QUANTUM PERMUTATIONS

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Abstract:
In the past decades a kind of „quantum mathematics“ has evolved as a more and more coherent theory. It contains, amongst others, C*-algebras (aka noncommutative topology), von Neumann algebras (aka noncommutative measure theory), Connes's noncommutative (differential) geometry, Voiculescu's free probability theory and many more. In this mostly analytic setting, Woronowicz's quantum groups provide a suitable notion of quantum symmetry.

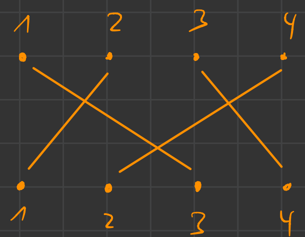
In my talk, I will sketch this broader context before introducing quantum permutations as a particular kind of quantum symmetry. I will then survey recent developments in the realm of quantum symmetries of graphs, quantum isomorphisms of graphs, quantum information theory and representation theory.

The talk will contain analytic, algebraic and combinatorial aspects.

NYUAD Mathematics Colloquium Series, 10 May 2021

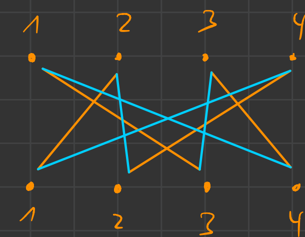
MOTIVATION

permutation



$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

quantum permutation



$$\begin{pmatrix} 0 & \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & 0 & \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\ 0 & \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} & 0 & \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & 0 & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & 0 \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & 0 & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & 0 \end{pmatrix}$$

MOTIVATION

quantum permutations

⇝
- symmetry concept in "quantum math" (analysis)
- planar diagrams (combinatorics)
- nice representation categories (algebra)
- quantum isomorphism of graphs (quantum information)

CONTEXT

CLASSICAL	NONCOMMUTATIVE
TOPOLOGY	C^*-ALGEBRAS
MEASURE THEORY	VON NEUMANN ALG.
PROBABILITY TH.	FREE PROB., QU. PROB
DIFF. GEOMETRY	NONCOMM. GEOMETRY
(LOC. COMP.) GROUPS	(LOC. COMP.) QU. GROUPS
INFORMATION TH.	QU. INFORMATION TH.
COMPLEX ANALYSIS	FREE ANALYSIS

CONTEXT

A **C^*-algebra** [Gelfand–Naimark 1940's] $:\Longleftrightarrow$ A (assoc.) algebra over $\mathbb{C}$,

$\exists\ {}^*: A \longrightarrow A$ antilin., $(xy)^* = y^* x^*$, $(x^*)^* = x$,

$\exists\ \|\cdot\|$ with $\|xy\| \leq \|x\|\|y\|$, $\|x^*x\| = \|x\|^2$,

complete w.r.t. $\|\cdot\|$ (Banach algebra)

Ex.: (a) $C(X) := \{ f: X \longrightarrow \mathbb{C} \text{ continuous} \}$, X comp. Hausdorff

(b) $B(H) := \{ T: H \longrightarrow H \text{ bounded, linear} \}$, H Hilbert space ($M_N(\mathbb{C})$)

[Gelfand–Naimark 1940's]: A unital C^*-algebra.

A commutative $\Longleftrightarrow$ $\exists X$ comp. Hausdorff: $A \cong C(X)$

CONTEXT

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Philosophy:

$$\begin{array}{ccc} \text{commutative} & \longleftrightarrow & \text{classical} \\ \cap & & \cap \\ \text{noncommutative} & \longleftrightarrow & \text{quantum} \end{array}$$

CONTEXT

$N \in \mathbb{N}$. (A, u) compact matrix quantum group [Woronowicz 1980s] $:\Leftrightarrow$

