Projective and Injective Resolutions

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Roadmap for this talk:

- Recap on projective and injective modules

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- Projective and injective resolutions

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- Lift of module homomorphisms to chain maps and homotopy properties of lift

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- Recap on projective and injective modules
- Projective and injective resolutions
- Lift of module homomorphisms to chain maps and homotopy properties of lift
- No examples


DEFINITIONS
BEST PART: EVEN MORE
DEFINITIONS
memegenerator.net

Examples in talk 4, promise!

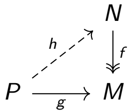
Section 1

Projective and Injective Modules

Projective Modules

Definition 1

Let P be a module. If for every surjective homomorphism $f: M \to N$ and every homomorphism $g: P \to M$ there is a homomorphism $h: P \to N$ that renders commutative the diagram



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$$\begin{array}{ccc} & & N \\ & \overset{h}{\nearrow} & \big\downarrow f \\ P & \xrightarrow[g]{} & M \end{array}$$

then P is called *projective*.

Note: P is projective iff every short exact sequence

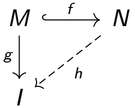
$$0 \longrightarrow A \overset{\iota}{\hookrightarrow} B \overset{\pi}{\twoheadrightarrow} P \longrightarrow 0$$

splits, i.e. there is a homomorphism $\varphi : P \to B$ such that $\pi \circ \varphi = \mathrm{id}_P$. In this case, $B = \operatorname{im} \varphi \oplus \ker \pi$.

Injective Modules

Codefinition 2

Let M, N and I be modules. If for every injective homomorphism $f\colon M \to N$ and any homomorphism $g\colon M \to I$ there is a homomorphism $h\colon N \to I$ rendering commutative the diagram



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Central Fact on Projective Modules

Theorem 3

Every module M is the homomorphic image of a projective module.

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Proof.

- Define $\varphi_m \colon R \to M$, $1 \mapsto m$ and obtain $(\varphi_m \colon R \to M)_{m \in M}$

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Proof.

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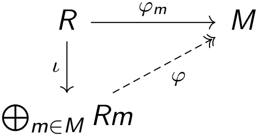
- By universal property of coproduct have commutative diagrams



□

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Cotheorem 4

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- Regard arbitrary unital ring R as $\mathbb{Z}$-module and "lift" statement to R-modules

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- Show that each $\mathbb{Z}$-module embedds into injective $\mathbb{Z}$-module
- Regard arbitrary unital ring R as $\mathbb{Z}$-module and “lift” statement to R-modules

Let M be module over principal ideal domain R. If for every $r \in R - \{0\}$ it holds $M = rM$, then M is called *divisible*.

Section 2

Projective and Injective Resolutions

Cochain Complexes

Codefinition 5

Let $\mathbf{C} = (M^n, \alpha^n)_{n\in\mathbb{Z}}$ be a family of modules and linear maps of the form

$$\mathbf{C}\colon \ \ldots \longrightarrow M^{n-1} \xrightarrow{\alpha^{n-1}} M^n \xrightarrow{\alpha^n} M^{n+1} \longrightarrow \ldots$$

If for all integers n it holds $\alpha^{n+1} \circ \alpha^n = 0$, then $\mathbf{C}$ is called a *cochain complex*. For the integer n, the quotient

$$H^n(\mathbf{C}) = \ker \alpha^n / \operatorname{im} \alpha^{n-1}$$

is called the *n-th cohomology module of* $\mathbf{C}$.

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If for all integers n it holds $\alpha^{n+1} \circ \alpha^n = 0$, then $\mathbf{C}$ is called a *cochain complex*. For the integer n, the quotient

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is called the *n-th cohomology module of* $\mathbf{C}$.

- By "reflection", make chain complex into cochain complex: For chain complex $\mathbf{M} = (M_n, \alpha_n)_{n\in\mathbb{Z}}$ put $N^n := M_{-n}$, $\beta^n := \alpha_{-n}$ and obtain $\mathbf{N} = (N^n, \alpha^n)_{n\in\mathbb{Z}}$.

