Homological Algebra

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For every talk, we assume R to be some fixed unital ring and any module is assumed to be an R-module. If not explicitly stated otherwise, any linear map is assumed to be R-linear.

1 About Chains and Snakes

Definition 1.1: A *left module* M is an abelian group $(M, +)$ together with a map

$$R \times M \to M, \ (r, m) \mapsto r \cdot m,$$

such that for all $r, r_1, r_2 \in R, \ m, m_1, m_2 \in M$ the following hold:

- $r_1 \cdot (r_1 \cdot m) = (r_1 \cdot r_2) \cdot m$
- $(r_1 + r_2) \cdot m = r_1 \cdot m + r_2 \cdot m$
- $r \cdot (m_1 + m_2) = r \cdot m_1 + r \cdot m_2$
- $1 \cdot m = m.$

Definition 1.2: A family $\mathbf{M} = \{M_n, \alpha_n\}_{n \in \mathbb{Z}}$ of modules and module homomorphisms

$$\mathbf{M}\colon \ \ldots \longrightarrow M_{n+1} \xrightarrow{\alpha_{n+1}} M_n \xrightarrow{\alpha_n} M_{n-1} \xrightarrow{\alpha_{n-1}} \ldots$$

such that $\alpha_n \circ \alpha_{n+1} = 0$ for each $n \in \mathbb{Z}$, i.e. $\operatorname{im} \alpha_{n+1} \subset \ker \alpha_n$, is called *chain complex.* Each mapping $\alpha_n \colon M_n \to M_{n-1}$ is called a *boundary mapping* or *differential operator.*

Definition 1.3: We call a chain complex

$$\mathbf{M}: \ \cdots \to M_{n+1} \xrightarrow{\alpha_{n+1}} M_n \xrightarrow{\alpha_n} M_{n-1} \xrightarrow{\alpha_{n-1}} M_{n-2} \to \cdots$$

- *exact at* M_n if $\operatorname{im}(\alpha_{n+1}) = \ker(\alpha_n)$.
- *exact* if it is exact at M_n for each $n \in \mathbb{Z}$.
- *short exact sequence* if it is an exact complex of the form

$$0 \to M_2 \xrightarrow{\alpha_2} M_1 \xrightarrow{\alpha_1} M_0 \to 0.$$

Definition 1.4: Let

$$\mathbf{M}:\ \cdots \to M_{n+1} \xrightarrow{\alpha_{n+1}} M_n \xrightarrow{\alpha_n} M_{n-1} \xrightarrow{\alpha_{n-1}} M_{n-2} \to \cdots$$

be a chain complex. We call $H_n(\mathbf{M}) = \ker \alpha_n / \operatorname{im} \alpha_{n+1}$ the *n-th homology module* of $\mathbf{M}$.

Definition 1.5: Let $\mathbf{M}$ and $\mathbf{N}$ be two chain complexes. A family $\mathbf{f} = \{f_n : M_n \to N_{n+k}\}_{n\in\mathbb{Z}}$ of linear mappings such that the diagram

$$\begin{array}{ccccccc}
\ldots & \xrightarrow{\alpha_{n+1}} & M_n & \xrightarrow{\alpha_n} & M_{n-1} & \xrightarrow{\alpha_{n-1}} & \ldots \\
 & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\ldots & \xrightarrow{\beta_{n+k+1}} & N_{n+k} & \xrightarrow{\beta_{n+k}} & N_{n+k-1} & \xrightarrow{\beta_{n+k-1}} & \ldots
\end{array}$$

is commutative for each $n \in \mathbb{Z}$ is called *chain map of degree k*.

Definition 1.6: Let $\mathbf{M}$ and $\mathbf{N}$ be two chain complexes and let $\mathbf{f}, \mathbf{g}\colon \mathbf{M} \to \mathbf{N}$ be two chain maps of degree 0. A chain map $\boldsymbol{\varphi} = \{\varphi_n\colon M_n \to N_{n+1}\}$ of degree 1 such that $f_n - g_n = \beta_{n+1}\varphi_n + \varphi_{n-1}\alpha_n$ for each $n \in \mathbb{Z}$ is called a *homotopy*.

