Examples are Manifold de Rham Cohomology

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Aims of this Talk

- Sketching physical origins

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- Cochain complexes,
- Homology,
- Chain maps,
- Induced Maps on cohomology.

Section 1

Physical motivation

Coulomb Theory

- Consider negative point charge q located in $0 \in \mathbb{R}^3$

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- Using assignment

$$\mathbf{E} \colon \mathbb{R}^3 - \{0\} \longrightarrow \mathbb{R}^3 - \{0\}, \qquad x_0 \longmapsto \frac{1}{4\pi\varepsilon_0} q \frac{x_0}{\|x_0\|_2^3}$$

the force on negative point charge Q at x_0 can be expressed as $Q\mathbf{E}(x_0)$. $\mathbf{E}$ is eletrical field of point charge q

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- Work W (work $=$ force $\times$ distance) required to move negative point charge Q along (smooth) path γ throught $\mathbf{E}$ is given by

$$W = \int_\gamma (Q\mathbf{E}|\, d\mathbf{x}) := Q\int_a^b (\mathbf{E}(\gamma(t))|\dot{\gamma}(t))\, dt$$

- Intuition for this formula: For partition $a = t_0 < \cdots < t_n = b$ of $[a, b]$, consider

$$\sum_{i=1}^{n}(\mathbf{E}(\gamma(t_{i-1})|\gamma(t_i) - \gamma(t_{i-1})),$$

each summand gives work required to move Q (at constant force) along cycle $\gamma(t_i) - \gamma(t_{i-1})$. Passing to limit gives $\int_\gamma (\mathbf{E}|\, d\mathbf{x})$

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- Think of $(\mathbf{E}(\gamma(t)) | \dot{\gamma}(t))$ as part of $\mathbf{E}(\gamma(t))$ tangential γ in t
- Because $\mathbf{E} = -\operatorname{grad} \phi$, have

$$(\mathbf{E}(\gamma(t)) | \dot{\gamma}(t)) = -(\operatorname{grad} \phi(\gamma(t)) | \dot{\gamma}(t)) = -(\phi \circ \gamma)'(t)$$

i.e. W only depends on start and end point of path γ

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- Natural questions: How to determine if given field is conservative? Classification of conservative/non-conservative fields possible? Construction of potential possible in case field is conservative?
- Answers to all questions: Yes
- In the following will concern with dual concept of differential forms instead of fields, i.e. will be studying Ulomb theory

Section 2

Basics from Linear Algebra and Analysis

Finite-dimensional speciality

Remark 1

For finite-dimensional K-vector space V have

$$\bigwedge^n V^* \cong \mathrm{Alt}^n_K(V, K) = \{\mu\colon V^n \to K \text{ alternating, multilinear}\}.$$

Will regard n-forms on V^* as n-times multilinear forms on V when convenient

The Functor "$\bigwedge^n$"

Lemma 2

Let $n \in \mathbb{N}$, let V and W be finite-dimensional K-vector spaces and let $f \in \mathrm{Hom}_K(V, W)$. Have linear map $\bigwedge^n f \colon \bigwedge^n V \to \bigwedge^n W$ such that for all $v_1, \ldots, v_n \in V$:

$$\bigwedge^n f(v_1 \wedge \cdots \wedge v_n) = f(v_1) \wedge \cdots \wedge f(v_n).$$

Pullback of an n-form

Definition 3

In the situation of Lemma 2: Have dual map $f^*\colon W^* \to V^*$, $\psi \mapsto \psi \circ f$ and linear map $\bigwedge^n f^*\colon \bigwedge^n W^* \to \bigwedge^n V^*$ uniquely determined by

$$\bigwedge^n f^*(\psi^1 \wedge \cdots \wedge \psi^n) = f^*(\psi^1) \wedge \cdots \wedge f^*(\psi^n).$$

For n-form $\omega \in \bigwedge^n W^*$, $\bigwedge^n f^*(\omega)$ is called *pullback of* ω *along* f.

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For n-form $\omega \in \bigwedge^n W^*$, $\bigwedge^n f^*(\omega)$ is called *pullback of* ω *along* f.

Note: Often $f^*\omega$ instead of $\bigwedge^n f^*(\omega)$ by abuse of notation.

Differentiability for $f\colon \mathbb{R}^N \to \mathbb{R}$

Let $U \subseteq \mathbb{R}^N$ be open, let $f\colon U \to \mathbb{R}$ and let $p_0 \in U$.

- If there is linear $L\colon \mathbb{R}^N \to \mathbb{R}$ with

$$\lim_{p \to p_0} \frac{\|f(p) - f(p_0) - L(p_0)(p - p_0)\|}{\|p - p_0\|} = 0,$$

then f is differentiable at p_0. $df(p_0) := L$ is called *differential of f at p_0*.

