Enriched Categories

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Definitions

Definition (Monoidal Category): A monoidal category consists of

- A category $\mathcal{V}$
- A functor $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ („tensor product“)
- An element $\mathbb{1} \in \mathrm{Ob}(\mathcal{V})$ („unit“)

such that the following laws hold for $A, B, C \in \mathrm{Ob}(\mathcal{V})$:

- Associativity: $A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$
- Unit law: $A \otimes \mathbb{1} \cong A \cong \mathbb{1} \otimes A$

A monoidal category is

- *symmetric* if $\otimes$ is commutative: $A \otimes B \cong B \otimes A$
- *cartesian* if $\otimes$ is the category-theoretic product: $A \otimes B = A \prod B$
- *closed* if for every X, the functor $\_ \otimes X$ has a right adjoint $[X, \_]$

For the coherence laws, see the previous talk.

Definition (Enriched Category): Let $(\mathcal{V}, \otimes, \mathbb{1})$ be a monoidal category.
A $\mathcal{V}$-Category $\mathcal{C}$ consists of

- A collection of objects $\mathrm{Ob}(\mathcal{C})$
- For $A, B \in \mathrm{Ob}(\mathcal{C})$ a *hom-object* $\mathcal{C}(A, B) \in \mathcal{V}$
- For $A, B, C \in \mathrm{Ob}(\mathcal{C})$ a *composition morphism*

$$\circ_{ABC} : \mathcal{C}(A, B) \otimes \mathcal{C}(B, C) \to \mathcal{C}(A, C)$$

- For $A \in \mathrm{Ob}(\mathcal{C})$ an *identity selector* $j_A : \mathbb{1} \to \mathcal{C}(A, A)$

such that the associativity law 1 and unit law 2 (as well as the symmetrical unit law) hold.

Definition (Enriched Functor): Let $(\mathcal{V}, \otimes, \mathbb{1})$ be a monoidal category and $\mathcal{C}$, $\mathcal{D}$ $\mathcal{V}$-categories. An enriched $\mathcal{V}$-functor $F : \mathcal{C} \to \mathcal{D}$ consists of

- A map $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$
- For $A, B \in \mathrm{Ob}(C)$ a morphism $F_{AB} : \mathcal{C}(A, B) \to \mathcal{D}(FA, FB)$

such that it respects compositions 3 and identity selections 4.

Definition (Enriched Natural Transformation): Let $(\mathcal{V}, \otimes, \mathbb{1})$ be a monoidal category, $\mathcal{C}$, $\mathcal{D}$ $\mathcal{V}$-categories and $F, G : \mathcal{C} \to \mathcal{D}$ $\mathcal{V}$-functors. A $\mathcal{V}$-natural transformation between F and G consists of a morphism $\alpha_A : \mathbb{1} \to \mathcal{D}(FA, GA)$ for every $A \in \mathrm{Ob}(\mathcal{C})$ such that the naturality law 5 holds.

Diagrams

In order to avoid cluttering, some edge labels are simplified, e.g. F instead of F_{AB}, if the actual variant is clear from the context. All of these diagrams are in $\mathcal{V}$.

$$\begin{array}{ccc} (\mathcal{C}(A,B)\otimes\mathcal{C}(B,C))\otimes\mathcal{C}(C,D) & \leftrightarrow & \mathcal{C}(A,B)\otimes(\mathcal{C}(B,C)\otimes\mathcal{C}(C,D)) \\ \downarrow{\scriptstyle \circ\otimes id} & & \downarrow{\scriptstyle id\otimes\circ} \\ \mathcal{C}(A,C)\otimes\mathcal{C}(C,D) & & \mathcal{C}(A,B)\otimes\mathcal{C}(B,D) \\ \downarrow{\scriptstyle\circ} & \swarrow{\scriptstyle\circ} & \\ \mathcal{C}(A,D) & & \end{array} \tag{1}$$

$$\begin{array}{ccc} \mathcal{C}(A,B)\otimes\mathbb{1} & \xrightarrow{id\otimes j_A} & \mathcal{C}(A,B)\otimes\mathcal{C}(A,A) \\ & {\scriptstyle\cong}\searrow \quad \swarrow{\scriptstyle\circ} & \\ & \mathcal{C}(A,B) & \end{array} \tag{2}$$

$$\begin{array}{ccc} \mathcal{C}(A,B)\otimes\mathcal{C}(B,C) & \xrightarrow{\circ} & \mathcal{C}(A,C) \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} \\ \mathcal{D}(FA,FB)\otimes\mathcal{D}(FB,FC) & \xrightarrow{\circ} & \mathcal{D}(FA,FC) \end{array} \tag{3}$$

$$\begin{array}{ccc} & \mathbb{1} & \\ {\scriptstyle j_A}\swarrow & & \searrow{\scriptstyle j_{FA}} \\ \mathcal{C}(A,A) & \xrightarrow{F} & \mathcal{D}(FA,FA) \end{array} \tag{4}$$

$$\begin{array}{cccc} \mathbb{1}\otimes\mathcal{C}(A,B) & \xrightarrow{\alpha_A\otimes G} & \mathcal{D}(FA,GA)\otimes\mathcal{D}(GA,GB) & \\ \uparrow & & & \searrow{\scriptstyle\circ} \\ \mathcal{C}(A,B) & & & \mathcal{D}(FA,GB) \\ \downarrow & & & \nearrow{\scriptstyle\circ} \\ \mathcal{C}(A,B)\otimes\mathbb{1} & \xrightarrow{F\otimes\alpha_B} & \mathcal{D}(FA,FB)\otimes\mathcal{D}(FB,GB) & \end{array} \tag{5}$$

Examples

- An ordinary category as a $(\mathbf{Set}, \times, \{\cdot\})$-category
- k-**Vec** as a $(k\text{-}\mathbf{Vec}, \oplus, \{0\})$-category
- k-**Vec** as a $(k\text{-}\mathbf{Vec}, \otimes, k)$-category
- Generalized (Lawvere-) metric spaces as $(\overline{\mathbb{R}}_+, +, 0)$-categories
- Preorders as categories enriched over $\circlearrowleft 0 \longrightarrow 1 \circlearrowright$ with multiplication as $\otimes$
- **Cat** as a self enriched **Cat**-category
- **Graph** as a self enriched **Graph**-category
- Simply typed λ-calculus (with product types) as a self enriched category