Derived Functors

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- Development of the left and right derived functors of an additive covariant functor $\mathcal{F} : \mathbf{Mod}_R \to \mathbf{Ab}$.
- The dual case of an additive contravariant functor can be obtained from the covariant case simply by reversing the arrows and making dual arguments.

PROP.: Let R, S be rings with identity and let $\mathcal{F} : \mathbf{Mod}_R \to \mathbf{Mod}_S$ be an additive covariant functor.

(i) If M is a chain complex in $\mathbf{Chain}_R$, then $\mathcal{F}(M)$ is a chain complex in $\mathbf{Chain}_S$.

(ii) If $f : M \to N$ is a chain map of degree k in $\mathbf{Chain}_R$, then $\mathcal{F}(f) : \mathcal{F}(M) \to \mathcal{F}(N)$ is a chain map of degree k in $\mathbf{Chain}_S$.

(iii) If $f, g: M \to N$ are homotopic chain maps in $\mathbf{Chain}_R$, then $\mathcal{F}(f), \mathcal{F}(g): \mathcal{F}(M) \to \mathcal{F}(N)$ are homotopic chain maps in $\mathbf{Chain}_S$, and therefore

$$H_n(\mathcal{F}(f)) = H_n(\mathcal{F}(g)): H_n(\mathcal{F}(M)) \longrightarrow H_n(\mathcal{F}(N))$$

for every $n \in \mathbb{Z}$.

$\Longrightarrow$ Every homotopy class $[f]$ of a chain map $f: M \to N$ produces exactly one homology mapping $H_n(\mathcal{F}(M)) \to H_n(\mathcal{F}(N))$.

- There are analogues of the previous PROP for
  - cochain complexes and cochain maps
  - an additive contravariant functor.

Let $\mathcal{F} : \mathbf{Mod}_R \to \mathbf{Ab}$ be an additive covariant functor. If

$$\mathbf{P} : \cdots \to P_n \xrightarrow{\alpha_n} P_{n-1} \to \cdots \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} M \to 0$$

is a projective resolution of $M \in \mathbf{Mod}_R$, then

$$\mathcal{F}(\mathbf{P}) : \cdots \to \mathcal{F}(P_n) \xrightarrow{\mathcal{F}(\alpha_n)} \mathcal{F}(P_{n-1}) \to \cdots \xrightarrow{\mathcal{F}(\alpha_1)} \mathcal{F}(P_0) \xrightarrow{\mathcal{F}(\alpha_0)} \mathcal{F}(M) \to 0$$

is a chain complex in $\mathbf{Ab}$. However, $\mathcal{F}(\mathbf{P})$ may not be exact and there is no reason to expect that $\mathcal{F}(P_n)$ is projective.

If $\mathbf{P}$ and $\mathbf{Q}$ are projective resolutions of M and N, respectively, and $f : M \to N$ is an R-mod. homomorphism, then there is a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P_n \arrow[r, "\alpha_n"] \arrow[d, "f_n"'] & P_{n-1} \arrow[r] \arrow[d, "f_{n-1}"'] & \cdots \arrow[r] & P_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"'] & P_0 \arrow[r, "\alpha_0"] \arrow[d, "f_0"'] & M \arrow[r] \arrow[d, "f"'] & 0 \\
\cdots \arrow[r] & Q_n \arrow[r, "\beta_n"] & Q_{n-1} \arrow[r] & \cdots \arrow[r] & Q_1 \arrow[r, "\beta_1"] & Q_0 \arrow[r, "\beta_0"] & N \arrow[r] & 0
\end{tikzcd}$$

where $\boldsymbol{f}: \boldsymbol{P}_M \to \boldsymbol{Q}_N$ is a chain map generated by f. By applying $\mathcal{F}$, we get a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{F}(P_n) \arrow[r, "\mathcal{F}(\alpha_n)"] \arrow[d, "\mathcal{F}(f_n)"'] & \mathcal{F}(P_{n-1}) \arrow[r] \arrow[d, "\mathcal{F}(f_{n-1})"'] & \cdots \arrow[r] & \mathcal{F}(P_1) \arrow[r, "\mathcal{F}(\alpha_1)"] \arrow[d, "\mathcal{F}(f_1)"'] & \mathcal{F}(P_0) \arrow[r] \arrow[d, "\mathcal{F}(f_0)"'] & 0 \\
\cdots \arrow[r] & \mathcal{F}(Q_n) \arrow[r, "\mathcal{F}(\beta_n)"] & \mathcal{F}(Q_{n-1}) \arrow[r] & \cdots \arrow[r] & \mathcal{F}(Q_1) \arrow[r, "\mathcal{F}(\beta_1)"] & \mathcal{F}(Q_0) \arrow[r] & 0
\end{tikzcd}$$