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- For $v \in \mathbb{R}^N$ the limit (if it exists)

$$D_v f(p_0) = \lim_{t \to 0} \frac{f(p_0 + tv) - f(p_0)}{t}$$

  is called *directional derivative of f in direction v at p_0*. For $1 \leq i \leq N$ and $v = e_i$, $\partial_i f(p_0) := D_v f(p_0)$ are called *partial derivatives of f at p_0*.

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- If f is differentiable at $p_0 \in U$, then for any $v \in \mathbb{R}^N$:

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- The transformation matrix of $df(p_0)$ w.r.t the canonical basis of $\mathbb{R}^N$ is

$$\left(\frac{\partial f}{\partial x^1}(p_0), \ldots, \frac{\partial f}{\partial x^n}(p_0)\right).$$

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- If f is differentiable on U then

$$df\colon U \longrightarrow (\mathbb{R}^N)^*, \qquad p \longmapsto (df)(p)$$

  is called *Fréchet derivative of f* or *derivative of f*.

Section 3

The Spaces

Definition 4 (Differential Form of Degree n)

Let $n \in \mathbb{N}$, let $U \subseteq \mathbb{R}^N$ be open. A map

$$\omega\colon U \longrightarrow \bigwedge^n (\mathbb{R}^N)^*,$$
$$u \longmapsto \sum_{1 \leq i_1 < \cdots < i_n \leq N} f_{(i_1,\ldots,i_n)}(u)\, dx^{i_1} \wedge \cdots \wedge dx^{i_n},$$

where $f_{(i_1,\ldots,i_n)}\colon U \to \mathbb{R}$, is *differential form of degree n on U*; briefly *n-form*. Put

$$\Omega^n(U) := \{\omega\colon U \to \bigwedge^n (\mathbb{R}^N)^* \text{ smooth differential form}\}.$$

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- $\Omega^n(U)$ is free $C^\infty(U)$-module of dimension $\binom{N}{n}$,
- $\Omega^0(U) = C^\infty(U)$.

Definition 5 (Wedge Product for Differential Forms)

Let $U \subseteq \mathbb{R}^N$ be open, let $n, m \in \mathbb{N}$ and let $\omega \in \Omega^n(U)$, $\eta \in \Omega^m(U)$.

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$\Omega(U) := \bigoplus_{n=0}^{\infty} \Omega^n(U)$ together with "$\wedge$" is a graded algebra.

Section 4

The maps

Fréchet Derivative Reinterpreted

Remark 6

Let $U \subseteq \mathbb{R}^N$ be open, let $f \in C^\infty(U)$. The derivative df of f is 1-form on U and w.r.t. the canonical basis of $\mathbb{R}^N$:

$$df(p) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(p)\, dx^i \,.$$

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Note: The Fréchet derivative of $\mathrm{pr}_i\colon \mathbb{R}^N \to \mathbb{R}$, $(x^1, \dots, x^N)^t \mapsto x^i$ is $d\mathrm{pr}_i = \sum_{j=1}^N \delta_{ij}\varepsilon^j = \varepsilon^i$, hence $dx^i := \varepsilon^i$.

Exterior Derivative

Definition 7

Let $U \subseteq \mathbb{R}^N$ be open and let $n \in \mathbb{N}$. For $\omega = \sum_{1 \leq i_1 < \cdots < i_n \leq N} f_{(i_1,\ldots,i_n)}\, dx^{i_1} \wedge \cdots \wedge dx^{i_n} \in \Omega^n(U)$,

$$d\omega := \sum_{1 \leq i_1 < \cdots < i_n \leq N} \Big(\sum_{j=1}^{N} \frac{\partial f_{(i_1,\ldots,i_n)}}{\partial x^j}\, dx^j \Big) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_n}$$

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- Definition of exterior derivative does not depend on choice of coordinates
- Exterior derivative is linear
- $d \circ d = 0$, i.e. d is differential for a certain cochain complex

Pullback of Differential Form

Definition 8

Let $U \subseteq \mathbb{R}^N$, $V \subseteq \mathbb{R}^M$ be open, let $\varphi\colon U \to V$ be smooth and let $\omega \in \Omega^n(V)$. Then,

$$(\varphi^*\omega)\colon U \longrightarrow \bigwedge^n (\mathbb{R}^N)^*,$$
$$u \longmapsto \big[(v_1, \ldots, v_n) \mapsto \omega(\varphi(u))[(D\varphi)(u)(v_1), \ldots, (D\varphi)(u)(v_n)]\big]$$

is called *pullback of* ω *along* φ.

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- $\varphi^*(\omega + \eta) = \varphi^*(\omega) + \varphi^*(\eta)$,
- $\varphi^*(\omega \wedge \eta) = \varphi^*(\omega) \wedge \varphi^*(\eta)$,
- $\varphi^*(d\omega) = d\varphi^*(\omega)$.

