Block Seminar On Category Theory

About Chains And Snakes - Chain Complexes And Homology

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UNIVERSITÄT DES SAARLANDES

A Little Reminder

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In the following, R is a ring with 1_R.

Definition

A *left R-module M* is an abelian group $(M, +)$ together with a map

$$R \times M \to M, \ (r, m) \mapsto r \cdot m,$$

such that for all $r, r_1, r_2 \in R, \ m, m_1, m_2 \in M$ the following hold:

- $r_1 \cdot (r_1 \cdot m) = (r_1 \cdot r_2) \cdot m$
- $(r_1 + r_2) \cdot m = r_1 \cdot m + r_2 \cdot m$
- $r \cdot (m_1 + m_2) = r \cdot m_1 + r \cdot m_2$
- $1 \cdot m = m.$

Examples

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- Let $R = \mathbb{Z}$. Then $M = 2\mathbb{Z} = \{2x \mid x \in \mathbb{Z}\}$ is an R-module.
- Let R be a ring and $I \subset R$ be an ideal. Then I is an R-module.

Chain Complexes

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$$\mathbf{M}: \ \ldots \to M_{n+1} \xrightarrow{\alpha_{n+1}} M_n \xrightarrow{\alpha_n} M_{n-1} \xrightarrow{\alpha_{n-1}} M_{n-2} \to \ldots$$

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A family $\mathbf{M}=\{M_n, \alpha_n\}_{n\in\mathbb{Z}}$ of R-modules and R-module homomorphisms such that $\alpha_n \circ \alpha_{n+1} = 0$ for each $n \in \mathbb{Z}$ is called *chain complex*.

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Each mapping $\alpha_n\colon M_n \to M_{n-1}$ is called a *boundary mapping* or *differential operator*.

Chain Complexes

Example

Let $R = \mathbb{Z}$. Consider

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- *exact at* M_n if $\mathrm{Im}(\alpha_{n+1}) = \mathrm{Ker}(\alpha_n)$.
- *exact* if it is exact at M_n for each $n \in \mathbb{Z}$.

Chain Complexes

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Remark

- *Exactness on the left means injectivity.*

Chain Complexes

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Remark

- *Exactness on the left means injectivity.*
- *Exactness on the right means surjectivity.*

Chain Complexes

Definition

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- We call the R-module $H_n(\mathbf{M}) = \operatorname{Ker}\alpha_n / \operatorname{Im}\alpha_{n+1}$ the *n-th homology module* of $\mathbf{M}$.

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Remark

The chain complex $\mathbf{M}$ *is exact at* n *if and only if* $H_n(\mathbf{M}) = \{0\}$.

Chain Maps

Chain Maps

Definition

Let **M** and **N** be two chain complexes of R-modules and R-module homomorphisms. A family $\mathbf{f} = \{f_n \colon M_n \to N_{n+k}\}_{n \in \mathbb{Z}}$ of R-linear mappings such that the diagram

$$\begin{array}{ccccccc}
\dots & \xrightarrow{\alpha_{n+1}} & M_n & \xrightarrow{\alpha_n} & M_{n-1} & \xrightarrow{\alpha_{n-1}} & \dots \\
 & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\dots & \xrightarrow{\beta_{n+k+1}} & N_{n+k} & \xrightarrow{\beta_{n+k}} & N_{n+k-1} & \xrightarrow{\beta_{n+k-1}} & \dots
\end{array}$$

is commutative for each $n \in \mathbb{Z}$ is called *chain map of degree k*.

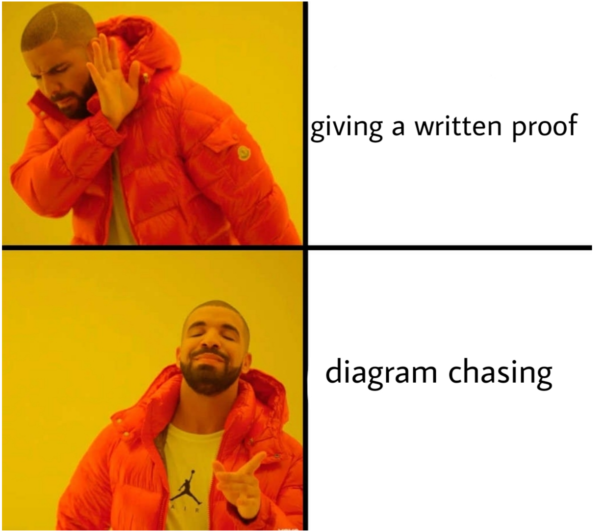
Chain Maps

Proposition

Let $\mathbf{M}$ *and* $\mathbf{N}$ *be two chain complexes of R-modules and R-module homomorphisms and let* $\mathbf{f}\colon \mathbf{M} \to \mathbf{N}$ *be a chain map between them. Then for each* $n \in \mathbb{Z}$ *there exists an R-linear mapping* $H_n(\mathbf{f})\colon H_n(\mathbf{M}) \to H_n(\mathbf{N})$ *which is defined by*

$$H_n(\mathbf{f})(x + \operatorname{Im} \alpha_{n+1}) = f_n(x) + \operatorname{Im} \beta_{n+1}$$

for all $x + \operatorname{Im} \alpha_{n+1} \in H_n(\mathbf{M})$.