where the rows are chain complexes and $\mathcal{F}(\boldsymbol{f}): \mathcal{F}(\boldsymbol{P}_M) \to \mathcal{F}(\boldsymbol{Q}_N)$ is a chain map, and thus for each $n \in \mathbb{N}$ there is an induced homology map

$$H_n(\mathcal{F}(\boldsymbol{f})): H_n(\mathcal{F}(\boldsymbol{P}_M)) \longrightarrow H_n(\mathcal{F}(\boldsymbol{Q}_N)).$$

PROP.:

(i) If $\mathcal{P}$ and $\mathcal{P}'$ are projective resolutions of M, then

$$H_n(\mathcal{F}(\mathcal{P}_M)) \cong H_n(\mathcal{F}(\mathcal{P}'_M)),\ \forall n \in \mathbb{N}.$$

(ii) If $f: M \longrightarrow N$ is an R-module homomorphism and $\mathcal{P}$ and $\mathcal{Q}$ are projective resolutions of M and N, respectively, then for each $n \in \mathbb{N}$ the induced group homomorphism (n-th homology mapping)

$$H_n(\mathcal{F}(f)): H_n(\mathcal{F}(\mathcal{P}_M)) \longrightarrow H_n(\mathcal{F}(\mathcal{Q}_N))$$

does not depend on the choice of the chain map $f: \mathcal{P}_M \longrightarrow \mathcal{Q}_N$ generated by f.

- There are analogous statements (with similar proofs) about
  - injective resolutions
  - contravariant functors

PROOF:

(i) We saw earlier that P_M and P'_M are of the same homotopy type; in particular, if $f: P_M \to P'_M$ and $g: P'_M \to P_M$ are chain maps generated by Id_M, then $fg \simeq Id_{P'_M}$ and $gf \simeq Id_{P_M}$. By previous PROP we infer

$$\mathcal{F}(fg) \simeq \mathcal{F}(Id_{P'_M}) = Id_{\mathcal{F}(P'_M)}$$
$$\mathcal{F}(gf) \simeq \mathcal{F}(Id_{P_M}) = Id_{\mathcal{F}(P_M)}.$$

This implies

$$H_n(\mathcal{F}(f))\, H_n(\mathcal{F}(g)) = H_n(\mathcal{F}(f)\,\mathcal{F}(g)) = H_n(\mathcal{F}(fg))$$
$$= H_n(Id_{\mathcal{F}(P'_M)}) = Id_{H_n(\mathcal{F}(P'_M))}$$

and similarly $H_n(\mathcal{F}(g))\, H_n(\mathcal{F}(f)) = Id_{H_n(\mathcal{F}(P_M))}$, which yields the claim.

(ii) If $\mathbf{f}, \mathbf{g} : P_M \longrightarrow Q_N$ are chain maps generated by f, then $\mathbf{f} \simeq \mathbf{g}$, so $\mathcal{F}(\mathbf{f}) \simeq \mathcal{F}(\mathbf{g})$, and therefore $H_n(\mathcal{F}(\mathbf{f})) = H_n(\mathcal{F}(\mathbf{g}))$, $\forall n \in \mathbb{N}$, which proves the claim. ∎

The above PROP (and its various analogues) provide the tools necessary to establish the n-th left and n-th right derived functors of an additive (covariant or contravariant) functor $\mathcal{F} : \mathbf{Mod}_R \longrightarrow \mathbf{Ab}$.

Left Derived Functors of an Additive Covariant Functor

$\mathcal{F} : \mathcal{M}od_R \longrightarrow \mathcal{A}b$: additive, covariant

Choose and fix a projective resolution of each R-module.

- If M is an R-module and

$$\underline{\mathcal{P}} : \cdots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

is the chosen projective resolution of M, then

$$\mathcal{F}(\underline{\mathcal{P}}) : \cdots \longrightarrow \mathcal{F}(P_n) \longrightarrow \mathcal{F}(P_{n-1}) \longrightarrow \cdots \longrightarrow \mathcal{F}(P_0) \longrightarrow \mathcal{F}(M) \longrightarrow 0$$

is a chain complex in $\mathcal{A}b$. Consider the deleted complex $\mathcal{F}(\mathcal{P}_M)$, take homology and set $\mathcal{L}_n \mathcal{F}(M) := H_n(\mathcal{F}(\mathcal{P}_M))$, $n \in \mathbb{N}$.

