$
- Preorders as categories enriched over $\circlearrowright 0 \longrightarrow 1 \circlearrowleft$
- (Generalized) metric spaces as categories over $\overline{\mathbb{R}}_+$

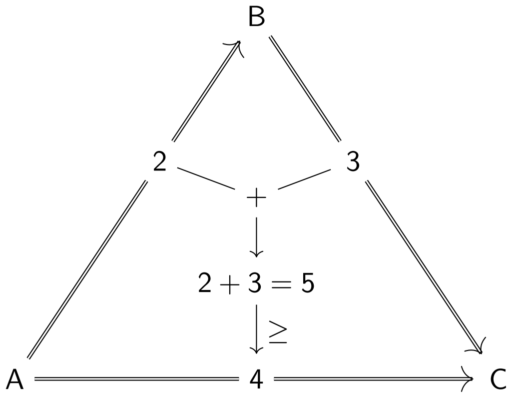
A closer look: Lawvere Metric Spaces

Let $\mathcal{V} := \overline{\mathbb{R}}_+$ with $x \to y \iff x \geq y$, $\otimes := +$ and $\mathbb{1} := 0$:

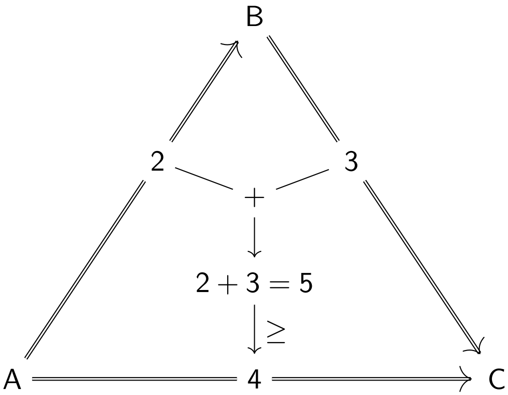


Let $\mathcal{C}$ be a $\mathcal{V}$-Category. What can we say about its structure?

Example diagram in a $\overline{\mathbb{R}}_+$-Category



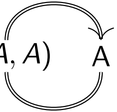
Example diagram in a $\overline{\mathbb{R}}_+$-Category



This is exactly the $\triangle$-inequality!

What about the identity selector?

$$\mathbb{1} = 0 \xrightarrow{\geq} \mathcal{C}(A, A)$$



This implies $\mathcal{C}(A, A) = 0$!

What about the identity selector?

$$\mathbb{1} = 0 \xrightarrow{\geq} \mathcal{C}(A, A) \qquad A$$

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A $\overline{\mathbb{R}}_+$-category is thus a generalized metric space:

- The hom-objects represent the metric ("distance" between objects)
- The triangle inequality holds
- $\mathcal{C}(A, A) = 0$

Enriched Functors

A regular functor is a mapping between objects and hom-sets. An *enriched* functor is a mapping between objects and hom-objects:

$$\begin{array}{ccccc} A & = & \mathcal{C}(A,B) & \Longrightarrow & B \\ \downarrow F & & \downarrow F_{AB} & & \downarrow F \\ FA & = & \mathcal{D}(FA,FB) & \Longrightarrow & FB \end{array}$$

$\mathcal{V}$-Functor $F : \mathcal{C} \to \mathcal{D}$

A $\mathcal{V}$-Functor has to respect composition...

$$\begin{array}{ccc}
\mathcal{C}(A, B) \otimes \mathcal{C}(B, C) & \xrightarrow{\ \circ\ } & \mathcal{C}(A, C) \\
\downarrow F & & \downarrow F \\
\mathcal{D}(FA, FB) \otimes \mathcal{D}(FB, FC) & \xrightarrow{\ \circ\ } & \mathcal{D}(FA, FC)
\end{array}$$

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...and identity selections

$$\begin{array}{ccc} & \mathbb{1} & \\ \swarrow j_A & & j_{FA} \searrow \\ \mathcal{C}(A,A) & \xrightarrow{F} & \mathcal{D}(FA,FA) \end{array}$$

Enriched natural transformations

An ordinary natural transformation $\alpha : F \to G$ ($F, G : \mathcal{C} \to \mathcal{D}$) assigns to each object A a morphism

$$FA \xrightarrow{\alpha_A} GA$$

In other words: We select an object of $\mathrm{Hom}(FA, GA)$.