giving a written proof
diagram chasing

Proof

Proof

- For each $n \in \mathbb{Z}$ we have the following commutative diagram

$$\begin{array}{ccccccccc} \dots & \xrightarrow{\alpha_{n+2}} & M_{n+1} & \xrightarrow{\alpha_{n+1}} & M_n & \xrightarrow{\alpha_n} & M_{n-1} & \xrightarrow{\alpha_{n-1}} & \dots \\ & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\ \dots & \xrightarrow{\beta_{n+2}} & N_{n+1} & \xrightarrow{\beta_{n+1}} & N_n & \xrightarrow{\beta_n} & N_{n-1} & \xrightarrow{\beta_{n-1}} & \dots . \end{array}$$

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(i) $H_n(\mathbf{f})$ maps $\operatorname{Ker}(\alpha_n)/\operatorname{Im}(\alpha_{n+1})$ to $\operatorname{Ker}(\beta_n)/\operatorname{Im}(\beta_{n+1})$:

Chain Maps

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(i) $H_n(\mathbf{f})$ maps $\mathrm{Ker}(\alpha_n)/\mathrm{Im}(\alpha_{n+1})$ to $\mathrm{Ker}(\beta_n)/\mathrm{Im}(\beta_{n+1})$:
Let $x \in \mathrm{Ker}\,\alpha_n$. We have

$$\beta_n(f_n(x)) = f_{n-1}(\alpha_n(x)) = 0,$$

i.e. $f_n(x) \in \mathrm{Ker}\,\beta_n$.

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$$f_n(x) - f_n(x') = f_n(x - x') = f_n(\alpha_{n+1}(y)) = \beta_{n+1}(f_{n+1}(y)).$$

Thus $f_n(x) - f_n(x') \in \operatorname{Im} \beta_{n+1}$.

Chain Maps

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Clear, since all f_n are R-linear.

□

Homotopy Equivalence

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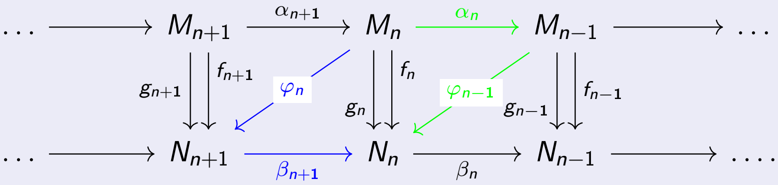
Definition

Let $\mathbf{M}$ and $\mathbf{N}$ be two chain complexes of R-modules and Rmodule homomorphisms and let $\mathbf{f}, \mathbf{g} \colon \mathbf{M} \to \mathbf{N}$ be two chain maps of degree 0. A chain map $\varphi = \{\varphi_n \colon M_n \to N_{n+1}\}$ of degree 1 such that $f_n - g_n = \beta_{n+1} \circ \varphi_n + \varphi_{n-1} \circ \alpha_n$ for each $n \in \mathbb{Z}$ is called a *homotopy*.



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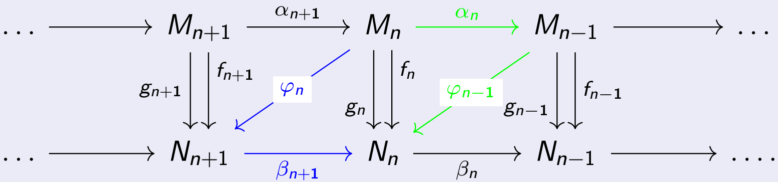

We denote it by $\varphi \colon \mathbf{f} \to \mathbf{g}$ ($\mathbf{f} \approx \mathbf{g}$) and say that $\mathbf{f}$ and $\mathbf{g}$ are *homotopic chain maps*.