- If $f: M \to N$ is an R-module homomorphism and $\mathbf{P}$ and $\mathbf{Q}$ are the chosen projective resolutions of M and N, respectively, then consider a chain map $\mathbf{f}: \mathbf{P}_M \longrightarrow \mathbf{Q}_N$ generated by f and the induced n-th homology mapping $H_n(\mathcal{F}(\mathbf{f})): H_n(\mathcal{F}(\mathbf{P}_M)) \longrightarrow H_n(\mathcal{F}(\mathbf{Q}_N))$. By the previous PROP, $H_n(\mathcal{F}(\mathbf{f}))$ depends only on f and not on the chain map $\mathbf{f}$ generated by f, and thus we set $\mathcal{L}_n\mathcal{F}(f) := H_n(\mathcal{F}(\mathbf{f}))$, $n \in \mathbb{N}$.

$$: \mathcal{L}_n\mathcal{F}(M) \longrightarrow \mathcal{L}_n\mathcal{F}(N)$$

$\Longrightarrow$ For each $n \in \mathbb{N}$ we have a functor

$$\mathcal{L}_n\mathcal{F}: \mathbf{Mod}_R \longrightarrow \mathbf{Ab}.$$

<u>Q</u>: Does its construction depend on the chosen projective resolutions?

If the projective resolutions are chosen in a different way, then we obtain another functor $\overline{\mathcal{L}_n}\mathcal{F}: \mathbf{Mod}_R \longrightarrow \mathbf{Ab}$ constructed in exactly the same fashion as $\mathcal{L}_n\mathcal{F}$. However:

PROP.: The functors $\mathcal{L}_n\mathcal{F}$ and $\overline{\mathcal{L}_n}\mathcal{F}$ are naturally equivalent.

PROOF: If $\mathbf{P}$ and $\mathbf{P}'$ are projective resolutions of M, then we saw in (the proof of) the previous PROP that $H_n(\mathcal{F}(\mathbf{f})): H_n(\mathcal{F}(\mathbf{P}_M)) \xrightarrow{\simeq} H_n(\mathcal{F}(\mathbf{P}'_M))$ is an isomorphism, where $\mathbf{f}: \mathbf{P}_M \longrightarrow \mathbf{P}'_M$ is a chain map generated by Id_M, and it is actually unique, so we set $\eta_M := H_n(\mathcal{F}(\mathbf{f}))$.

If $f: M \longrightarrow N$ is an R-module homomorphism and $\mathbf{Q}$ and $\mathbf{Q}'$ are projective resolutions of N, then we obtain a commutative diagram

$$\begin{tikzcd} H_n(\mathcal{F}(P_M)) \arrow[r, "\eta_M"] \arrow[d, "H_n(\mathcal{F}(f))"'] & H_n(\mathcal{F}(P'_M)) \arrow[d, "H_n(\mathcal{F}(g))"] \\ H_n(\mathcal{F}(Q_N)) \arrow[r, "\eta_N"] & H_n(\mathcal{F}(Q'_N)) \end{tikzcd}$$

Where $f : P_M \longrightarrow Q_N$ and $g : P'_M \longrightarrow Q'_N$ are chain maps generated by f. Hence we have a natural isomorphism $\eta : \mathcal{L}_n\mathcal{F} \longrightarrow \overline{\mathcal{L}}_n\mathcal{F}$, which finishes the proof. $\blacksquare$

$\Longrightarrow$ The functor $\mathcal{L}_n\mathcal{F}$ does not depend on the projective resolution chosen for its development.

$\rightsquigarrow$ $\boxed{\mathcal{L}_n\mathcal{F} : \text{"n-th left derived functor of } \mathcal{F}\text{"}}$

Right Derived Functors of an Additive Covariant Functor

$\mathcal{F} : \mathcal{M}od_R \longrightarrow \mathcal{A}b$: additive, covariant

Choose and fix an injective resolution of each R-module.