Section 5

de Rham Cohomology

Definition de Rham Cohomology

Definition 9

Let $U \subseteq \mathbb{R}^N$ be open. Have the cochain complex

$$0 \longrightarrow \Omega^0(U) \xrightarrow{d^0} \Omega^1(U) \xrightarrow{d^1} \dots \xrightarrow{d^{N-1}} \Omega^N(U) \longrightarrow 0$$

Put $H^n(U) := \ker d^n / \operatorname{im} d^{n-1}$, the *n-th de Rham cohomology of* U, and put $H^\bullet(U) := \bigoplus_{n=0}^N H^n(U)$, the *de Rham Cohomology of* U.

Definition de Rham Cohomology

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Note: Let $[\omega] \in H^n(U)$, $[\eta] \in H^m(U)$. Then

$$[\omega] \wedge [\eta] := [\omega \wedge \eta] \in H^{n+m}(U)$$

is well-defined making $H^\bullet(U)$ a graded algebra.

Pullback as Chain Map

Example 10

Let $U, V \subseteq \mathbb{R}^N$ be open and let $\varphi \colon U \to V$ be smooth. For each n, obtain $\varphi^* \colon \Omega^n(V) \to \Omega^n(U)$ and the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^0(V) & \xrightarrow{d} & \Omega^1(V) & \xrightarrow{d} & \dots & \xrightarrow{d} & \Omega^N(V) & \longrightarrow & 0 \\
 & & \downarrow \varphi^* & & \downarrow \varphi^* & & & & \downarrow \varphi^* & & \\
0 & \longrightarrow & \Omega^0(U) & \xrightarrow[d]{} & \Omega^1(U) & \xrightarrow[d]{} & \dots & \xrightarrow[d]{} & \Omega^N(U) & \longrightarrow & 0
\end{array}$$

Because of $\varphi^*(d\omega) = d\varphi^*\omega$, the pullback is a chain map. Also get induced map $\varphi^* \colon H^n(V) \to H^n(U)$, $[\omega] \mapsto [\varphi^*\omega]$.

Hands on Pullback

Example 11

- Consider $\omega \in \Omega^1(U)$ (and fix ω in the following) with

$$\omega(x, y) = \frac{x}{x^2 + y^2}\, dy - \frac{y}{x^2 + y^2}\, dx\,.$$

(Of interest for differential geometry, “winding form”)

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(Of interest for differential geometry, "winding form")

- This form is closed (i.e. $d\omega = 0$):

$$d\omega(x,y) = \frac{y^2-x^2}{(x^2+y^2)^2}\,dx \wedge dy - \frac{x^2-y^2}{(x^2+y^2)^2}\,dy \wedge dx = 0$$

Hands on Pullback

Example 11

- Consider $\omega \in \Omega^1(U)$ (and fix ω in the following) with

$$\omega(x,y) = \frac{x}{x^2+y^2}\,dy - \frac{y}{x^2+y^2}\,dx\,.$$

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- Is ω exact? *No, but how do we see this?*

Example 11 (Continuation)

- **Reminder:** For $\eta \in \Omega^1(U)$ and $\gamma\colon [a,b] \to U$,

$$\int_\gamma \eta := \int_a^b \gamma^*\eta(t)\,dt = \int_a^b \langle \eta(\gamma(t)), \gamma'(t)\rangle\,dt$$

is called *line integral of η along γ*.

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- **Fact:** If $\eta = df$ for some $f \in \Omega^0(U)$, then

$$\int_\gamma df = \int_a^b \gamma^*(df)$$

$$= \int_a^b d(\gamma^* f) = \int_a^b (f \circ \gamma)'(t)\,dt = f(\gamma(b)) - f(\gamma(a))$$

In particular, for closed paths γ it holds $\int_\gamma df = 0$.

Example 11 (Continuation)

- For pullback of 1-form, we need to understand two things:

$$\gamma^* f := f \circ \gamma, \qquad \gamma^* \, dx^i = d(\gamma^* \circ \mathrm{pr}_i) = d(\mathrm{pr}_i \circ \gamma) = d\gamma^i$$

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- For $\gamma\colon [0, 2\pi] \to \mathbb{R}^2 - \{0\}$, $t \mapsto (\cos(t), \sin(t))^\top$ and ω have

$$\gamma^*\, dx = d\gamma^1 = -\sin t\, dt, \qquad \gamma^*\, dy = d\gamma^2 = \cos t\, dt,$$
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- Alternatively,

$$\langle \omega(\gamma(t)), \gamma'(t) \rangle = \cos(t) \langle dy, -\sin(t) e_x + \cos(t) e_y \rangle$$
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- Get $\int_\gamma \omega = \int_0^{2\pi} dt = 2\pi \neq 0$, i.e. ω can't be exact.

Conclusion

- In general, the form

$$\omega = \frac{1}{\|x\|^N} \sum_{i=1}^{N} (-1)^{i-1} x_i \, dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^N$$

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- It can be shown that the non-trivial cohomology spaces of $\mathbb{R}^N - \{0\}$ are

$$H^0(\mathbb{R}^N - \{0\}) = \mathbb{R}, \qquad H^{N-1}(\mathbb{R}^N - \{0\}) = \mathbb{R}\omega \cong \mathbb{R}$$