Definitions abelian categories

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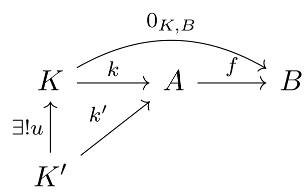
Definition 1:
In a category $\mathcal{C}$ an object is called *null object* or *zero object* $\mathbf{0}$ if it is both initial and terminal. A category with zero objects is called *pointed category.*

Definition 2:
Let $\mathcal{C}$ be a category with a zero object. The morphism $f : a \mapsto b$ is called *null morphism* if it factors through **0.**

Definition 3:
For a pointed category $\mathcal{C}$ the *kernel* of a morphism $f : A \mapsto B$ is the equalizer of f and $0_{a,b}$. The *cokernel* is the dual concept.



Definition 4:
A category is called *pre-additive* or *Ab-category* if every hom-set $C(a,b)$ is an additive abelian group and composition is bilinear in a way that the it distributes over the addition.

$$h \circ (f+g) = h \circ f + h \circ g \text{ and } (f+g) \circ h = f \circ h + g \circ h$$

Proposition 1: *Let z be an object of an Ab-category A. The following conditions are equivalent:*

(i) *z is initial.*

(ii) *z is terminal.*

(iii) *A has a zero object.*

Definition 5:
Let A be an Ab-category. For objects $a, b \in A$ a biproduct is diagram $a \underset{p_1}{\overset{i_1}{\rightleftarrows}} c \underset{i_2}{\overset{p_2}{\rightleftarrows}} b$ that fulfils $p_1 i_1 = 1_a$, $p_2 i_2 = 1_b$, $i_1 p_1 + i_2 p_2 = 1_c$.

Proposition 2: *In an Ab-category the following statements are equivalent:*

(i) *The product of a, b exists.*

(ii) *The coproduct of a, b exists.*

(iii) The biproduct of a, b exists.

Definition 6:
An *additive category* is defined as a pre-additive category with a zero object and a biproduct for each pair of objects.

Definition 7:
If A, B are *Ab*-categories, a functor $F : A \mapsto B$ is *additive* when for all parallel arrows $f, f' : b \mapsto c$ in A

$$F(f + f') = F(f) + F(f')$$

holds, so F is a group homomorphism. The composite of of additive functors is additive.

Proposition 3: *F is additive if and only if F preserves biproducts.*

Definition 8:
An Ab-category C is *abelian* if the following conditions are satisfied:

(i) C has a zero object.

(ii) C has a biproduct for each pair of objects.

(iii) Every arrow has a kernel and a cokernel.

(iv) Every monomorphism is a kernel and every epi is cokernel.

if it only fulfils the conditions (i)-(iii) it is *pre-abelian*.

Definition 9:
The *image* and *coimage* of a function is defined as

$$\operatorname{im} f = \ker(\operatorname{coker}(f)) \text{ and } \textit{coim } f = \operatorname{coker}(\ker(f))$$

Theorem 1: *For an abelian category $\mathcal{A}$, every morphism has a factorization, $f = me$, where m is a monomorphism and e is an epimorphism. Moreover $m = \ker(\operatorname{coker}(f))$ and $e = coker(ker(f))$.*

Definition 10:
A composable pair of arrows $\cdot \xrightarrow{f} b \xrightarrow{g} \cdot$ is called *exact at b* if $\operatorname{im} f = \ker g$. A diagram

$$0 \longrightarrow a \xrightarrow{f} b \xrightarrow{g} c \longrightarrow 0$$

is called *short exact sequence* when it is exact at a, b and c.

Theorem 2 (five-lemma): *If the rows of (1) are exact and f_1, f_2, f_4, f_5 are isomorphisms, so is f_3.*

$$
\begin{array}{ccccccccc}
a_1 & \xrightarrow{g_1} & a_2 & \xrightarrow{g_2} & a_3 & \xrightarrow{g_3} & a_4 & \xrightarrow{g_4} & a_5 \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\
b_1 & \xrightarrow{h_1} & b_2 & \xrightarrow{h_2} & b_3 & \xrightarrow{h_3} & b_4 & \xrightarrow{h_4} & a_5.
\end{array}
$$

Theorem 3 (The Freyd-Mitchell embedding theorem): *Every small abelian category admits a full, faithful and exact functor to the category Mod_R for some Ring R.*