# $C^*$-tensor categories and Woronowicz' Tannaka-Krein reconstruction

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# Motivation: Group Representations

## Definition (Classical point of view)

A finite dimensional **representation** of a group $G$ is a grouphomomorphism $G \to GL_n(\mathbb{K})$ for some field $\mathbb{K}$.

## Definition

$T : \mathbb{K}^n \to \mathbb{K}^m$ such that for two representations $\rho, \sigma$ we have $\rho(g)T = T\sigma(g) \forall g \in G$ is called **intertwiner**.

What do we know from a categorical point of view about representations ?

# Categorical view on representations

## Definition (Categorical point of view)

A representation of a group $G$ (as one-object category) is a functor $G \to Vect_{\mathbb{K}}$. Intertwiners are then natural transformations between two such functors.

Thus:

- Representations form a category,
- a representation sends $G$ to some $\mathbb{K}^n$,
- and thus any $g \in G$ to some Matrix in $GL_n(\mathbb{K})$

But this category has even more sturcture:

- $Rep_{\mathbb{K}}(G)$ forms a monoidal category.

# Category of representations

For $G$ a group, the following is a **monoidal** category, denoted $Rep(G)$

- Objects: Representations $\rho : G \to GL_n(\mathbb{C})$,
- Morphisms: Intertwiners,
- $\mathbb{C}$-linear,
- Direct sum $(\rho \oplus \sigma)(g) = \rho(g) \oplus \sigma(g) = \in GL_{n+m}$,
- Tensor product: $\rho \otimes \sigma(g) = \rho(g) \otimes \sigma(g) \in GL_{n \cdot m}$,
- Unit: trivial representation $\mathbb{C}$.

Furthermore if we deal with compact topological groups we get the notion of a rigid $C^*$-tensor category.

# Woronowicz's Tannaka Krein duality

**Theorem (Woronowicz's Tannaka Krein duality [NT13])**

*Let $\mathcal{C}$ be a rigid $C^*$-tensor category, $F : \mathcal{C} \to$ f.d.- Hilb be a unitary fiber functor. Then there exist*

- *a compact quantum group $G$ and*
- *a unitary monoidal equivalence $E : \mathcal{C} \to Rep(G)$*
- *such that $F$ is naturally unitarily monoidally isomorphic to the composition of the canonical fiber functor with $E$.*
- *Furthermore, the Hopf *-algebra $(\mathbb{C}[G], \Delta)$ for such a $G$ is uniquely determined up to isomorphism.*

# $C^*$-categories

**Definition ($C^*$-category [NT13, Web17])**

A category $\mathcal{C}$ is called $C^*$-category provided that:

1. $\forall U, V \in Ob(\mathcal{C}) Mor(U, V)$ is a Banach space, composition of morphisms is bilinear and $\|ST\| \leq \|S\|\|T\|$
2. we have an antilinear contravariant functor "involution" $* : \mathcal{C} \to \mathcal{C}$: $* = \text{id}$ on objects, and on morphisms $T \in Mor(U, V)$:
   1. $T^{**} = T$,
   2. $\|T^*T\| = \|T\|^2$, and $End(U)$ is a unital $C^*$-algebra $\forall U \in Ob(\mathcal{C})$,
   3. $T^*T \in End(U) = Mor(U, U)$ is positive.

Via the $*$- operation we can introduce the notions of "projection, partial isometry, unitary".

# Examples of $C^*$-categories

## Example

Category of finite-dimensional Hilbert spaces:

- Objects: finite dimensional Hilbert spaces,
- Morphisms: bounded linear maps.

## Example

Any **unital** $C^*$-algebra is a one-object $C^*$-category.

# $C^*$-tensor categories

**Definition ($C^*$-tensor category [NT13])**

A $C^*$-category is called $C^*$-tensor category provided that it is a tensor category and if furthermore:

1. The associativity morphisms are **unitary**: $uu^* = \mathrm{id}, u^*u = \mathrm{id}$ and $L_1, R_1 : X \to X \otimes 1, 1 \otimes X$ are unitary.
2. $(S \otimes T)^* = S^* \otimes T^*$ for all morphisms $S, T$.

# Examples of $C^*$-tensor categories

**Example**

Category of finite-dimensional Hilbert spaces.

Idea : Topology $\leftrightarrow$ $C^*$- structure

**Example**

Finite dimensional unitary representations of a CG:

- $Rep(G)$ has a faithful "fiber functor",
- thus we can also view $Rep(G)$ as subcategory of f.d.-Hilb.

Notation: For $G$ a compact group we can identify a representation with an element $U \in \mathcal{C}(G) \otimes M_n(\mathbb{C})$.

### Definition

A representation $U$ of a CG is **irreducible** if it has no invariant subspaces.