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & M_{n+1} & \xrightarrow{\alpha_{n+1}} & M_n & \xrightarrow{\alpha_n} & M_{n-1} & \longrightarrow & \ldots \\
 & & {\scriptstyle g_{n+1}}\downarrow\downarrow{\scriptstyle f_{n+1}} & \swarrow{\scriptstyle \varphi_n} & {\scriptstyle g_n}\downarrow\downarrow{\scriptstyle f_n} & \swarrow{\scriptstyle \varphi_{n-1}} & {\scriptstyle g_{n-1}}\downarrow\downarrow{\scriptstyle f_{n-1}} & & \\
\ldots & \longrightarrow & N_{n+1} & \xrightarrow[\beta_{n+1}]{} & N_n & \xrightarrow[\beta_n]{} & N_{n-1} & \longrightarrow & \ldots .
\end{array}$$

We denote it by $\boldsymbol{\varphi}\colon \mathbf{f} \to \mathbf{g}$ ($\mathbf{f} \approx \mathbf{g}$) and say that $\mathbf{f}$ and $\mathbf{g}$ are *homotopic chain maps*. If there exist two chain maps $\mathbf{f}\colon \mathbf{M} \to \mathbf{N}$ and $\mathbf{g}\colon \mathbf{N} \to \mathbf{M}$ such that $\mathbf{gf} \approx \mathrm{id}_{\mathbf{M}}$ and $\mathbf{fg} \approx \mathrm{Id}_{\mathbf{N}}$ then $\mathbf{M}$ and $\mathbf{N}$ are said to be of the *same homotopy type*. The chain maps $\mathbf{f}\colon \mathbf{M} \to \mathbf{N}$ and $\mathbf{g}\colon \mathbf{N} \to \mathbf{M}$ are called *homotopy equivalences*.

Definition 1.7: A category $\mathcal{C}$ is called *(pre-)additive* if for all objects $A, B \in \mathcal{C}$ the set $\mathrm{Mor}_{\mathcal{C}}(A, B)$ has the structure of an additive abelian group and if for all $f, f_1, f_2 \in \mathrm{Mor}_{\mathcal{C}}(A, B)$ and all $g, g_1, g_2 \in \mathrm{Mor}_{\mathcal{C}}(B, C)$ it holds

$$g \circ (f_1 + f_2) = g \circ f_1 + g \circ f_2$$

and

$$(g_1 + g_2) \circ f = g_1 \circ f + g_2 \circ f.$$

Definition 1.8: Let $\mathcal{C}$ and $\mathcal{D}$ be (pre-)additive categories. A (covariant or contravariant) functor $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ is called *additive* if for all objects $A, B \in \mathcal{C}$ the induced map

$$\begin{aligned} \mathrm{Mor}_{\mathcal{C}}(A,B) &\to \mathrm{Mor}_{\mathcal{D}}\big(\mathcal{F}(A), \mathcal{F}(B)\big) \\ f &\mapsto \mathcal{F}(f) \end{aligned}$$

is a group homomorphism.

Chain complexes and chain maps form an additive category. We denote it by $\mathbf{Chain}_R$.

Definition 1.9: Let $\mathbf{L}, \mathbf{M}$ and $\mathbf{N}$ be three chain complexes. A sequence

$$\mathbf{0} \to \mathbf{L} \xrightarrow{\mathbf{f}} \mathbf{M} \xrightarrow{\mathbf{g}} \mathbf{N} \to \mathbf{0}$$

of chain complexes is said to be a *short exact sequence*, if for each $n \in \mathbb{Z}$ the sequence $0 \to L_n \xrightarrow{f_n} M_n \xrightarrow{g_n} N_n \to 0$ is a short exact sequence in $\mathbf{Mod}_R$. Note that a short exact sequence of chain complexes is a 2-dimensional commutative diagram of the form