Homotopy Equivalence

If there exist two chain maps $\mathbf{f}: \mathbf{M} \to \mathbf{N}$ and $\mathbf{g}: \mathbf{N} \to \mathbf{M}$ such that $\mathbf{g} \circ \mathbf{f} \approx \mathrm{Id}_{\mathbf{M}}$ and $\mathbf{f} \circ \mathbf{g} \approx \mathrm{Id}_{\mathbf{N}}$ then $\mathbf{M}$ and $\mathbf{N}$ are said to be of the *same homotopy type*. The chain maps $\mathbf{f}: \mathbf{M} \to \mathbf{N}$ and $\mathbf{g}: \mathbf{N} \to \mathbf{M}$ are called *homotopy equivalences*.

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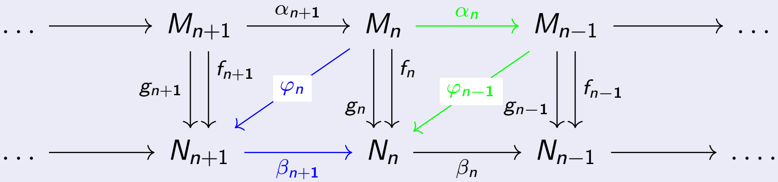
... M_{n+1} M_n M_{n-1} ...
α_{n+1}
α_n
g_{n+1}
f_{n+1}
phi_n
g_n
f_n
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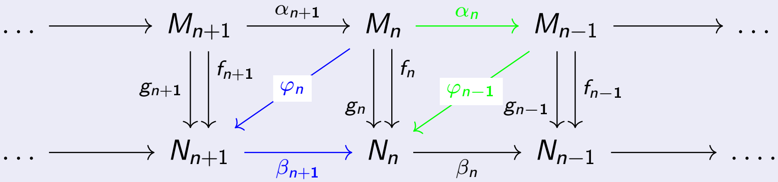
β_{n+1}
β_n



- If $x + \operatorname{Im} \alpha_{n+1} \in H_n(\mathbf{M})$, where $x \in \operatorname{Ker} \alpha_n$ then

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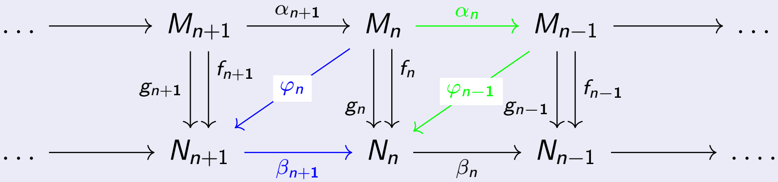

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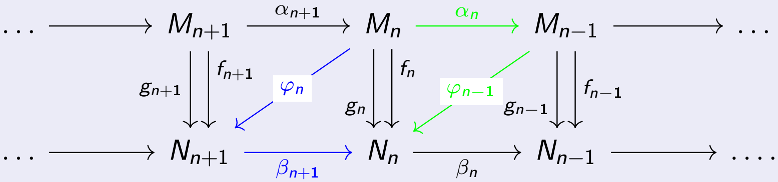

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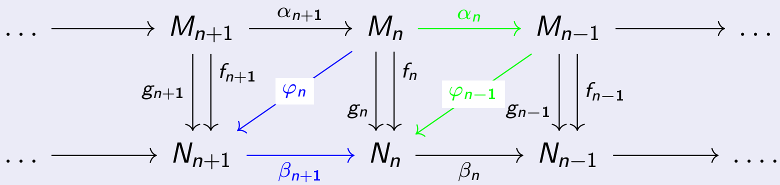
- If $x \in \operatorname{Ker}\alpha_n$ then

  $$\beta_n(f_n(x)) = f_{n-1}(\alpha_n(x)) = 0,$$

  so $f_n(x) \in \operatorname{Ker}\beta_n$.



- If $x + \operatorname{Im} \alpha_{n+1} \in H_n(\mathbf{M})$, where $x \in \operatorname{Ker} \alpha_n$ then

  $$f_n(x) - g_n(x) = \beta_{n+1}(\varphi_n(x)) + \varphi_{n-1}(\alpha_n(x)) = \beta_{n+1}(\varphi_n(x)) \in \operatorname{Im} \beta_{n+1},$$

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  The same works for g_n.

□

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Chain complexes and chain maps form a preadditive category. We denote it by $\mathbf{Chain}_R$.

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Let $\mathbf{f}, \mathbf{f}' \colon \mathbf{L} \to \mathbf{M}$ and $\mathbf{g} \colon \mathbf{M} \to \mathbf{N}$ be chain maps. We define

- $\mathbf{g} \circ \mathbf{f} = \{g_n \circ f_n \colon L_n \to N_n\}_{n \in \mathbb{Z}}$.
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Proposition

Let $\mathbf{M}$ and $\mathbf{N}$ be two chain complexes of R-modules and R-module homomorphisms. The relation "$\approx$" on $\mathrm{Mor}(\mathbf{M}, \mathbf{N})$ given by $\mathbf{f} \approx \mathbf{g}$, if there is a homotopy $\varphi \colon \mathbf{f} \to \mathbf{g}$ is an equivalence relation on $\mathrm{Mor}(\mathbf{M}, \mathbf{N})$ in $\mathbf{Chain}_R$.

