C*-tensor categories and Woronowicz’ Tannaka-Krein reconstruction

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1. Introduction and Motivation

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4. Tannaka-Krein and quantum groups
Motivation: Group Representations

**Definition (Classical point of view)**

A finite dimensional **representation** of a group $G$ is a group homomorphism $G \to GL_n(\mathbb{K})$ for some field $\mathbb{K}$.

**Definition**

$T : \mathbb{K}^n \to \mathbb{K}^m$ such that for two representations $\rho, \sigma$ we have $\rho(g)T = T\sigma(g) \forall g \in G$ is called **intertwiner**.

What do we know from a categorical point of view about representations?
Categorical view on representations

Definition (Categorical point of view)

A representation of a group $G$ (as one-object category) is a functor $G \to Vect_K$. Intertwiners are then natural transformations between two such functors.

Thus:

- Representations form a category,
- a representation sends $G$ to some $K^n$,
- and thus any $g \in G$ to some Matrix in $GL_n(K)$

But this category has even more structure:

- $Rep_K(G)$ forms a monoidal category.
Category of representations

For $G$ a group, the following is a **monoidal** category, denoted $\text{Rep}(G)$

- **Objects:** Representations $\rho : G \rightarrow GL_n(\mathbb{C})$,
- **Morphisms:** Intertwiners,
- $\mathbb{C}$-linear,
- Direct sum $(\rho \oplus \sigma)(g) = \rho(g) \oplus \sigma(g) \in GL_{n+m},$
- Tensor product: $\rho \otimes \sigma(g) = \rho(g) \otimes \sigma(g) \in GL_{n \cdot m},$
- **Unit:** trivial representation $\mathbb{C}$.

Furthermore if we deal with compact topological groups we get the notion of a rigid $C^*$-tensor category.
Theorem (Woronowicz’s Tannaka Krein duality [NT13])

Let $\mathcal{C}$ be a rigid $C^*$-tensor category, $F : \mathcal{C} \to \text{f.d.-Hilb}$ be a unitary fiber functor. Then there exist

- a compact quantum group $G$ and
- a unitary monoidal equivalence $E : \mathcal{C} \to \text{Rep}(G)$
- such that $F$ is naturally unitarily monoidally isomorphic to the composition of the canonical fiber functor with $E$.
- Furthermore, the Hopf *-algebra $(\mathbb{C}[G], \Delta)$ for such a $G$ is uniquely determined up to isomorphism.
**$C^*$-categories**

**Definition ($C^*$-category [NT13, Web17])**

A category $\mathcal{C}$ is called $C^*$-category provided that:

1. $\forall U, V \in \text{Ob}(\mathcal{C})\text{Mor}(U, V)$ is a Banach space, composition of morphisms is bilinear and $\|ST\| \leq \|S\|\|T\|$

2. we have an antilinear contravariant functor “involution” $\ast : \mathcal{C} \to \mathcal{C}$: $\ast = \text{id}$ on objects, and on morphisms $T \in \text{Mor}(U, V)$:
   
   1. $T^{**} = T$,
   2. $\|T^* T\| = \|T\|^2$, and $\text{End}(U)$ is a unital $C^*$-algebra
   3. $T^* T \in \text{End}(U) = \text{Mor}(U, U)$ is positive.

Via the $\ast$- operation we can introduce the notions of “projection, partial isometry, unitary”.

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Examples of $C^*$-categories

Example

Category of finite-dimensional Hilbert spaces:
- Objects: finite dimensional Hilbert spaces,
- Morphisms: bounded linear maps.

Example

Any **unital** $C^*$-algebra is a one-object $C^*$-category.
**Definition (C*-tensor category [NT13])**

A C*-category is called C*-tensor category provided that it is a tensor category and if furthermore:

1. The associativity morphisms are **unitary**: \( uu^* = \text{id}, u^* u = \text{id} \) and \( L_1, R_1 : X \rightarrow X \otimes 1, 1 \otimes X \) are unitary.
2. \((S \otimes T)^* = S^* \otimes T^*\) for all morphisms \(S, T\).
Examples of $C^*$-tensor categories

Example

Category of finite-dimensional Hilbert spaces.

Idea: Topology $\leftrightarrow C^*$-structure

Example

Finite dimensional unitary representations of a CG:

- $\text{Rep}(G)$ has a faithful “fiber functor”,
- thus we can also view $\text{Rep}(G)$ as subcategory of f.d.-Hilb.

Notation: For $G$ a compact group we can identify a representation with an element $U \in \mathcal{C}(G) \otimes M_n(\mathbb{C})$. 
Definition

A representation $U$ of a CG is **irreducible** if it has no invariant subspaces.

Lemma (Schur’s Lemma [NT13])

*Two irreducible unitary representations $U, V$ are either unitarily equivalent and $\text{Mor}(U, V)$ one-dimensional or $\text{Mor}(U, V) = 0$. Every f.d. rep is equivalent to a unitary one, and every f.d. rep is direct sum of irreducible ones.*
Definition ((Semi-)Simplicity [NT13])

An object in a $C^*$-tensor category $U \in \mathcal{C}$ is called \textbf{simple} if $\text{End}(U) \cong \mathbb{C}1$. $\mathcal{C}$ is called \textbf{semisimple} provided that every object can be decomposed into a direct sum of simple objects.