Deleted (Co)Chain Complexes

Definition 6

Let **C** be a positive chain complex

$$\mathbf{C}\colon \; \ldots \longrightarrow M_n \longrightarrow \ldots \longrightarrow M_0 \longrightarrow M \longrightarrow 0$$

Then $\mathbf{C}_M$, the so called *deleted chain complex*, denotes the complex where M is omitted.

Deleted (Co)Chain Complexes

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$$\mathbf{C}: \ldots \longrightarrow M_n \longrightarrow \ldots \longrightarrow M_0 \longrightarrow M \longrightarrow 0$$

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Codefinition 7

Let $\mathbf{C}$ be the positive cochain complex

$$\mathbf{C}: 0 \longrightarrow M \longrightarrow M^1 \longrightarrow \ldots \longrightarrow M^n \longrightarrow \ldots$$

$\mathbf{C}^M$ denotes the cochain complex where M is omitted.

Projective Resolutions

Definition 8

Let $\mathbf{P}' = (P_n, \alpha_n)_{n\in\mathbb{N}_0}$ be an exact positive chain complex. If all P_n are projective and if the chain complex

$$\mathbf{P}\colon \ \ldots \longrightarrow P_n \xrightarrow{\alpha_n} \ldots \longrightarrow P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} M \longrightarrow 0$$

is exact, then $\mathbf{P}$ is called a *projective resolution of* M. In this case, $\mathbf{P}_M := \mathbf{P}'$ is called the *deleted projective resolution*.

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is exact, then $\mathbf{P}$ is called a *projective resolution of M*. In this case, $\mathbf{P}_M := \mathbf{P}'$ is called the *deleted projective resolution*.

Note: Since

$$H_0(\mathbf{P}_M) = \ker(P_0 \to 0)/\operatorname{im}\alpha_1 = P_0/\operatorname{im}\alpha_1 = P_0/\ker\alpha_0 \cong M$$

we do not suffer loss of information when deleting M.

Injective Resolutions

Codefinition 9

Let $\mathbf{E}' = (E^n, \alpha^n)_{n\in\mathbb{N}_0}$ be an exact positive cochain complex. If all E_n are injective and if the cochain complex

$$\mathbf{E}\colon 0 \longrightarrow M \xrightarrow{\alpha^{-1}} E^0 \xrightarrow{\alpha^0} E^1 \xrightarrow{\alpha^1} E^2 \longrightarrow \dots$$

is exact, then $\mathbf{E}$ is called an *injective resolution of M* and $\mathbf{E}^M := \mathbf{E}'$ is called the *deleted injective resolution*.

In the following: Concepts and proofs for chain complexes. Can be adapted to cochain complexes by "reflection".

Existence of Projective Resolutions

Theorem 10

Every module M has a projective resolution.

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Every module M has an injective resolution.

Proof.

We want to build fences out of short exact sequences:

- Use Theorem 3 to get a surjection $\pi_0 \colon P_0 \to M$ and thus the short exact sequence

$$0 \longrightarrow \ker \pi_0 \xrightarrow{\iota_0} P_0 \xrightarrow{\pi_0} M \longrightarrow 0$$

Proof.

- Continue inductively for the emerging kernels and construct short exact sequences

$$0 \longrightarrow \ker \pi_n \xrightarrow{\iota_n} P_n \xrightarrow{\pi_n} \ker \pi_{n-1} \longrightarrow 0 \qquad (n \in \mathbb{N})$$

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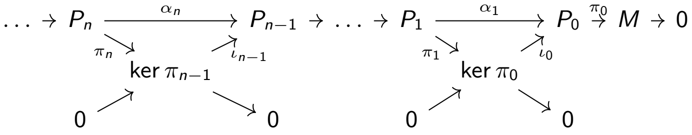
$$0 \longrightarrow \ker \pi_n \xrightarrow{\iota_n} P_n \xrightarrow{\pi_n} \ker \pi_{n-1} \longrightarrow 0 \qquad (n \in \mathbb{N})$$

- Bend each short exact sequence into a triangle, weave together a fence



and put $\alpha_n := \iota_{n-1} \circ \pi_n$ for $n \geq 1$, $\alpha_0 := \pi_0$.