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Let $\mathbf{M}$ *and* $\mathbf{N}$ *be two chain complexes of R-modules and R-module homomorphisms. The relation "*$\approx$*" on* $\mathrm{Mor}(\mathbf{M}, \mathbf{N})$ *given by* $\mathbf{f} \approx \mathbf{g}$*, if there is a homotopy* $\varphi \colon \mathbf{f} \to \mathbf{g}$ *is an equivalence relation on* $\mathrm{Mor}(\mathbf{M}, \mathbf{N})$ *in* $\mathbf{Chain}_R$*. We call the equivalence class* $[\mathbf{f}]$ *determined by this equivalence relation the homotopy class of* $\mathbf{f}$.

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- $\mathbf{g} \circ \mathbf{f} = \{g_n \circ f_n : L_n \to N_n\}_{n \in \mathbb{Z}}$.
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Proposition

For each $n \in \mathbb{Z}$, $H_n : \mathbf{Chain}_R \to \mathbf{Mod}_R$ is an additive functor.

What Is The Goal?

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Proposition

Corresponding to each short exact sequence

$$0 \to \mathbf{L} \xrightarrow{\mathbf{f}} \mathbf{M} \xrightarrow{\mathbf{g}} \mathbf{N} \to 0$$

of chain complexes of R- modules, there exists a long exact sequence in homology of the form

$$\dots \xrightarrow{\Phi_{n+1}} H(\mathbf{L}) \xrightarrow{H_n(\mathbf{f})} H_n(\mathbf{M}) \xrightarrow{H_n(\mathbf{g})} H_n(\mathbf{N}) \xrightarrow{\Phi_n} H_{n-1}(\mathbf{L}) \xrightarrow{H_{n-1}(\mathbf{f})} H_{n-1}(\mathbf{M}) \xrightarrow{H_{n-1}(\mathbf{g})} H_{n-1}(\mathbf{N}) \xrightarrow{\Phi_{n-1}} \dots$$

where Φ_n is a connecting homomorphism for each $n \in \mathbb{Z}$.

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We need some preparation. . . .

Proof Part One: Exact Sequences Of Chain Complexes

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Definition

Let $\mathbf{L}, \mathbf{M}$ and $\mathbf{N}$ be three chain complexes of R-modules and R-module homomorphisms. Let

$$\mathbf{L} \xrightarrow{\mathbf{f}} \mathbf{M} \xrightarrow{\mathbf{g}} \mathbf{N}$$

be a sequence.

- A sequence

$$\mathbf{0} \to \mathbf{L} \xrightarrow{\mathbf{f}} \mathbf{M} \xrightarrow{\mathbf{g}} \mathbf{N} \to \mathbf{0}$$

  is said to be a *short exact sequence*, if $0 \to L_n \xrightarrow{f_n} M_n \xrightarrow{g_n} N_n \to 0$ is a short exact sequence in $\mathbf{Mod}_R$ for each $n \in \mathbb{Z}$.

Proof Part One: Exact Sequences Of Chain Complexes

Actually, a short exact sequence of chain complexes is a 2-dimensional commutative diagram of the form

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L_{n+1} & \xrightarrow{f_{n+1}} & M_{n+1} & \xrightarrow{g_{n+1}} & N_{n+1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_{n+1}} & & \downarrow{\scriptstyle \beta_{n+1}} & & \downarrow{\scriptstyle \gamma_{n+1}} & & \\
0 & \longrightarrow & L_{n} & \xrightarrow{f_{n}} & M_{n} & \xrightarrow{g_{n}} & N_{n} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_{n}} & & \downarrow{\scriptstyle \beta_{n}} & & \downarrow{\scriptstyle \gamma_{n}} & & \\
0 & \longrightarrow & L_{n-1} & \xrightarrow{f_{n-1}} & M_{n-1} & \xrightarrow{g_{n-1}} & N_{n-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

Proof Part One: Exact Sequences Of Chain Complexes

Remark

Note that if the diagram

$$\begin{array}{ccc} M_1 & \xrightarrow{\alpha} & M_2 \\ \downarrow f & & \downarrow g \\ N_1 & \xrightarrow{\beta} & N_2 \end{array}$$

of R-modules and R-module homomorphisms is commutative, then there are induced mappings $\bar{f}\colon \operatorname{Ker}\alpha \to \operatorname{Ker}\beta$ *and* $\bar{g}\colon \operatorname{Coker}\alpha \to \operatorname{Coker}\beta$, *which are defined by* $\bar{f}(x) = f(x)$ *and* $\bar{g}(x + \operatorname{Im}\alpha) = g(x) + \operatorname{Im}\beta$.