Corollary

For a compact group $G$, $\text{Rep}_\mathbb{C}(G)$ is semisimple.
Conjugate Objects

**Definition (Conjugate Object[NT13])**

\( \bar{U} \in \text{Ob}(\mathcal{C}) \) is **conjugate** to \( U \in \text{Ob}(\mathcal{C}) \) if there exist morphisms \( R : 1 \to \bar{U} \otimes U \), \( \bar{R} : 1 \to U \otimes \bar{U} \) such that the **conjugate equations** (CEQ) hold:

\[
(\bar{R}^* \otimes \text{id})(\text{id} \otimes R) = \text{id} \\
(R^* \otimes \text{id})(\text{id} \otimes \bar{R}) = \text{id}
\]

If every object in a \( C^* \)-tensor category has a conjugate, then we call it **rigid**.

Conjugate objects are, if they exist, unique up to isomorphism.
Conjugate Objects

Example (f.d.-Hilb [NT13])

\( H \in \text{f.d.-Hilb} \) the dual space \( H^* = \overline{H} \) is its conjugate (dual space).

\[
r : \mathbb{C} \rightarrow \overline{H} \otimes H, \quad r(1) = \sum_i \overline{e_i} \otimes e_i = \sum_i J(e_i) \otimes e_i
\]

Example (Representation of CG [NT13])

For a f.d. unitary representation \( U \) the conjugate representation is given by complex conjugation: \( \overline{U}(g) = U(\overline{g}) \).
Properties of conjugates

Proposition ([NT13])

If $U$ has a conjugate, then $\text{End}(U)$ is finite dimensional and every object with a conjugate decomposes into a finite direct sum of simple objects (subobjects are required).

Corollary ([NT13])

The class of objects having a conjugate form a (semisimple) $C^*$-tensor subcategory.
Theorem (Frobenius reciprocity[NT13])

If an object $U$ in a $C^*$-tensor category $\mathcal{C}$ has a conjugate, then we have for $V, W \in \text{Ob}(\mathcal{C})$ an isomorphism:

$\text{Mor}(U \otimes V, W) \cong \text{Mor}(V, \bar{U} \otimes W)$

Taking conjugates can be extended to a contravariant functor via Frobenius reciprocity:

Theorem ([NT13])

The maps $U \mapsto \bar{U}$, $\text{Mor}(U, V) \ni T \mapsto T^v \in \text{Mor}(\bar{V}, \bar{U})$ define a contravariant endofunctor.
**Definition**

Let $U$ be a simple object in a $C^*$-tensor category. The number

$$d_i(U) = \| R \| \cdot \| \bar{R} \|$$

is called intrinsic dimension. For general $U$ decompose into simple objects $U_k$ and define $d_i(U) := \sum_k d_i(U_k)$.

**Example (f.d.-Hilb[NT13])**

In this case we recover the usual notion of dimension, since $C^n = \bigoplus_{i=1}^n C$.

Uses: eg: classify $C^*$-tensor category generated by an object with certain intrinsic dimension: see work of Froelich: Quantum Groups, Quantum Categories.
intrinsinc/statistical dimension

**Definition**

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Fiber functors

**Definition (Fiber functor [NT13])**

A tensor functor $F : C \to \text{f.d.-Hilb}$ is called **fiber functor** if it is faithful and exact.

In case we are dealing with a rigid $C^*$-tensor category this is the same as tensor functor!

**Example (canonical fiber functor [NT13])**

$$F(U \xrightarrow{S} V) = H_U \xrightarrow{S} H_V$$
Monoidal equivalence

**Definition (Monoidal equivalence [EGNO15])**

A monoidal functor is called monoidal equivalence if it is an equivalence seen as functor. Monoidal equivalence is an equivalence relation for tensor categories. In that case there are two functors $F, G$, with natural isomorphisms $FG \cong \text{id}$ and $GF \cong \text{id}$.

**Proposition ([NT13])**

Let $F : \mathcal{C} \to \mathcal{C}'$ be a tensor functor between semisimple $C^*$-tensor categories. $F$ is a monoidal equivalence iff $F(U_\alpha)$ are simple for all simple $U_\alpha$, pairwise non-isomorphic and $F$ is isomorphism-dense.
**Definition**

A functor $F$ between $C^*$-tensor categories is called unitary if:

- $F(T^*) = F(T)^*$ and
- $F(U) \otimes F(V) \to F(U \otimes V)$ is unitary.