- $M \in \mathcal{M}od_R \rightsquigarrow \mathbf{I}$: (the chosen) injective resolution of M

  $\rightsquigarrow \mathcal{F}(\mathbf{I}_M)$: cochain complex in $\mathcal{A}b$

  $\rightsquigarrow \mathcal{R}^n\mathcal{F}(M) := H^n(\mathcal{F}(\mathbf{I}_M))$, $n \in \mathbb{N}$

- $f : M \longrightarrow N \in \mathcal{M}od_R \rightsquigarrow \mathbf{f} : \mathbf{I}_M \longrightarrow \mathbf{J}_N$: cochain map generated by f, where $\mathbf{J}$ is (the chosen) injective resolution of N

  $\rightsquigarrow \mathcal{R}^n\mathcal{F}(f) := H^n(\mathcal{F}(\mathbf{f})) : H^n(\mathcal{F}(\mathbf{I}_M)) \longrightarrow H^n(\mathcal{F}(\mathbf{J}_N))$

$\Longrightarrow$ For each $n \in \mathbb{N}$ we have a functor

$$\mathcal{R}^n\mathcal{F} : \mathbf{Mod}_R \longrightarrow \mathbf{Ab}$$

which does not depend on the injective resolution chosen for its development.

$\rightsquigarrow$ $\boxed{\mathcal{R}^n\mathcal{F} : \text{"n-th right derived functor of } \mathcal{F}\text{"}}$

- If $\mathcal{F} : \mathbf{Mod}_R \longrightarrow \mathbf{Ab}$ is an additive contravariant functor, then
  - $\mathcal{L}_n\mathcal{F}$ is constructed using injective resolutions.
  - $\mathcal{R}^n\mathcal{F}$ is constructed using projective resolutions.
- $\mathcal{L}_n\mathcal{F}$ and $\mathcal{R}^n\mathcal{F}$ are additive functors.

Exactness of Functors

DEF.: A covariant functor $\mathcal{F}: \mathcal{Mod}_R \longrightarrow \mathcal{Mod}_S$ is called

(i) left exact if for every exact sequence

$$0 \longrightarrow M_1 \xrightarrow{f} M \xrightarrow{g} M_2$$

in $\mathcal{Mod}_R$, the sequence

$$0 \longrightarrow \mathcal{F}(M_1) \xrightarrow{\mathcal{F}(f)} \mathcal{F}(M) \xrightarrow{\mathcal{F}(g)} \mathcal{F}(M_2)$$

is exact in $\mathcal{Mod}_S$.

(ii) right exact if for every exact sequence

$$M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \longrightarrow 0$$

in $\mathcal{Mod}_R$, the sequence

$$\mathcal{F}(M_1) \xrightarrow{\mathcal{F}(f)} \mathcal{F}(M) \xrightarrow{\mathcal{F}(g)} \mathcal{F}(M_2) \longrightarrow 0$$

is exact in $\mathcal{M}od_S$.

(iii) exact if it both left exact and right exact.

DEF.: A contravariant functor $\mathcal{F} : \mathcal{M}od_R \longrightarrow \mathcal{M}od_S$ is called

(i) left exact if for every exact sequence

$$M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \longrightarrow 0$$

in $\mathcal{M}od_R$, the sequence

$$0 \longrightarrow \mathcal{F}(M_2) \xrightarrow{\mathcal{F}(g)} \mathcal{F}(M) \xrightarrow{\mathcal{F}(f)} \mathcal{F}(M_1)$$

is exact in $\mathcal{M}od_S$.

(ii) right exact if for every exact sequence

$$0 \longrightarrow M_1 \xrightarrow{f} M \xrightarrow{g} M_2$$

in $\mathcal{M}od_R$, the sequence

$$\mathcal{F}(M_2) \xrightarrow{\mathcal{F}(g)} \mathcal{F}(M) \xrightarrow{\mathcal{F}(f)} \mathcal{F}(M_1) \longrightarrow 0$$

is exact in $\mathcal{M}od_S$.

(iii) exact if it is both left exact and right exact.

- A functor $\mathcal{F} : \mathcal{M}od_R \longrightarrow \mathcal{M}od_S$ is exact (i.e. preserves s.e.s.) iff for every exact complex

$$\mathbf{M} : \cdots \longrightarrow M_{n-1} \longrightarrow M_n \longrightarrow M_{n+1} \longrightarrow \cdots$$

in $\mathcal{M}od_R$, the induced sequence

$$\mathcal{F}(\mathbf{M}) : \cdots \longrightarrow \mathcal{F}(M_{n-1}) \longrightarrow \mathcal{F}(M_n) \longrightarrow \mathcal{F}(M_{n+1}) \longrightarrow \cdots$$

is an exact complex in $\mathcal{M}od_S$.

- If $(\mathcal{F}, \mathcal{G})$ is an adjoint pair, then $\mathcal{F}$ is right exact and $\mathcal{G}$ is left exact.