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We obtain the following diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \operatorname{Ker}\alpha & \longrightarrow & M_1 & \xrightarrow{\alpha} & M_2 & \longrightarrow & \operatorname{Coker}\alpha & \longrightarrow & 0 \\ & & \big\downarrow \bar{f} & & \big\downarrow f & & \big\downarrow g & & \big\downarrow \bar{g} & & \\ 0 & \longrightarrow & \operatorname{Ker}\beta & \longrightarrow & N_1 & \xrightarrow{\beta} & N_2 & \longrightarrow & \operatorname{Coker}\beta & \longrightarrow & 0 \end{array}$$

Proof Part Two: What About The Snakes?

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Lemma (Snake Lemma)

Let

$$\begin{array}{ccccccc} & & M_1 & \xrightarrow{f_1} & M & \xrightarrow{g_1} & M_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & N_1 & \xrightarrow{f_2} & N & \xrightarrow{g_2} & N_2 & & \end{array}$$

be a commutative diagram of R-modules and R-module homomorphisms with exact rows. Then there is an R-linear mapping $\Phi\colon \operatorname{Ker}\gamma \to \operatorname{Coker}\alpha$ *such that the sequence*

$$\operatorname{Ker}\alpha \xrightarrow{\bar{f_1}} \operatorname{Ker}\beta \xrightarrow{\bar{g_1}} \operatorname{Ker}\gamma \xrightarrow{\Phi} \operatorname{Coker}\alpha \xrightarrow{\bar{f_2}} \operatorname{Coker}\beta \xrightarrow{\bar{g_2}} \operatorname{Coker}\gamma$$

is exact.

Proof

$$\begin{array}{ccccccccc} & & M_1 & \xrightarrow{f_1} & M & \xrightarrow{g_1} & M_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & N_1 & \xrightarrow{f_2} & N & \xrightarrow{g_2} & N_2 & & \end{array}$$

(i) Existence of Φ:

Proof

$$\begin{array}{ccccccccc} & & M_1 & \xrightarrow{f_1} & M & \xrightarrow{g_1} & M_2 & \longrightarrow & 0 \\ & & \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \\ 0 & \longrightarrow & N_1 & \xrightarrow{f_2} & N & \xrightarrow{g_2} & N_2 & & \end{array}$$

(i) Existence of Φ:
For $z \in \operatorname{Ker}\gamma$ take $x \in M$ such that $g_1(x) = z$.

Proof

$$\begin{array}{ccccccccc}
 & & M_1 & \xrightarrow{f_1} & M & \xrightarrow{g_1} & M_2 & \longrightarrow & 0 \\
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(i) Existence of Φ:
For $z \in \operatorname{Ker} \gamma$ take $x \in M$ such that $g_1(x) = z$.
Then $\gamma(g_1(x)) = 0$, so $g_2(\beta(x)) = 0$.

Proof

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For $z \in \operatorname{Ker} \gamma$ take $x \in M$ such that $g_1(x) = z$.
Then $\gamma(g_1(x)) = 0$, so $g_2(\beta(x)) = 0$. Thus $\beta(x) \in \operatorname{Ker} g_2 = \operatorname{Im} f_2$.

Proof

$$
\begin{array}{ccccccccc}
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f_2 is an injection, so there is a unique $y \in N_1$ such that $f_2(y) = \beta(x)$.

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f_2 is an injection, so there is a unique $y \in N_1$ such that $f_2(y) = \beta(x)$.
Define Φ by

$$\Phi(z) = y + \operatorname{Im} \alpha.$$

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(ii) Well-definedness of Φ:

$$\begin{array}{ccccccccc} & & M_1 & \xrightarrow{f_1} & M & \xrightarrow{g_1} & M_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & N_1 & \xrightarrow{f_2} & N & \xrightarrow{g_2} & N_2 & & \end{array}$$

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Let $x, x' \in M$ such that $g_1(x) = g_1(x') = z$.

$$\begin{array}{ccccccccc} & & M_1 & \xrightarrow{f_1} & M & \xrightarrow{g_1} & M_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & N_1 & \xrightarrow{f_2} & N & \xrightarrow{g_2} & N_2 & & \end{array}$$