A unital C^*-algebra, $A = C^*(1, u_{ij} \mid i,j = 1, \ldots, N)$,

$u = (u_{ij})_{i,j=1,\ldots N}$, $\bar{u} = (u_{ij}^*) \in M_N(A)$ invertible,

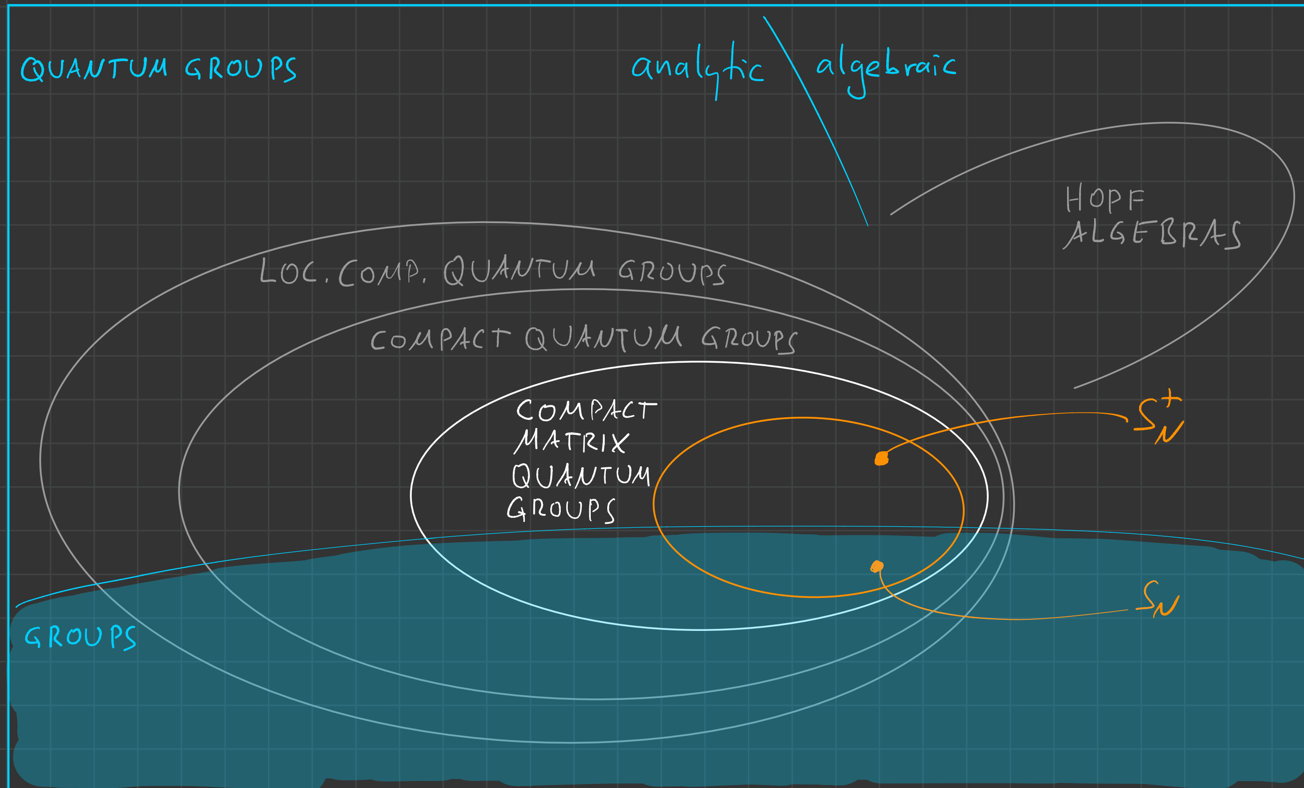
$\exists\, \Delta : A \longrightarrow A \otimes_{\min} A$, $u_{ij} \longmapsto \sum_{k=1}^{N} u_{ik} \otimes u_{kj}$ $*$-hom.

[Woronowicz 1980s]: (A, u) compact matrix quantum group.

A commutative $\Longleftrightarrow$ $\exists\, G \subseteq GL_N(\mathbb{C})$ comp. group : $A \cong C(G)$

Philosophy:	commutative	$\longleftrightarrow$	classical
	$\cap$		$\cap$
	noncommutative	$\longleftrightarrow$	quantum

CONTEXT


QUANTUM GROUPS
analytic
algebraic
HOPF ALGEBRAS
LOC. COMP. QUANTUM GROUPS
COMPACT QUANTUM GROUPS
COMPACT MATRIX QUANTUM GROUPS
S_N^+
S_N
GROUPS

QUANTUM PERMUTATIONS: INTRO

A $*$-algebra, $u = (u_{ij})_{i,j=1,\dots,N} \in M_N(A)$. u magic unitary $:\Longleftrightarrow$

$$u_{ij} = u_{ij}^* = u_{ij}^2, \quad \sum_k u_{ik} = \sum_k u_{kj} = 1, \quad u_{ik}u_{jk} = u_{ki}u_{kj} = 0, \; i \neq j$$