Back to Derived Functors

PROP.: Let $\mathcal{F} : \mathcal{M}od_R \longrightarrow \mathcal{A}b$ be an additive covariant functor.

(i) If $\mathcal{F}$ is right exact, then $\mathcal{L}_0\mathcal{F}(M) \cong \mathcal{F}(M)$, so that $\mathcal{L}_0\mathcal{F}$ and $\mathcal{F}$ are naturally equivalent functors, and if P is a projective R-module, then $\mathcal{L}_n\mathcal{F}(P) = 0$, $\forall n \geq 1$. ("P is $\mathcal{F}$-acyclic")

(ii) If $\mathcal{F}$ is left exact, then $\mathcal{R}^0\mathcal{F}(M) \cong \mathcal{F}(M)$ for each R-module M, so that $\mathcal{R}^0\mathcal{F}$ and $\mathcal{F}$ are naturally equivalent functors, and if I is an injective R-module, then $\mathcal{R}^n\mathcal{F}(I) = 0$, $\forall n \geq 1$ ("I is $\mathcal{F}$-acyclic")

- There is an analogous statement (with similar proof) for a left/right exact additive contravariant functor.

PROOF:

(i) Let $M \in \mathbf{Mod}_R$. If

$$\mathbf{P}: \dots \longrightarrow P_n \xrightarrow{\alpha_n} P_{n-1} \longrightarrow \dots \longrightarrow P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} M \longrightarrow 0$$

is a projective resolution of M, then

$$\mathcal{F}(\mathbf{P}): \dots \longrightarrow \mathcal{F}(P_n) \xrightarrow{\mathcal{F}(\alpha_n)} \mathcal{F}(P_{n-1}) \longrightarrow \dots \longrightarrow \underbrace{\mathcal{F}(P_1) \xrightarrow{\mathcal{F}(\alpha_1)} \mathcal{F}(P_0) \xrightarrow{\mathcal{F}(\alpha_0)} \mathcal{F}(M) \longrightarrow 0}$$

is a chain complex such that

$$\mathcal{F}(P_1) \xrightarrow{\mathcal{F}(\alpha_1)} \mathcal{F}(P_0) \xrightarrow{\mathcal{F}(\alpha_0)} \mathcal{F}(M) \longrightarrow 0$$

is exact, because $\mathcal{F}$ is right exact. Hence

$$\mathcal{L}_0\mathcal{F}(M) = H_0(\mathcal{F}(\mathbf{P}_M)) = \mathcal{F}(P_0) / \operatorname{Im} \mathcal{F}(\alpha_1) = \mathcal{F}(P_0) / \operatorname{Ker} \mathcal{F}(\alpha_0)$$

$$\cong \mathcal{F}(M)$$

Thus there is an isomorphism $\eta_M : \mathcal{L}_0\mathcal{F}(M) \longrightarrow \mathcal{F}(M)$. We will show now that the family $\eta = \{\eta_M\}_M$ is a natural isomorphism. Indeed, let $f: M \to N$ be an R-module homomorphism and let $\mathbf{Q}$ be a projective resolution of N. If $\mathbf{f}: \mathbf{P}_M \to \mathbf{Q}_N$ is a chain map generated by f:

$$
\begin{tikzcd}
\mathbf{P}: \cdots \arrow[r] & P_1 \arrow[r] \arrow[d, "f_1"'] & P_0 \arrow[r] \arrow[d, "f_0"'] & M \arrow[r] \arrow[d, "f"'] & 0 \\
\mathbf{Q}: \cdots \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & N \arrow[r] & 0
\end{tikzcd}
$$

then we obtain a commutative diagram

$$
\begin{tikzcd}
L_0\mathcal{F}(M) \arrow[r, "\eta_M"] \arrow[d, "L_0\mathcal{F}(\mathbf{f})"'] & \mathcal{F}(M) \arrow[d, "\mathcal{F}(f)"] \\
L_0\mathcal{F}(N) \arrow[r, "\eta_N"'] & \mathcal{F}(N)
\end{tikzcd}
$$

where η_M and η_N are isomorphisms. This yields the claim about η.

Finally, if P is a projective R-module, then

$$\mathcal{P}: \cdots \longrightarrow 0 \longrightarrow P \xrightarrow{Id_P} P \longrightarrow 0$$

is clearly a projective resolution of P and therefore $\mathcal{L}_n\mathcal{F}(P) = 0, \forall n \geq 1$, since

$$\mathcal{F}(\mathcal{P}_P): \cdots \longrightarrow 0 \longrightarrow P \longrightarrow 0.$$

(ii) Similar to (i).