(ii) Well-definedness of Φ:
Let $x, x' \in M$ such that $g_1(x) = g_1(x') = z$.
Suppose that there are $y, y' \in N_1$ with $f_2(y) = \beta(x)$ and $f_2(y') = \beta(x')$.

$$
\begin{array}{ccccccccc}
 & & M_1 & \xrightarrow{f_1} & M & \xrightarrow{g_1} & M_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
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Then $x - x' \in \operatorname{Ker} g_1 = \operatorname{Im} f_1$, so there is $w \in M_1$ such that $f_1(w) = x - x'$.

$$\begin{array}{ccccccc} & & M_1 & \xrightarrow{f_1} & M & \xrightarrow{g_1} & M_2 & \longrightarrow & 0 \\ & & \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \\ 0 & \longrightarrow & N_1 & \xrightarrow{f_2} & N & \xrightarrow{g_2} & N_2 & & \end{array}$$

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$$f_2(y) - f_2(y') = \beta(x) - \beta(x') = \beta(f_1(w)) = f_2(\alpha(w)),$$

$$
\begin{array}{ccccccc}
 & & M_1 & \xrightarrow{f_1} & M & \xrightarrow{g_1} & M_2 & \longrightarrow & 0 \\
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$$f_2(y) - f_2(y') = \beta(x) - \beta(x') = \beta(f_1(w)) = f_2(\alpha(w)),$$

so $y - y' - \alpha(w) \in \operatorname{Ker} f_2 = 0$. Thus $y - y' = \alpha(w) \in \operatorname{Im} \alpha$, so $y + \operatorname{Im} \alpha = y' + \operatorname{Im} \alpha$.

$$
\begin{array}{ccccccccc}
 & & M_1 & \xrightarrow{f_1} & M & \xrightarrow{g_1} & M_2 & \longrightarrow & 0 \\
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\end{array}
$$

(iii) Exactness at $\operatorname{Ker}\gamma$:

$$\begin{array}{ccccccccc} & & M_1 & \xrightarrow{f_1} & M & \xrightarrow{g_1} & M_2 & \longrightarrow & 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\ 0 & \longrightarrow & N_1 & \xrightarrow{f_2} & N & \xrightarrow{g_2} & N_2 & & \end{array}$$

(iii) Exactness at $\operatorname{Ker}\gamma$:
$\operatorname{Ker}\Phi \subset \operatorname{Im}\bar{g_1}$: Let $z \in \operatorname{Ker}\gamma$ such that $z \in \operatorname{Ker}\Phi$, $x \in M$, $y \in N_1$ as above.

$$\begin{array}{ccccccccc} & & M_1 & \xrightarrow{f_1} & M & \xrightarrow{g_1} & M_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & N_1 & \xrightarrow{f_2} & N & \xrightarrow{g_2} & N_2 & & \end{array}$$

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$$0 = \Phi(z) = y + \operatorname{Im}\alpha,$$

thus $y \in \operatorname{Im}\alpha$; there is $u \in M_1$ such that $\alpha(u) = y$.

$$\begin{array}{ccccccc} & & M_1 & \xrightarrow{f_1} & M & \xrightarrow{g_1} & M_2 & \longrightarrow & 0 \\ & & \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \\ 0 & \longrightarrow & N_1 & \xrightarrow{f_2} & N & \xrightarrow{g_2} & N_2 & & \end{array}$$

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$$\beta(f_1(u)) = f_2(\alpha(u)) = f_2(y) = \beta(x),$$

$$
\begin{array}{ccccccccc}
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$$0 = \Phi(z) = y + \operatorname{Im}\alpha,$$

thus $y \in \operatorname{Im}\alpha$; there is $u \in M_1$ such that $\alpha(u) = y$. Thus

$$\beta(f_1(u)) = f_2(\alpha(u)) = f_2(y) = \beta(x),$$

so $x - f_1(u) \in \operatorname{Ker}\beta$.

$$
\begin{array}{ccccccccc}
 & & M_1 & \xrightarrow{f_1} & M & \xrightarrow{g_1} & M_2 & \longrightarrow & 0 \\
 & & \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \\
0 & \longrightarrow & N_1 & \xrightarrow{f_2} & N & \xrightarrow{g_2} & N_2 & &
\end{array}
$$