Proof.

- Continue inductively for the emerging kernels and construct short exact sequences

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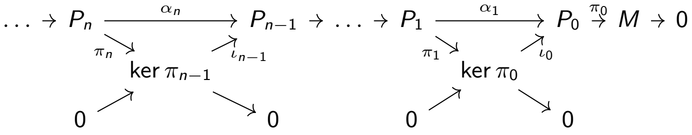

and put $\alpha_n := \iota_{n-1} \circ \pi_n$ for $n \geq 1$, $\alpha_0 := \pi_0$.

- Obtain $\operatorname{im} \alpha_n = \ker \alpha_{n-1} = \ker \pi_{n-1}$ for $n \geq 1$. □

Section 3

Induced Chain Maps

Lift to Chain Map

Lemma 12

Let $\mathbf{P}\colon \ldots \to P_n \to \ldots \to P_0 \to M \to 0$ *be a chain complex (with differentials* $\alpha_n\colon P_n \to P_{n-1}$*) such that the* P_n *are projective and let* $\mathbf{Q}\colon \ldots \to N_n \to \ldots \to N_0 \to N \to 0$ *be an exact chain complex (with differentials* $\beta_n\colon N_n \to N_{n-1}$*). Then for any module homomorphism* $f\colon M \to N$ *there is a chain map* $\mathbf{f}\colon \mathbf{P}_M \to \mathbf{Q}_N$ *such that the following diagram commutes:*

$$\begin{array}{ccccccccccccc}
\ldots & \longrightarrow & P_n & \xrightarrow{\alpha_n} & P_{n-1} & \xrightarrow{\alpha_{n-1}} & \ldots & \xrightarrow{\alpha_1} & P_0 & \xrightarrow{\alpha_0} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \\
\ldots & \longrightarrow & N_n & \xrightarrow[\beta_n]{} & N_{n-1} & \xrightarrow[\beta_{n-1}]{} & \ldots & \xrightarrow[\beta_1]{} & N_0 & \xrightarrow[\beta_0]{} & N & \longrightarrow & 0
\end{array}$$

Proof.

We want to work inductively and for commutativity we want to use projectivity.

- In the first step, we stage the use of projectivity:

$$\begin{array}{ccccc} P_0 & \xrightarrow{\alpha_0} & M & \longrightarrow & 0 \\ & & \downarrow f & & \\ N_0 & \xrightarrow[\beta_0]{} & N & \longrightarrow & 0 \end{array}$$

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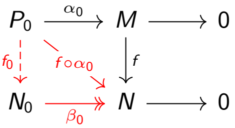

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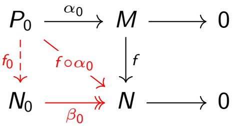
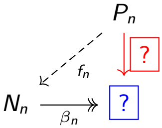


- Suppose we have $f_0, \ldots, f_{n-1}$. We need $f_n \colon P_n \to N_n$ with $\beta_n \circ f_n = f_{n-1} \circ \alpha_n$ and for projectivity, we need



Proof.

- Easiest shot for ? is $\operatorname{im} \beta_n$ and easiest shot for ? is $f_{n-1} \circ \alpha_n$. Is this legal?