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In other words: We select an object of $\text{Hom}(FA, GA)$.
But we already saw how to pick items of the hom-object: We use morphisms from $\mathbb{1}$:

$$\begin{array}{ccccc} & & \mathbb{1} & & \\ & & \downarrow{\scriptstyle \alpha_A} & & \\ FA & = & \mathcal{D}(FA, GA) & \Longrightarrow & GA \end{array}$$

Self-Enriched Categories

Set can be thought as being enriched over itself. Can we generalize this idea?

Ordinary Category $\mathcal{V}$

- Objects
- Hom-Sets

$\mathcal{V}$-Category $\mathcal{C}$

- Objects
- Hom-Objects in $\mathcal{V}$

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Ordinary Category $\mathcal{V}$	$\mathcal{V}$-Category $\mathcal{V}$
▶ Objects	▶ Objects
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$\Longrightarrow$ We need to express hom-sets in $\mathcal{V}$ as objects within $\mathcal{V}$!.

Theorem

Any closed symmetrical monoidal category $\mathcal{V}$ can be self-enriched, i.e. it is equivalent to a $\mathcal{V}$-Category.

Reminder

$(\mathcal{V}, \otimes, \mathbb{1})$ is *closed* if every functor $\_ \otimes A$ has a right adjoint $[A, \_]$.

$$\mathrm{Hom}(A \otimes B, C) \cong \mathrm{Hom}(A, [B, C])$$

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$$\begin{aligned} & \mathrm{Hom}(A \otimes B, C) \cong \mathrm{Hom}(A, [B, C]) \\ \implies & \mathrm{Hom}(B, C) \cong \mathrm{Hom}(\mathbb{1}, [B, C]) \end{aligned}$$

How to turn $\mathcal{V}$ into a $\mathcal{V}$-Category

This gives us an ideal candidate for the hom-object:

$$[A, B] \in Ob(\mathcal{V})$$

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Thus we define the $\mathcal{V}$-category $\overline{\mathcal{V}}$ as:

$$Ob(\overline{\mathcal{V}}) := Ob(\mathcal{V}) \qquad \overline{\mathcal{V}}(A, B) := [A, B] \in Ob(V)$$

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$$Ob(\overline{\mathcal{V}}) := Ob(\mathcal{V}) \qquad \overline{\mathcal{V}}(A, B) := [A, B] \in Ob(V)$$

Now we need to provide a concatenation and a unit selector

$$\circ : [A, B] \otimes [B, C] \to [A, C] \tag{1}$$

$$j_A : \mathbb{1} \to [A, A] \tag{2}$$

(2) is easy: Just set $X := \mathbb{1}$, $Y, Z := A$ in the adjunction property

$$\text{Hom}(X \otimes Y, Z) \cong \text{Hom}(X, [Y, Z])$$

Similarly, for $\circ$, we use

$$\mathrm{Hom}([A,B]\otimes[B,C],[A,C]) \cong \mathrm{Hom}(([A,B]\otimes[B,C])\otimes A, C)$$

and define

$$\begin{array}{c} ([A,B]\otimes[B,C])\otimes A \\ \big\downarrow \text{By associativity and commutativity} \\ [B,C]\otimes([A,B]\otimes A) \\ \big\downarrow \text{By ``almost invertability'' of adjoints} \\ [B,C]\otimes B \\ \big\downarrow \text{By ``almost invertability'' of adjoints} \\ C \end{array}$$

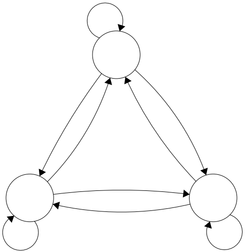
Examples of Self-Enrichment

The categories

- **Set** with $[A, B] = \{f : A \to B\}$
- $k - \textbf{Vec}$ with the usual tensor product $\otimes$ and $[A, B] = \{f : A \to B \text{ linear }\}$
- **Cat** with the categorical product where $[A, B]$ is the functor category B^A.
- **Graph** with the tensor product of graphs and a kind of "exponential graph" as the internal hom
- $\textbf{SimplyTyped}-\lambda$

are all cartesian (and thus symmetrically) closed and hence are enriched over themselves.

Thank you for listening!



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