If

$$x - f_1(u) = w \in \operatorname{Ker} \beta,$$

then $z = g_1(x) = g_1(f_1(u)) = g_1(w)$;

$$
\begin{array}{ccccccccc}
 & & M_1 & \xrightarrow{f_1} & M & \xrightarrow{g_1} & M_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & N_1 & \xrightarrow{f_2} & N & \xrightarrow{g_2} & N_2 & &
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$$
\begin{array}{ccccccccc}
 & & M_1 & \xrightarrow{f_1} & M & \xrightarrow{g_1} & M_2 & \longrightarrow & 0 \\
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\end{array}
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then $z = g_1(x) = g_1(f_1(u)) = g_1(w)$; $z \in \operatorname{Im}\bar{g_1}$.
$\operatorname{Im}\bar{g_1} \subset \operatorname{Ker}\Phi$: Let $z \in \operatorname{Ker}\gamma$ such that $z \in \operatorname{Im}\bar{g_1} \subset \operatorname{Im} g_1$.

$$\begin{array}{ccccccccc} & & M_1 & \xrightarrow{f_1} & M & \xrightarrow{g_1} & M_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & N_1 & \xrightarrow{f_2} & N & \xrightarrow{g_2} & N_2 & & \end{array}$$

If

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then $z = g_1(x) = g_1(f_1(u)) = g_1(w)$; $z \in \operatorname{Im} \bar{g_1}$.
$\operatorname{Im} \bar{g_1} \subset \operatorname{Ker} \Phi$: Let $z \in \operatorname{Ker} \gamma$ such that $z \in \operatorname{Im} \bar{g_1} \subset \operatorname{Im} g_1$.
There is $x \in \operatorname{Ker} \beta$ such that $g_1(x) = z$ and $y \in N_1$ such that $f_2(y) = \beta(x) = 0$.

$$\begin{array}{ccccccccc}
 & & M_1 & \xrightarrow{f_1} & M & \xrightarrow{g_1} & M_2 & \longrightarrow & 0 \\
 & & \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \\
0 & \longrightarrow & N_1 & \xrightarrow{f_2} & N & \xrightarrow{g_2} & N_2 & &
\end{array}$$

If

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There is $x \in \operatorname{Ker}\beta$ such that $g_1(x) = z$ and $y \in N_1$ such that $f_2(y) = \beta(x) = 0$.
Thus $y = 0$ by injectivity of f_2, so $\Phi(z) = y + \operatorname{Im}\alpha = 0$.

$$\begin{array}{ccccccc} & & M_1 & \xrightarrow{f_1} & M & \xrightarrow{g_1} & M_2 & \longrightarrow & 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\ 0 & \longrightarrow & N_1 & \xrightarrow{f_2} & N & \xrightarrow{g_2} & N_2 & & \end{array}$$

(iii) Exactness at Coker α:

$$\begin{array}{ccccccc}
 & & M_1 & \xrightarrow{f_1} & M & \xrightarrow{g_1} & M_2 & \longrightarrow & 0 \\
 & & \downarrow{\alpha} & & \downarrow{\beta} & & \downarrow{\gamma} & & \\
0 & \longrightarrow & N_1 & \xrightarrow{f_2} & N & \xrightarrow{g_2} & N_2 & &
\end{array}$$

(iii) Exactness at Coker α:
Analogously to the proof of the exactness at Ker γ.

(iv) R-linearity of Φ:

$$
\begin{array}{ccccccccc}
 & & M_1 & \xrightarrow{f_1} & M & \xrightarrow{g_1} & M_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\
0 & \longrightarrow & N_1 & \xrightarrow{f_2} & N & \xrightarrow{g_2} & N_2 & &
\end{array}
$$

(iii) Exactness at $\operatorname{Coker}\alpha$:
Analogously to the proof of the exactness at $\operatorname{Ker}\gamma$.

(iv) R-linearity of Φ:
Since all mappings which are involved in the definition of the map Φ are R- linear, via an easy computation Φ is R-linear.

$$\begin{array}{ccccccc} & & M_1 & \xrightarrow{f_1} & M & \xrightarrow{g_1} & M_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & N_1 & \xrightarrow{f_2} & N & \xrightarrow{g_2} & N_2 & & \end{array}$$

(iii) Exactness at $\operatorname{Coker} \alpha$:
Analogously to the proof of the exactness at $\operatorname{Ker} \gamma$.

(iv) R-linearity of Φ:
Since all mappings which are involved in the definition of the map Φ are R- linear, via an easy computation Φ is R-linear.

(v) Exactness of the sequences

$$\operatorname{Ker} \alpha \xrightarrow{\bar{f_1}} \operatorname{Ker} \beta \xrightarrow{\bar{g_1}} \operatorname{Ker} \gamma$$

and

$$\operatorname{Coker} \alpha \xrightarrow{\bar{f_2}} \operatorname{Coker} \beta \xrightarrow{\bar{g_2}} \operatorname{Coker} \gamma :$$

Easy.