$u = (u_{ij})_{i,j=1,\dots,N} \in M_N(M_m(\mathbb{C}))$ magic unitary "quantum permutation matrix"

$m=1$: $\quad u_{ij} = \overline{u_{ij}} \Rightarrow u_{ij} \in \mathbb{R}, \quad u_{ij} = u_{ij}^2 \Rightarrow u_{ij} \in \{0,1\}$

$u_{ik}u_{jk} = u_{ki}u_{kj} = 0, \; i \neq j \quad \Rightarrow$ max. one $u_{ij} \neq 0$ per row/column

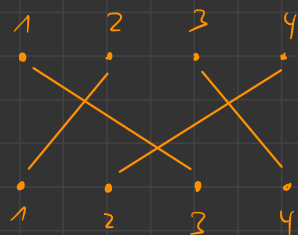
$\sum_k u_{ik} = \sum_k u_{kj} = 1 \quad \Rightarrow$ exactly one $u_{ij} \neq 0$ per row/column

$\Longrightarrow u \in M_N(\{0,1\})$ permutation matrix

QUANTUM PERMUTATIONS : INTRO

$\boxed{m=1}$

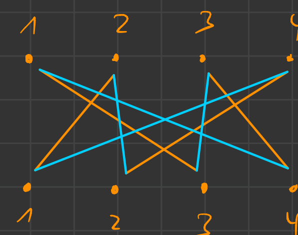
permutation



$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

quantum permutation

$\boxed{m=2}$



$$\begin{pmatrix} 0 & \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & 0 & \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\ 0 & \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} & 0 & \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & 0 & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & 0 \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & 0 & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & 0 \end{pmatrix} \quad \begin{matrix} \Sigma = 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \Sigma = 1 \\ \Sigma = 1 \\ \Sigma = 1 \end{matrix}$$

$\Sigma = 1 \quad \Sigma = 1 \quad \Sigma = 1 \quad \Sigma = 1$

QUANTUM PERMUTATIONS : INTRO

$S_N^+ := (A_s(N), u)$ quantum permutation group $:\Leftrightarrow$ [Sh. Wang 1990's]

$$A_s(N) := C^*(1, u_{ij}, \; i,j=1,\dots,N \mid u=(u_{ij})_{i,j=1,\dots,N} \text{ magic unitary})$$

"free symmetric quantum group"

$N \in \mathbb{N}$. (A,u) compact matrix quantum group $:\Leftrightarrow$ [Woronowicz 1980s]
A unital C^*-algebra, $A = C^*(1, u_{ij} \mid i,j=1,\dots,N)$,
$u=(u_{ij})_{i,j=1,\dots,N}$, $\bar{u} = (u_{ij}^*) \in M_N(A)$ invertible,
$\exists \, \Delta : A \to A \otimes_{\min} A$, $u_{ij} \mapsto \sum_{k=1}^{N} u_{ik} \otimes u_{kj}$ $*$-hom.

Check: S_N^+ compact matrix quantum group

$S_N \subseteq S_N^+$: $\quad A_s(N) \twoheadrightarrow C(S_N)$ surjective $*$-homomorphism, $S_N \subseteq M_N(\mathbb{C})$

$$u_{ij} \longmapsto (\sigma \mapsto \sigma_{ij})$$

have more ways of quantum permuting points! $(N \geq 4)$

SURVEY

- S_N^+ as quantum symmetry group
- representation theory of S_N^+
- quantum isomorphisms/automorphisms of graphs

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QUANTUM SYMMETRIES OF N POINTS

N points "quantized": $\quad X_N = \{1,\dots,N\}$ compact set

$$C(X_N) \cong C^*\left(p_1,\dots,p_N \mid p_i = p_i^* = p_i^2,\ \sum_k p_k = 1,\ p_i p_j = 0,\ i\neq j\right)$$

$$p_i : \{1,\dots,N\} \to \mathbb{C}, \quad p_i(t) = \begin{cases} 1 & t=i \\ 0 & t\neq i \end{cases}$$

$S_N \curvearrowright X_N$: $\quad \alpha : S_N \times X_N \longrightarrow X_N, \quad (\sigma, i) \mapsto \sigma(i)$