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L_{n+1} & \xrightarrow{f_{n+1}} & M_{n+1} & \xrightarrow{g_{n+1}} & N_{n+1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_{n+1}} & & \downarrow{\scriptstyle \beta_{n+1}} & & \downarrow{\scriptstyle \gamma_{n+1}} & & \\
0 & \longrightarrow & L_n & \xrightarrow{f_n} & M_n & \xrightarrow{g_n} & N_n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_n} & & \downarrow{\scriptstyle \beta_n} & & \downarrow{\scriptstyle \gamma_n} & & \\
0 & \longrightarrow & L_{n-1} & \xrightarrow{f_{n-1}} & M_{n-1} & \xrightarrow{g_{n-1}} & N_{n-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Definition 1.10: Let M and N be modules and let $f\colon M \to N$ be a linear map. Then $\operatorname{coker} f := N/\operatorname{im} f$ is called *cokernel of f*.

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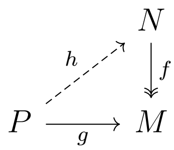
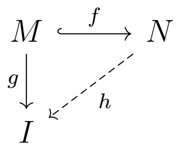
2 Projective and injective resolutions

Definition 2.1: Let P be a module. If for every surjective homomorphism $f\colon M \to N$ and every homomorphism $g\colon P \to M$ there is a homomorphism $h\colon P \to N$ that renders commutative the diagram



then P is called *projective.*

Codefinition 2.2: Let M, N and I be modules. If for every injective homomorphism $f\colon M \to N$ and any homomorphism $g\colon M \to I$ there is a homomorphism $h\colon N \to I$ rendering commutative the diagram



then I is called *injective.*

Codefinition 2.3: Let $\mathbf{C} = (M^n, \alpha^n)_{n\in\mathbb{Z}}$ be a family of modules and linear maps of the form

$$\mathbf{C}\colon \; \ldots \longrightarrow M^{n-1} \overset{\alpha^{n-1}}{\longrightarrow} M^n \overset{\alpha^n}{\longrightarrow} M^{n+1} \longrightarrow \ldots$$

If for all integers n it holds $\alpha^{n+1} \circ \alpha^n = 0$, then $\mathbf{C}$ is called a *cochain complex.* For the integer n, the quotient

$$H^n(\mathbf{C}) = \ker \alpha^n / \operatorname{im} \alpha^{n-1}$$

is called the *n-th cohomology module of* $\mathbf{C}$.

Definition 2.4: Let $\mathbf{C}$ be a positive chain complex

$$\mathbf{C}\colon \; \ldots \longrightarrow M_n \longrightarrow \ldots \longrightarrow M_0 \longrightarrow M \longrightarrow 0$$

Then $\mathbf{C}_M$, the so called *deleted chain complex*, denotes the complex where M is omitted.

Definition 2.5: Let $\mathbf{P}' = (P_n, \alpha_n)_{n \in \mathbb{N}_0}$ be an exact positive chain complex. If all P_n are projective and if the chain complex

$$\mathbf{P}\colon \ \ldots \longrightarrow P_n \xrightarrow{\alpha_n} \ldots \longrightarrow P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} M \longrightarrow 0$$

is exact, then $\mathbf{P}$ is called a *projective resolution of* M. In this case, $\mathbf{P}_M := \mathbf{P}'$ is called the *deleted projective resolution of* M.

Codefinition 2.6: Let $\mathbf{I}$ be the exact positive cochain complex

$$\mathbf{I}\colon 0 \longrightarrow M \xrightarrow{\alpha^{-1}} I^0 \xrightarrow{\alpha^0} I^1 \longrightarrow \ldots \longrightarrow I^{n-1} \xrightarrow{\alpha^{n-1}} I^n \longrightarrow \ldots$$

If for every natural number n the module I_n is injective, then $\mathbf{I}$ is called *injective resolution of* M. Again, $\mathbf{I}^M$ denotes the deleted injective resolution of M.