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- Because f_{n-1} satisifes $\beta_{n-1} \circ f_{n-1} = f_{n-2} \circ \alpha_{n-1}$ we get $\beta_{n-1} \circ f_{n-1} \circ \alpha_n = f_{n-2} \circ \alpha_{n-1} \circ \alpha_n = 0$ and thus $\operatorname{im} f_{n-1} \circ \alpha_n \subseteq \ker \beta_{n-1} = \operatorname{im} \beta_n$

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- Projectivity yields desired $f_n \colon P_n \to N_n$. □

Proof.

- Easiest shot for ? is im β_n and easiest shot for ? is $f_{n-1} \circ \alpha_n$. Is this legal?
- Because f_{n-1} satisifes $\beta_{n-1} \circ f_{n-1} = f_{n-2} \circ \alpha_{n-1}$ we get $\beta_{n-1} \circ f_{n-1} \circ \alpha_n = f_{n-2} \circ \alpha_{n-1} \circ \alpha_n = 0$ and thus $\operatorname{im} f_{n-1} \circ \alpha_n \subseteq \ker \beta_{n-1} = \operatorname{im} \beta_n$
- Projectivity yields desired $f_n \colon P_n \to N_n$. □

Note: In particular, we get from Lemma 12: If $f\colon M \to N$ is linear and $\mathbf{P}_M$ and $\mathbf{Q}_N$ are deleted projective resolutions for M resp. N, then there is an induced chain map $\mathbf{f}\colon \mathbf{P}_M \to \mathbf{Q}_N$. We call $\mathbf{f}$ the *chainmap generated by f*.

Generated Chain Maps Induce the same map on Homology

Lemma 13

Let $\mathbf{P}$ and $\mathbf{Q}$ be projective resolutions of M resp. N. If $f : M \to N$ is a homomorphism and $\mathbf{f}, \mathbf{g} : \mathbf{P}_M \to \mathbf{Q}_N$ are chain maps generated by f, then $\mathbf{f}$ and $\mathbf{g}$ are homotopic, i.e. induce the same map between the homologies.

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Let $\mathbf{P}$ *and* $\mathbf{Q}$ *be projective resolutions of* M *resp.* N*. If* $f\colon M \to N$ *is a homomorphism and* $\mathbf{f}, \mathbf{g}\colon \mathbf{P}_M \to \mathbf{Q}_N$ *are chain maps generated by* f*, then* $\mathbf{f}$ *and* $\mathbf{g}$ *are homotopic, i.e. induce the same map between the homologies.*

Proof.

Again, we want to work inductively to construct family $(\varphi_n\colon P_n \to N_n)_{n\in\mathbb{Z}}$ of homomorphisms such that for all $n \in \mathbb{Z}$

$$f_n - g_n = \beta_{n+1} \circ \varphi_n + \varphi_{n-1} \circ \alpha_n.$$

Proof.

Because P is positive, $\varphi_n = 0$ for $n < 0$.

- Consider

$$\begin{array}{ccccc} & & P_0 & \xrightarrow{\alpha_0} & M \\ & & g_0 \downarrow\downarrow f_0 & & \downarrow f \\ N_1 & \xrightarrow[\beta_1]{} & N_0 & \xrightarrow[\beta_0]{} & N \end{array}$$

Because of $\beta_0 \circ g_0 = f \circ \alpha_0 = \beta_0 \circ f_0$ we have

$$\beta_0 \circ (f_0 - g_0) = f \circ (\alpha_0 - \alpha_0) = 0,$$

thus $\operatorname{im} f_0 - g_0 \subseteq \ker \beta_0 = \operatorname{im} \beta_1$. Same trick as before yields $\varphi\colon P_0 \to N_1$ with $f_0 - g_0 = \beta_1 \circ \varphi_0$.

Proof.

- Suppose we have $\varphi_0, \ldots, \varphi_{n-1}$. We need $\varphi_n \colon P_n \to N_{n+1}$ such that

$$f_n - g_n = \beta_{n+1} \circ \varphi_n + \varphi_{n-1} \circ \alpha_n$$

Proof.

