Abelian Categories

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10. September 2020

Null object and null morphism

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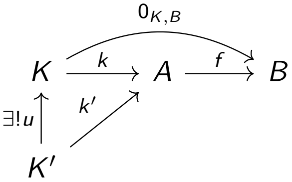
Example

In **Gr** the zero object is the trivial group and the zero morphisms are exactly the morphisms, that map all elements to the neutral element.

Kernels and cokernels

Definition

For a pointed category $\mathcal{C}$ the *kernel* of a morphism $f : A \mapsto B$ is the equalizer of f and $0_{a,b}$. The *cokernel* is the dual concept.



So

(i) $f \circ k = 0 \circ k = 0$

(ii) for any other k' with $f \circ k' = 0$, k' factors uniquely to $k' = k \circ u$.

pre-additive categories

Definition

A category is called *pre-additive* or *Ab-category* if every hom-set $C(a, b)$ is an additive abelian group and composition is bilinear in a way that the it distributes over the addition.

$$h \circ (f + g) = h \circ f + h \circ g \text{ and } (f + g) \circ h = f \circ h + g \circ h$$

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Proposition

Let z be an object of an Ab-category A. The following conditions are equivalent:

(i) z is initial.

(ii) z is terminal.

(iii) A has a zero object.

pre-additive categories

Proof.

(iii)$\Rightarrow$(i) and (iii)$\Rightarrow$(ii) by definition. By duality it suffices to proof (i) $\Rightarrow$ (ii). As z is initial there exists a unique map $1_z : z \mapsto z$, thus $Mor_A(z, z)$ has only one element, the zero element. For any other arrow $f : b \mapsto z$ the composition $1_z \circ f$ has to be zero as well. And therefore z is terminal. □

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Definition

Let A be an Ab-category. For objects $a, b \in A$ a biproduct is diagram

$a \underset{p_1}{\overset{i_1}{\rightleftarrows}} c \underset{i_2}{\overset{p_2}{\rightleftarrows}} b$ that fulfils $p_1 i_1 = 1_a$, $p_2 i_2 = 1_b$, $i_1 p_1 + i_2 p_2 = 1_c$.

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Example

In the abelian groups the biproduct is exactly the direct sum.

pre-additive categories

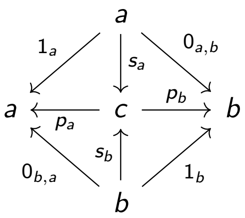
Proposition

In an Ab-category the following statements are equivalent:

(i) *The product of* a, b *exists.*

(ii) *The coproduct of* a, b *exists.*

(iii) *The biproduct of* a, b *exists.*



Proof.

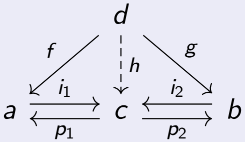
By duality we only have to proof the equivalence of (i) and (iii). Let us assume we have a product $a \xleftarrow{p_a} c \xrightarrow{p_b} b$. By the definition of the product there exists a unique morphism $s_a : a \mapsto c$, such that $p_a \circ s_a = 1_a,\ p_b \circ s_b = 0$ and vice versa for s_b. Then

$$p_a \circ (s_a \circ p_a + s_b \circ p_b) = p_a + 0 = p_a$$
$$p_b \circ (s_a \circ p_a + s_b \circ p_b) = 0 + p_b = p_b$$

and $s_a \circ p_a + s_b \circ p_b = 1_c$.

Proof.

We now have a biproduct as in our definition.



Define $h : d \mapsto c$ as $h = i_1 \circ f + i_2 \circ g$. Then $p_1 \circ h = f$ and $p_2 \circ h = g$.
For $h' d \mapsto c$ with $p_1 \circ h' = f$ and $p_2 \circ h' = g$ we deduce

$$h' = 1_c h' = (i_1 p_1 + i_2 p_2) h' = i_1 p_1 h' + i_2 p_2 h' = i_1 f + i_2 g = h$$

and therefore $a \xleftarrow{p_1} c \xrightarrow{p_2} b$ is indeed a product. □

Additive category

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If A, B are *Ab*-categories, a functor $F : A \mapsto B$ is *additive* when for all parallel arrows $f, f' : b \mapsto c$ in A

$$F(f + f') = F(f) + F(f')$$

holds, so F is a group homomorphism. The composite of of additive functors is additive.

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Proposition

F is additive if and only if F preserves biproducts.

Additive Categories

Example

Let $\mathcal{A}$ be pre-additive and $a \in \mathcal{A}$. The representable functor

$$\mathcal{A}(a, -) : \mathcal{A} \mapsto Ab, \, \mathcal{A}(a, -)(b) = \mathcal{A}(a, b)$$

is additive. Let $f, g : b \mapsto c$ be parallel arrows. As $Hom_A(b, c)$ is an abelian group and thus

$$\mathcal{A}(A, -)(f - g) : \mathcal{A}(a, b) \mapsto \mathcal{A}(a, c)$$

is well defined, as

$$\mathcal{A}(a, f - g)(h) = (f - g) \circ h = f \circ h - g \circ h = \mathcal{A}(a, f)(h) - \mathcal{A}(a, g)(h)$$

and with the zero morphism $\mathcal{A}(a, -)(0)(h) = 0 \circ h = 0$ the functor $\mathcal{A}(a, -)$ is additive.

Additive Yoneda

Theorem

If $\mathcal{A}$ is a pre additive category,$A \in \mathcal{A}$, $F : A \mapsto Ab$ is an additive functor, there exist isomorphisms of abelian groups such that:

$$Nat(\mathcal{A}(A, -), F) \simeq F(A)$$

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Proof.