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3 Derived Functors

Definition 3.1: Let R and S be rings with identity.

(a) A covariant functor $\mathcal{F}\colon \mathbf{Mod}_R \to \mathbf{Mod}_S$ is called

(i) *left exact* if for every exact sequence

$$0 \longrightarrow M_1 \xrightarrow{f} M \xrightarrow{g} M_2$$

in $\mathbf{Mod}_R$, the sequence

$$0 \longrightarrow \mathcal{F}(M_1) \xrightarrow{\mathcal{F}(f)} \mathcal{F}(M) \xrightarrow{\mathcal{F}(g)} \mathcal{F}(M_2)$$

is exact in $\mathbf{Mod}_S$.

(ii) *right exact* if for every exact sequence

$$M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \longrightarrow 0$$

in $\mathbf{Mod}_R$, the sequence

$$\mathcal{F}(M_1) \xrightarrow{\mathcal{F}(f)} \mathcal{F}(M) \xrightarrow{\mathcal{F}(g)} \mathcal{F}(M_2) \longrightarrow 0$$

is exact in $\mathbf{Mod}_S$.

(iii) *exact* if it is both left exact and right exact.

(b) A contravariant functor $\mathcal{F}\colon \mathbf{Mod}_R \to \mathbf{Mod}_S$ is called

(i) *left exact* if for every exact sequence

$$M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \longrightarrow 0$$

in $\mathbf{Mod}_R$, the sequence

$$0 \longrightarrow \mathcal{F}(M_2) \xrightarrow{\mathcal{F}(g)} \mathcal{F}(M) \xrightarrow{\mathcal{F}(f)} \mathcal{F}(M_1)$$

is exact in $\mathbf{Mod}_S$.

(ii) *right exact* if for every exact sequence

$$0 \longrightarrow M_1 \xrightarrow{f} M \xrightarrow{g} M_2$$

in $\mathbf{Mod}_R$, the sequence

$$\mathcal{F}(M_2) \xrightarrow{\mathcal{F}(g)} \mathcal{F}(M) \xrightarrow{\mathcal{F}(f)} \mathcal{F}(M_1) \longrightarrow 0$$

is exact in $\mathbf{Mod}_S$.

(iii) *exact* if it is both left exact and right exact.

Definition 3.2: Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $\mathcal{F}, \mathcal{G}\colon \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* η *from* $\mathcal{F}$ *to* $\mathcal{G}$ is a rule that associates a morphism $\eta_A\colon \mathcal{F}(A) \to \mathcal{G}(A)$ in $\mathcal{D}$ with every object $A \in \mathcal{C}$ is such a way that for every morphism $f\colon A \to B$ in $\mathcal{C}$ the diagram

$$\begin{array}{ccc} \mathcal{F}(A) & \xrightarrow{\eta_A} & \mathcal{G}(A) \\ {\scriptstyle \mathcal{F}(f)}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{G}(f)} \\ \mathcal{F}(B) & \xrightarrow{\eta_B} & \mathcal{G}(B) \end{array}$$

is commutative.

If, moreover, η_A is an isomorphism in $\mathcal{D}$ for every object $A \in \mathcal{C}$, then $\eta\colon \mathcal{F} \to \mathcal{G}$ is called a *natural isomorphism* and $\mathcal{F}$ and $\mathcal{G}$ are said to be *naturally equivalent* functors.