- Suppose we have $\varphi_0, \ldots, \varphi_{n-1}$. We need $\varphi_n \colon P_n \to N_{n+1}$ such that
$$f_n - g_n = \beta_{n+1} \circ \varphi_n + \varphi_{n-1} \circ \alpha_n$$
- For $0 \leq k \leq n-1$, we know we have
$$f_k - g_k - \beta_{k+1} \circ \varphi_k = \varphi_{k-1} \circ \alpha_k \iff f_k - g_k = \beta_{k+1} \circ \varphi_k + \varphi_{k-1} \circ \alpha_k.$$

Proof.

- Suppose we have $\varphi_0, \ldots, \varphi_{n-1}$. We need $\varphi_n \colon P_n \to N_{n+1}$ such that

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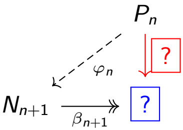
- To make use of projectivity, we need



Proof.

- Suppose we have $\varphi_0, \ldots, \varphi_{n-1}$. We need $\varphi_n \colon P_n \to N_{n+1}$ such that

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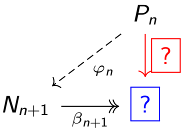

- We want to put ? to be $\operatorname{im} \beta_{n+1}$ and need to check, if we can put ? to be $f_n - g_n - \varphi_{n-1} \circ \alpha_n \colon P_n \to N_n$ (because we want $\beta_{n+1} \circ \varphi_n = f_n - g_n - \varphi_{n-1} \circ \alpha_n$)

Proof.

- We have

$$\begin{aligned}\beta_n \circ (f_n - g_n \circ \varphi_{n-1} \circ \alpha_n) \\ &= \beta_n \circ f_n - \beta_n \circ g_n - \beta_n \circ \varphi_{n-1} \circ \alpha_n \\ &= f_{n-1} \circ \alpha_n - g_{n-1} \circ \alpha_n - \beta_n \circ \varphi_{n-1} \circ \alpha_n \\ &= (f_{n-1} - g_{n-1} - \beta_n \circ \varphi_{n-1}) \circ \alpha_n \\ &= \varphi_{n-2} \circ \alpha_{n-1} \circ \alpha_n = 0,\end{aligned}$$

thus $\operatorname{im}(f_n - g_n - \varphi_{n-1} \circ \alpha_n) \subseteq \ker \beta_n = \operatorname{im} \beta_{n+1}$.

□

Proof.

- We have

$$\begin{aligned}
&\beta_n \circ (f_n - g_n \circ \varphi_{n-1} \circ \alpha_n) \\
&\quad = \beta_n \circ f_n - \beta_n \circ g_n - \beta_n \circ \varphi_{n-1} \circ \alpha_n \\
&\quad = f_{n-1} \circ \alpha_n - g_{n-1} \circ \alpha_n - \beta_n \circ \varphi_{n-1} \circ \alpha_n \\
&\quad = (f_{n-1} - g_{n-1} - \beta_n \circ \varphi_{n-1}) \circ \alpha_n \\
&\quad = \varphi_{n-2} \circ \alpha_{n-1} \circ \alpha_n = 0,
\end{aligned}$$

  thus $\operatorname{im}(f_n - g_n - \varphi_{n-1} \circ \alpha_n) \subseteq \ker \beta_n = \operatorname{im} \beta_{n+1}$.
- Now we are done by induction.

□

Propsition 14

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Proof.

Consider $\mathrm{id}_M \colon M \to M$. This map generates chain maps $\mathbf{f} \colon \mathbf{P}_M \to \mathbf{Q}_M$ and $\mathbf{g} \colon \mathbf{Q}_M \to \mathbf{P}_M$. By the previous lemma, $\mathbf{g} \circ \mathbf{f} = \mathbf{id}_{\mathbf{P}_M}$ and $\mathbf{f} \circ \mathbf{g} = \mathbf{id}_{\mathbf{Q}_M}$. □