$S_N^+ \curvearrowright X_N$: $\quad \alpha : C(X_N) \longrightarrow A_s(N) \otimes C(X_N), \quad p_i \mapsto \sum_k u_{ik} \otimes p_k =: p_i'$

check $\quad p_i'^2 = \sum_{k,l} u_{ik} u_{il} \otimes \underbrace{p_k p_l}_{=\delta_{kl} p_k} = \sum_k \underbrace{u_{ik}^2}_{=u_{ik}} \otimes p_k = p_i' \quad$ etc

$\leadsto$ S_N^+ is the quantum symmetry group of N points! [Sh. Wang 1990's]

QUANTUM SYMMETRIES OF FREE INDEPENDENCE

(classical) independence: $X, Y \in L^\infty(\Omega, \mathbb{P})$ random variables, $\mathbb{E} : L^\infty(\ldots) \to \mathbb{C}$

$\rightsquigarrow \mathbb{E}[X^n Y^m] = \mathbb{E}X^n \cdot \mathbb{E}Y^m \in \mathrm{Poly}(\mathbb{E}X^a, \mathbb{E}X^b \mid a, b \in \mathbb{N}_0)$

free independence [Voiculescu 1980's]: $x, y \in A$, A algebra, $\varphi : A \to \mathbb{C}$ lin.

$\rightsquigarrow \varphi(x^{n_1} y^{m_1} x^{n_2} y^{m_2}) \in \mathrm{Poly}(\varphi(x^a), \varphi(y^b) \mid a, b \in \mathbb{N}_0)$

classical de Finetti Thm: $(x_n)_{n \in \mathbb{N}} \subseteq L^\infty(\Omega, \mathbb{P})$ real random variables

$(x_n)_{n \in \mathbb{N}}$ indep., id. distr. $\iff$ distribution of (x_n) invar. under S_N

free de Finetti Thm. [Köstler–Speicher 2009]: $(x_n)_{n \in \mathbb{N}}$ free indep., id. distr. $\iff$ ... S_N^+

$\rightsquigarrow$ S_N^+ is the quantum symmetry group of free independence!

REPRESENTATION THEORY OF S_N^+

Tannaka-Krein duality for quantum groups [Woronowicz 1980's]:

(a) $G=(A,u)$ comp. matrix qu. group $\Longrightarrow$ $\mathrm{Rep}(G)$ "nice" tensor category

(b) $\exists G$ comp. matrix qu. group: $\mathcal{C}=\mathrm{Rep}(G) \Longleftarrow$ $\mathcal{C}$ "nice" tensor category

"easy" quantum groups [Banica-Speicher 09] [W.]

(quantum) group	rep. category	diagrams
U_N	permutations [Schur-Weyl]	
O_N	pair partitions [Brauer]	Brauer diagrams 
S_N	all partitions	
S_N^+	planar part. [Banica 1990's]	
O_N^+	planar pair p. [Banica 1990's]	Temperley-Lieb 

"categories of partitions"

REPRESENTATION THEORY OF S_N^+

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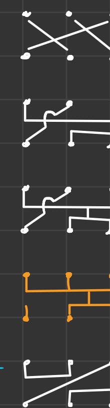


"categories of partitions"

Results: [Deligne 2007] interpolation categories $Rep(S_t)$, $t \in \mathbb{C}$ [Flake-Maaßen 2020] $Rep(\mathcal{C}_t)$

[Banica 1990s, Bichon, Collins, Vergnioux 2007/2009, …, Freslon-W. 2013] irrep. of "easy" q. groups, fusion rules

[Banica, Curran 2009/2010, Speicher, Tarrago 2015, Gromada, Mang 2018-, W.] classification of categories of part.

+ nice von Neumann algebras

QUANTUM SYMMETRIES OF GRAPHS

symmetries of Γ: $\Gamma = (\{1,\dots,N\}, E)$, $\varepsilon \in M_N(\{0,1\})$ adjacency matrix

$\mathrm{Aut}(\Gamma) := \{ \sigma \in S_N \mid \sigma\varepsilon = \varepsilon\sigma \}$ automorphism group

qu. symmetries of Γ: $A_\Gamma(N) := C^*(1, u_{ij}, i,j=1,\dots,N \mid u=(u_{ij}) \text{ magic unitary}, u\varepsilon = \varepsilon u)$

[Banica 2005]