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4 Examples are Manifold

Definition 4.1: Let V and W be finite-dimensional vector spaces, let $f\colon V \to W$ be linear and let $f^*\colon W^* \to V^*$, $\psi \mapsto \psi \circ f$ be its dual map. Then there is one and only one linear map $\bigwedge^n f^*\colon \bigwedge^n W^* \to \bigwedge^n V^*$ which is uniquely determined by

$$\bigwedge^n f^*(\psi^1 \wedge \cdots \wedge \psi^n) = f^*(\psi^1) \wedge \cdots \wedge f^*(\psi^n).$$

For an n-form $\omega \in \bigwedge^n W^*$, $\bigwedge^n f^*(\omega)$ is called *pullback of ω along f.*

Definition 4.2 (Differential Form of Degree n): Let n be a natural number and let $U \subseteq \mathbb{R}^N$ be open. A map

$$\omega\colon U \longrightarrow \bigwedge^n (\mathbb{R}^N)^*, \qquad u \longmapsto \sum_{1 \leq i_1 < \cdots < i_n \leq N} f_{(i_1,\ldots,i_n)}(u)\, dx^{i_1} \wedge \cdots \wedge dx^{i_n},$$

where $f_{(i_1,\ldots,i_n)}\colon U \to \mathbb{R}$ are functions, is a *differential form of degree n on U*; briefly called *n-form* (on U). The space

$$\Omega^n(U) := \{\omega\colon U \to \bigwedge^n (\mathbb{R}^N)^* \text{ smooth differential form}\}$$

is called *space of smooth n-forms on U.*

Definition 4.3 (Wedge Product for Differential Forms): Let n and m be natural numbers, let $U \subseteq \mathbb{R}^N$ be open and let $\omega \in \Omega^n(U)$, $\eta \in \Omega^m(U)$. Then the function

$$\omega \wedge \eta\colon U \longrightarrow \bigwedge^{n+m} (\mathbb{R}^N)^*, \qquad u \longmapsto \omega(u) \wedge \eta(u)$$

is called *wedge product of ω and η.*

Definition 4.4: Let n be a natural number and let $U \subseteq \mathbb{R}^N$ be open. For a differential form $\omega = \sum_{1 \leq i_1 < \cdots < i_n \leq N} f_{(i_1,\ldots,i_n)}\, dx^{i_1} \wedge \cdots \wedge dx^{i_n} \in \Omega^n(U)$, the differential form

$$d\omega := \sum_{1 \leq i_1 < \cdots < i_n \leq N} \left(\sum_{j=1}^N \frac{\partial f_{(i_1,\ldots,i_n)}}{\partial x^j}\, dx^j \right) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_n}$$

is called *exterior derivative of ω.*

Definition 4.5: Let $U \subseteq \mathbb{R}^N$ and $V \subseteq \mathbb{R}^M$ be open, let $\varphi\colon U \to V$ be smooth and let $\omega \in \Omega^n(V)$. Then,

$$\begin{aligned}(\varphi^*\omega)\colon U &\longrightarrow \bigwedge^n (\mathbb{R}^N)^*, \\ u &\longmapsto \Big[(v_1, \ldots, v_n) \mapsto \omega(\varphi(u))[(D\varphi)(u)(v_1), \ldots, (D\varphi)(u)(v_n)]\Big]\end{aligned}$$

is called *pullback of ω along φ.*

Definition 4.6: Let $U \subseteq \mathbb{R}^N$ be open. Then there is the cochain complex

$$0 \longrightarrow \Omega^0(U) \xrightarrow{d^0} \Omega^1(U) \xrightarrow{d^1} \dots \xrightarrow{d^{N-1}} \Omega^N(U) \longrightarrow 0$$

For a natural number n, the $\mathbb{R}$-vector space $H^n(U) := \ker d^n / \operatorname{im} d^{n-1}$ is called the *n-th de Rham cohomology of U*, and the direct sum $H^\bullet(U) := \bigoplus_{n=0}^N H^n(U)$ is called the *de Rham cohomology of U*.

Reminder 4.7 (Line Integral): Let $U \subseteq \mathbb{R}^N$ be open and let $\gamma\colon [a,b] \to U$ be a smooth path. For $\eta \in \Omega^1(U)$, the value

$$\int_\gamma \eta := \int_a^b \gamma^*\eta(t)\,dt = \int_a^b \langle \eta(\gamma(t)), \gamma'(t) \rangle\,dt$$

is called *line integral of η along γ*.