### Lemma (Schur's Lemma [NT13])

*Two irreducible unitary representations $U, V$ are either unitarily equivalent and $Mor(U, V)$ one-dimensional or $Mor(U, V) = 0$. Every f.d. rep is equivalent to a unitary one, and every f.d. rep is direct sum of irreducible ones.*

## Definition ((Semi-)Simplicity [NT13])

An object in a $C^*$-tensor category $U \in \mathcal{C}$ is called **simple** if $End(U) \cong \mathbb{C}1$. $\mathcal{C}$ is called **semisimple** provided that every object can be decomposed into a direct sum of simple objects.

## Corollary

*For a compact group $G$, $Rep_{\mathbb{C}}(G)$ is semisimple.*

# Conjugate Objects

**Definition (Conjugate Object[NT13])**

$\bar{U} \in Ob(\mathcal{C})$ is **conjugate** to $U \in Ob(\mathcal{C})$ if there exist morphisms $R : 1 \to \bar{U} \otimes U, \bar{R} : 1 \to U \otimes \bar{U}$ such that the **conjugate equations** (CEQ) hold:

$$(\bar{R}^* \otimes \mathrm{id})(\mathrm{id} \otimes R) = \mathrm{id}$$
$$(R^* \otimes \mathrm{id})(\mathrm{id} \otimes \bar{R}) = \mathrm{id}$$

If every object in a $C^*$-tensor category has a conjugate, then we call it **rigid**.

Conjugate objects are, if they exist, unique up to isomorphism.

# Conjugate Objects

**Example (f.d.-Hilb [NT13])**

$H \in$ f.d.-Hilb the dual space $H^* = \bar{H}$is its conjugate (dual space).

$$r : \mathbb{C} \to \bar{H} \otimes H, \quad r(1) = \sum_i \bar{e}_i \otimes e_i = \sum_i J(e_i) \otimes e_i$$

**Example (Representation of CG [NT13])**

For a f.d. unitary representation $U$ the conjugate representation is given by complex conjugation: $\bar{U}(g) = U\bar{(}g)$.

# Properties of conjugates

**Proposition ([NT13])**

*If $U$ has a conjugate, then $End(U)$ is finite dimensional and every object with a conjugate decomposes into a finite direct sum of simple objects (subobjects are required).*

**Corollary ([NT13])**

*The class of objects having a conjugate form a (semisimple) $C^*$-tensor subcategory.*

**Theorem (Frobenius reciprocity[NT13])**

*If an object $U$ in a $C^*$-tensor category $\mathcal{C}$ has a conjugate, then we have for $V, W \in Ob(\mathcal{C})$ an isomorphism:*

$$Mor(U \otimes V, W) \cong Mor(V, \bar{U} \otimes W)$$

Taking conjugates can be extended to a contravariant functor via Frobenius reciprocity:

**Theorem ([NT13])**

*The maps $U \mapsto \bar{U}$, $Mor(U, V) \ni T \mapsto T^{\vee} \in Mor(\bar{V}, \bar{U})$ define a contravariant endofunctor.*

# instrinsic/statistical dimension

## Definition

Let $U$ be a simple object in a $C^*$-tensor category. The number

$$d_i(U) = \|R\| \cdot \|\bar{R}\|$$

is called intrinsic dimension. For general $U$ decompose into simple objects $U_k$ and define $d_i(U) := \sum_k d_i(U_k)$.

## Example (f.d.-Hilb[NT13])

In this case we recover the usual notion of dimension, since $\mathbb{C}^n = \oplus_{i=1}^n \mathbb{C}$.

Uses: eg: classify $C^*$-tensor category generated by an object with certain intrinsic dimension: see work of Froelich: Quantum Groups, Quantum Categories

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# Fiber functors

**Definition (Fiber functor [NT13])**

A tensor functor $F : \mathcal{C} \to$ f.d.-Hilb is called **fiber functor** if it is faithful and exact.

In case we are dealing with a rigid $C^*$-tensor category this is the same as tensor functor!

**Example (canonical fiber functor [NT13])**

$$F(U \overset{S}{\to} V) = H_U \overset{S}{\to} H_V$$

# Monoidal equivalence

### Definition (Monoidal equivalence [EGNO15])

A monoidal functor is called monoidal equivalence if it is an equivalence seen as functor. Monoidal equivalence is an equivalence relation for tensor categories. In that case there are two functors $F, G$, with natural isomorphisms $FG \cong \mathrm{id}$ and $GF \cong \mathrm{id}$.

### Proposition ([NT13])

*Let $F : \mathcal{C} \to \mathcal{C}'$ be a tensor functor between semisimple $C^*$-tensor categories. $F$ is a monoidal equivalence iff $F(U_\alpha)$ are simple for all simple $U_\alpha$, pairwise non-isomorphic and $F$ is isomorphism-dense.*

## Definition

A functor $F$ between $C^*$-tensor categories is called unitary if:

- $F(T^*) = F(T)^*$ and
- $F(U) \otimes F(V) \to F(U \otimes V)$ is unitary.