As for the "normal" Yoneda Lemma. □

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An Ab-category C is *abelian* if the following conditions are satisfied:

(i) C has a zero object.

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(iv) Every monomorphism is a kernel and every epi is cokernel.

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Proposition

The dual notion of an abelian category is again the one of an abelian category.

A factorization theorem

Definition

The *image* and *coimage* of a function is defined as

$$\text{im } f = \ker(\text{coker}(f)) \text{ and } \textit{coim } f = \text{coker}(\ker(f))$$

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In an abelian category $\mathcal{A}$, f is and isomorphism, iff f is iso and monic.

Proof.

"$\Rightarrow$" $\checkmark$

""$\Leftarrow$" As f is monic, f is the kernel for some g, $f = \ker g$. But since f is epi $g \circ f = 0$ implies that $g = 0$. Then $0 \circ 1_A = 0$ and as f is the kernel of g, there exists a unique factorization such that $1 = f \circ h$. Thus f is and isomorphism. □

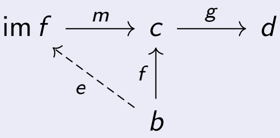
A factorization theorem

Theorem

For an abelian category $\mathcal{A}$, every morphism has a factorization, $f = me$, where m is a monomorphism and e is an epimorphism. Moreover $m = \ker(\text{coker}(f))$ and $e = coker(ker(f))$.

Proof.

Let us define the following morphisms:
$g = \text{coker}(f), m = \ker(g) = \ker(\text{coker}(f))$ and $f : b \mapsto c$.

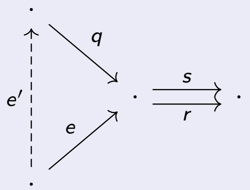


As $g \circ f = 0$ and m is the kernel of g there exists a unique e, such that the diagram commutes, and as m is monic for every $ft = 0 \Leftrightarrow et = 0$ and thus $ker(f) = ker(e)$. If we proof now that e is epi, $e = \text{coker}(\ker(e)) = \text{coker}(\ker(f))$ holds. □

A factorization theorem

Proof.

Now consider a pair of parallel arrows r, s such that $re = se$ and their equalizer q with $rq = sq$.



Now we can factor e through the equalizer as $e = qe'$ and $f = me = mqe'$. As q is monic as an equalizer, so is mq and as we are abelian, $m' = mq$ is a kernel. One can show that for two factorizations $f = me$ and $f = m'e'$, there exists an arrow t such that $m = m't$ and $e = te'$. Thus $m = m't = mqt \Rightarrow qt = 1$, so the monic q has a right inverse, therefore q is an isomorphism and $rq = sq \Rightarrow r = s$. □

Exact sequences

Definition

A composable pair of arrows $\cdot \xrightarrow{f} b \xrightarrow{g} \cdot$ is called *exact at b* if $\operatorname{im} f = \ker g$. A diagram

$$0 \longrightarrow a \xrightarrow{f} b \xrightarrow{g} c \longrightarrow 0$$

is called *short exact sequence* when it is exact at a, b and c.

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is called *short exact sequence* when it is exact at a, b and c.

Exactness at a giving us, that f is epi and conversly that g is monic.

Diagram Lemmas

Motivation

As for an arbitrary abelian category Proofs concerning exact sequences can be a lot easier when we perform them in a concrete category like $Mod_{\mathcal{R}}$ *for a Ring* $\mathcal{R}$ *as monomorphisms are just injections, epis are surjections. This leads to*

(i) $f : A \mapsto B$ *is monic*

$$\text{iff } \forall a \in A\, f(a) = 0 \Rightarrow a = 0$$
$$\text{iff } \forall a, a' \in A\, f(a) = f(a') \Rightarrow a = a'$$

(ii) $f : A \mapsto B$ *is epi*

$$\text{iff } \forall b \in B\ \exists a \in A;\ f(a) = b$$

(iii) $A \xrightarrow{f} B \xrightarrow{g} C$ *is exact iff*

$$\forall a \in A\ g(f(a)) = a \text{ and}$$
$$\forall b \in B\ g(b) = 0 \Rightarrow \exists a \in A \text{ such that } b = f(a)$$

This makes proving statements concerning exact sequences much easier.

Five Lemma

We are going to prove the five lemma in Mod_R. Consider the following diagram

$$
(1) \quad \begin{array}{ccccccccc}
a_1 & \xrightarrow{g_1} & a_2 & \xrightarrow{g_2} & a_3 & \xrightarrow{g_3} & a_4 & \xrightarrow{g_4} & a_5 \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\
b_1 & \xrightarrow{h_1} & b_2 & \xrightarrow{h_2} & b_3 & \xrightarrow{h_3} & b_4 & \xrightarrow{h_4} & a_5.
\end{array}
$$

Theorem

If the rows of (1) are exact and f_1, f_2, f_4, f_5 are isomorphisms, so is f_3.

Proof.

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The Freyd-Mitchell embedding theorem

Theorem (The Freyd-Mitchell embedding theorem)

Every small abelian category admits a full, faithful and exact functor to the category Mod_R for some Ring R.

Hereby we call a functor *exact* if it preserves finite limits.
This theorem lets us understand small abelian categories as full subcategories of Mod_R, therefore leading to some nice as the diagram chasing seen above. A proof can be found in Francis Borceuxs *Handbook of Categorical Algebra Volume 2*.