Proof Part Three: From Short To Long Exact Sequences

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Lemma

Let $\mathbf{M}$ *be a chain complex of R-modules and R-module homomorphisms. The map* $\alpha_n \colon M_n \to M_{n-1}$ *induces an R-linear mapping*

$$\bar{\alpha_n} \colon \operatorname{Coker} \alpha_{n+1} \to \operatorname{Ker} \alpha_{n-1}.$$

Moreover, $H_n(\mathbf{M}) = \operatorname{Ker} \bar{\alpha_n}$ *and* $H_{n-1}(\mathbf{M}) = \operatorname{Coker} \bar{\alpha_n}$*.*

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Moreover, $H_n(\mathbf{M}) = \operatorname{Ker} \bar{\alpha_n}$ *and* $H_{n-1}(\mathbf{M}) = \operatorname{Coker} \bar{\alpha_n}$.

Remark

In the Snake-Lemma, let f_1 be injective and g_2 be surjective. Then $\bar{f_1}$ is injective and $\bar{g_2}$ is surjective. Thus from the short exact sequence

$$\mathbf{0} \to \mathbf{L} \xrightarrow{\mathbf{f}} \mathbf{M} \xrightarrow{\mathbf{g}} \mathbf{N} \to \mathbf{0}$$

we obtain for each $n \in \mathbb{Z}$ the following row and column exact diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Ker}\alpha_n & \longrightarrow & \operatorname{Ker}\beta_n & \longrightarrow & \operatorname{Ker}\gamma_n & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L_n & \longrightarrow & M_n & \longrightarrow & N_n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_n} & & \downarrow{\scriptstyle \beta_n} & & \downarrow{\scriptstyle \gamma_n} & & \\
0 & \longrightarrow & L_{n-1} & \longrightarrow & M_{n-1} & \longrightarrow & N_{n-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \operatorname{Coker}\alpha_n & \longrightarrow & \operatorname{Coker}\beta_n & \longrightarrow & \operatorname{Coker}\gamma_n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}
$$

By using this diagram and one previous lemma, we obtain the following diagram

$$
\begin{array}{ccccccccc}
 & & & & & & H_n(\mathbf{N}) & & \\
 & & & & & & \| & & \\
 & & & & & & \operatorname{Ker} \bar{\gamma_n} & & \\
 & & & & & & \downarrow & & \\
 & & \operatorname{Coker} \alpha_{n+1} & \longrightarrow & \operatorname{Coker} \beta_{n+1} & \longrightarrow & \operatorname{Coker} \gamma_{n+1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \bar{\alpha_n}} & & \downarrow{\scriptstyle \bar{\beta_n}} & & \downarrow{\scriptstyle \bar{\gamma_n}} & & \\
0 & \longrightarrow & \operatorname{Ker} \alpha_{n-1} & \longrightarrow & \operatorname{Ker} \beta_{n-1} & \longrightarrow & \operatorname{Ker} \gamma_{n-1} & & \\
 & & \downarrow & & & & & & \\
 & & \operatorname{Coker} \bar{\alpha_n} & & & & & & \\
 & & \| & & & & & & \\
 & & H_{n-1}(\mathbf{L}) & & & & & &
\end{array}
$$

for each $n \in \mathbb{Z}$. The Snake Lemma gives us $\Phi_n \colon H_n(\mathbf{N}) \to H_{n-1}(\mathbf{L})$.

Last But Not Least: Find Your Purpose!

What category are you?

The category of

Length of your first name:		Month you were born:		Day you were born:			
1.	Commutative	1.	Pre-	1.	Groups	17.	Chains
2.	Infinitary	2.	Algebraic	2.	Manifolds	18.	Distributions
3.	Projective	3.	Quasi-	3.	Sheaves	19.	Bundles
4.	Preadditive	4.	Hilbert	4.	Complexes	20.	Sequences
5.	Opposite	5.	Differential	5.	Schemes	21.	Grassmanians
6.	Semi-	6.	p-adic	6.	Filtrations	22.	Surfaces
7.	Smooth	7.	Discrete	7.	Spaces	23.	Modules
8.	Injective	8.	Pointed	8.	Graphs	24.	Varieties
9.	Homogenous	9.	Coherent	9.	Monoids	25.	Curves
10.	Local	10.	Affine	10.	Functors	26.	Magmas
11.	Hyper-	11.	Complex	11.	Morphisms	27.	Languages
12+	Complete	12.	Simplicial	12.	Groupoids	28.	Fibrations
				13.	Algebras	29.	Knots
				14.	Diagrams	30.	Universes
				15.	Fields	31.	Lattices
				16.	Representations		