## Definition

Two $C^*$-tensor categories are called unitarily monoidally equivalent if they are monoidally equivalent and if F,G between them such that F,G and the natural isomorphisms $FG \cong \mathrm{id}$ and $GF \cong \mathrm{id}$ are unitary.

# Hopf *-algebras and CQGs

## Definition

A **Hopf *-algebra** is an involutive algebra $A$ together with algebra *-homomorphisms $\Delta\colon A \to A \otimes A$, $\varepsilon\colon A \to \mathbb{C}$, $S\colon A \to A^{\mathrm{opp}}$, such that the following conditions hold:

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta$$
$$(\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id} = (\varepsilon \otimes \mathrm{id}) \circ \Delta$$
$$m \circ (S \otimes \mathrm{id}) \circ \Delta = \eta \circ \varepsilon = m \circ (\mathrm{id} \otimes S) \circ \Delta$$

# CQGs

**Definition (Compact quantum group [NT13])**

A compact quantum group is a pair $(A, \Delta)$, where $A$ is a unital $C^*$-algebra and $\Delta : A \to A \otimes A$ is a unital *-homomorphism, called comultiplication:

1. $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$ coassociativity,
2. $(A \otimes 1)\Delta(A), (1 \otimes A)\Delta(A)$ are dense in $A \otimes A$ : cancellation property.

**Definition ([NT13])**

A representation of a compact quantum group $A$ is an invertible element $U \in M_n(A) \cong M_n(\mathbb{C}) \otimes A$. If $U$ is unitary, then we call the representation unitary.

# Woronowicz's Tannaka Krein duality

**Theorem (Woronowicz's Tannaka Krein duality [NT13])**

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# Sketch of proof: Tannaka-Krein

- Consider the *-algebra $End(F) = Nat(F, F)$.
- $\eta \in End(F)$ is determined by $\eta_{U_\alpha} : F(U_\alpha) \to F(U_\alpha)$ and thus:

$$End(F) \cong \Pi_\alpha B(F(U_\alpha))$$

Define $\delta : End(F) \to End(F^{\otimes 2})$ via:

$$\delta(\eta)_{U,V} = F_2^* \eta_{U \otimes V} F_2$$

- Coassociativity follows from the property of tensor functor.
- We construct a Hopf *-algebra from this data.

# Sketch of proof: Tannaka-Krein

- Set $\mathcal{A} := \oplus_\alpha B(F(U_\alpha))^* \subset End(F)^*$.
- Define $ab := (a \otimes b)\delta$,
- $\Delta(a)(\omega \otimes \eta) = a(\omega\eta)$ $\epsilon(a) = a(1)$,
- $S(a)(\eta) = a(\eta^v)$ and $a^*(\eta) := \bar{a}(\eta^{v*})$
- Then $(\mathcal{A}, \Delta, \varepsilon, S)$ forms a Hopf *-algebra.
- We can construct a tensor functor to $Rep(A^*)$ by defining for $U \in \mathcal{C}$ $X^U \in B(F(U)) \otimes End(F)^* : (\mathrm{id} \otimes \eta)(X^U) = \eta_U$ and such that for $T \in Mor(U, V), F(T)$ intertwines $X^U$ and $X^V$.

# Hopf *-algebras and CQGs

**Definition ([NT13, Web17])**

We denote by $\mathbb{C}[G] :=$ the linear span of matrix coefficients of all f.d. unitary representations of G = algebra generated by these.

**Theorem ([Web17, NT13])**

*For any CQG $\mathbb{C}[G], \Delta$ is a Hopf-*-algebra. Conversely, any Hopf-*-algebra generated by matrix coefficients of f.d. unitary corepresentations is of that form for some CQG.*

We have seen: $C^*$- tensor categories $\leftrightarrow$ Representation categories of compact (quantum) groups.

## Example (quantum permutation group)

$$\mathcal{C}(S_n^+) := C^*(u_{ij}, i, j = 1, \ldots, n \quad (1)$$

$$|u_{ij} \text{are projections and} \sum_k u_{ik} = \sum u_{kj} = 1) \quad (2)$$

- $\mathcal{C}(S_n^+)/\langle u_{ij}u_{kl} = u_{kl}u_{ij}\rangle \cong \mathcal{C}(S_n)$.
- But how is this related with the representation theory ?
- Answer: While the intertwiner space of $S_n$ is induced by partitions, the intertwiner space of $S_n^+$ is induced by noncrossing ones. $\Rightarrow$ "easy quantum groups".

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