**Definition**

Two $C^*$-tensor categories are called unitarily monoidally equivalent if they are monoidally equivalent and if $F, G$ between them such that $F, G$ and the natural isomorphisms $FG \cong \text{id}$ and $GF \cong \text{id}$ are unitary.
Hopf *-algebras and CQGs

Definition

A **Hopf *-algebra** is an involutive algebra $A$ together with algebra *-homomorphisms $\Delta: A \to A \otimes A$, $\varepsilon: A \to \mathbb{C}$, $S: A \to A^{\text{opp}}$, such that the following conditions hold:

$$(\text{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \text{id}) \circ \Delta$$

$$(\text{id} \otimes \varepsilon) \circ \Delta = \text{id} = (\varepsilon \otimes \text{id}) \circ \Delta$$

$$m \circ (S \otimes \text{id}) \circ \Delta = \eta \circ \varepsilon = m \circ (\text{id} \otimes S) \circ \Delta$$
CQGs

**Definition (Compact quantum group [NT13])**

A compact quantum group is a pair \((A, \Delta)\), where \(A\) is a unital C*-algebra and \(\Delta : A \rightarrow A \otimes A\) is a unital *-homomorphism, called comultiplication:

1. \((\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta\) coassociativity,
2. \((A \otimes 1)\Delta(A), (1 \otimes A)\Delta(A)\) are dense in \(A \otimes A\) : cancellation property.

**Definition ([NT13])**

A representation of a compact quantum group \(A\) is an invertible element \(U \in M_n(A) \cong M_n(\mathbb{C}) \otimes A\). If \(U\) is unitary, then we call the representation unitary.
Woronowicz’s Tannaka Krein duality

**Theorem (Woronowicz’s Tannaka Krein duality [NT13])**

Let $\mathcal{C}$ be a rigid $C^*$-tensor category, $F : \mathcal{C} \to \text{f.d.-Hilb}$ be a unitary fiber functor. Then there exist

- a compact quantum group $G$ and
- a unitary monoidal equivalence $E : \mathcal{C} \to \text{Rep}(G)$

such that $F$ is naturally unitarily monoidally isomorphic to the composition of the canonical fiber functor with $E$.

Furthermore, the Hopf $*$-algebra $(\mathbb{C}[G], \Delta)$ for such a $G$ is uniquely determined up to isomorphism.
Sketch of proof: Tannaka-Krein

- Consider the *-algebra $\text{End}(F) = \text{Nat}(F, F)$.
- $\eta \in \text{End}(F)$ is determined by $\eta_{U_\alpha} : F(U_\alpha) \to F(U_\alpha)$ and thus:

$$\text{End}(F) \cong \prod_\alpha B(F(U_\alpha))$$

Define $\delta : \text{End}(F) \to \text{End}(F \otimes^2)$ via:

$$\delta(\eta)_{U,V} = F_2^* \eta_{U \otimes V} F_2$$

- Coassociativity follows from the property of tensor functor.
- We construct a Hopf *-algebra from this data.
Sketch of proof: Tannaka-Krein

- Set $\mathcal{A} := \bigoplus_\alpha B(F(U_\alpha))^* \subset \text{End}(F)^*$.
- Define $ab := (a \otimes b)\delta$,
- $\Delta(a)(\omega \otimes \eta) = a(\omega\eta) \epsilon(a) = a(1)$,
- $S(a)(\eta) = a(\eta^v)$ and $a^*(\eta) := \bar{a}(\eta^{v*})$
- Then $(\mathcal{A}, \Delta, \epsilon, S)$ forms a Hopf $*$-algebra.
- We can construct a tensor functor to $\text{Rep}(A^*)$ by defining for $U \in \mathcal{C}$, $X^U \in B(F(U)) \otimes \text{End}(F)^*$ : $(\text{id} \otimes \eta)(X^U) = \eta_U$ and such that for $T \in \text{Mor}(U, V), F(T)$ intertwines $X^U$ and $X^V$. 

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Hopf $\ast$-algebras and CQGs

**Definition ([NT13, Web17])**

We denote by $\mathbb{C}[G] :=$ the linear span of matrix coefficients of all f.d. unitary representations of $G = \text{algebra generated by these.}$

**Theorem ([Web17, NT13])**

For any CQG $\mathbb{C}[G]$, $\Delta$ is a Hopf-$\ast$-algebra. Conversely, any Hopf-$\ast$-algebra generated by matrix coefficients of f.d. unitary corepresentations is of that form for some CQG.
We have seen: $C^*$-tensor categories $\leftrightarrow$ Representation categories of compact (quantum) groups.

Example (quantum permutation group)

\[
\mathcal{C}(S_n^+) := C^*(u_{ij}, i, j = 1, \ldots, n) \quad (1)
\]
\[
|u_{ij}\text{are projections and } \sum_k u_{ik} = \sum_k u_{kj} = 1) \quad (2)
\]

- $\mathcal{C}(S_n^+)/\langle u_{ij}u_{kl} = u_{kl}u_{ij} \rangle \cong \mathcal{C}(S_n)$.
- But how is this related with the representation theory?
- Answer: While the intertwiner space of $S_n$ is induced by partitions, the intertwiner space of $S_n^+$ is induced by noncrossing ones. $\Rightarrow$ “easy quantum groups”.

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