Recall: u magic $\Longleftrightarrow$ $u_{ij} = u_{ij}^* = u_{ij}^2$, $\sum_k u_{ik} = \sum_k u_{kj} = 1$, $u_{ik}u_{jk} = u_{ki}u_{kj} = 0$, $i \neq j$

$$\begin{array}{ccc} \mathrm{Aut}^+(\Gamma) := (A_\Gamma(N), u) & \subseteq & S_N^+ \\ \cup\!\! \mid & & \cup\!\! \mid \\ \mathrm{Aut}(\Gamma) & \subseteq & S_N \end{array}$$

$$\begin{array}{ccc} A_\Gamma(N) & \twoheadleftarrow & A_S(N) \\ \downarrow & & \downarrow \\ C(\mathrm{Aut}(\Gamma)) & \twoheadleftarrow & C(S_N) \end{array}$$

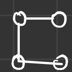
Γ has quantum symmetries $:\Longleftrightarrow$ $\mathrm{Aut}(\Gamma) \subsetneq \mathrm{Aut}^+(\Gamma)$ $\left(\Longleftrightarrow A_\Gamma(N) \text{ noncomm.}\right)$

QUANTUM SYMMETRIES OF GRAPHS

qu. symmetries of Γ: $A_\Gamma(N) := C^*(1, u_{ij}, i,j=1,\dots,N \mid u=(u_{ij})$ magic unitary, $u\varepsilon = \varepsilon u)$

Γ has quantum symmetries $:\Longleftrightarrow \operatorname{Aut}(\Gamma) \subsetneq \operatorname{Aut}^+(\Gamma)$ ($\Longleftrightarrow A_\Gamma(N)$ noncomm.)

Ex.: a) $\Gamma =$  , $\varepsilon = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}$, $\operatorname{Aut}(\Gamma) = S_4 \subsetneq S_4^+ = \operatorname{Aut}^+(\Gamma)$ has qu. sym.

b) $\Gamma =$  , $\varepsilon = \begin{pmatrix} 0&1&1&0\\1&0&0&0\\1&0&0&1\\0&0&1&0 \end{pmatrix}$, $u = \begin{pmatrix} u_{11}&0&1-u_{11}&0\\0&u_{11}&0&1-u_{11}\\1-u_{11}&0&u_{11}&0\\0&1-u_{11}&0&u_{11} \end{pmatrix}$, $\operatorname{Aut}(\Gamma) = \mathbb{Z}_2 = \operatorname{Aut}^+(\Gamma)$ no qu. sym.

Results: no q. sym [Banica 2007, Bichon, Chenevier, Schmidt 2018-, ...]: Petersen, Johnson $J(n,2)$, odd graphs, Hamming $H(n,3)$, ...

q. sym [...]: complete graphs, cycles, (folded) cube, Clebsch, Hamming $H(n,k)$, $k>3$, ...

Erdős–Renyi $(\mathbb{P}\to 1)$ [Lupini-Mancinska-Roberson 2017, Junk-Schmidt-W. 2019]: graphs no q. sym., trees q. sym.

[Bichon 2003] $\operatorname{Aut}^+(\Gamma \sqcup \dots \sqcup \Gamma) = \operatorname{Aut}^+(\Gamma) \wr_* S_N^+$ [...] computation of $\operatorname{Aut}^+(\Gamma)$

QUANTUM ISOMORPHISMS OF GRAPHS

$\Gamma_1 \cong \Gamma_2 :\Longleftrightarrow \exists\, \sigma$ permutation matrix : $\quad \sigma \varepsilon_1 = \varepsilon_2 \sigma \quad , \varepsilon_1, \varepsilon_2$ adj. matrices

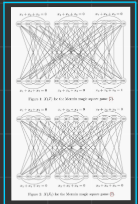
$\Gamma_1 \cong_q \Gamma_2 :\Longleftrightarrow \exists\, u$ quantum permutation matrix : $\quad u \varepsilon_1 = \varepsilon_2 u$

[Atserias-Mancinska-Roberson-Samal-Severini-Varvitsiotis 2019]

Recall: $u = (u_{ij})_{i,j=1,\dots,N} \in M_N(M_m(\mathbb{C}))$ magic unitary $(m=1: \; u \in S_N)$

$$u_{ij} = u_{ij}^* = u_{ij}^2 \,, \quad \sum_k u_{ik} = \sum_k u_{kj} = 1 \,, \quad u_{ik} u_{jk} = u_{ki} u_{kj} = 0 \,, \; i \neq j$$

[Lupini-Mancinska-Roberson 2017]: $\Gamma_1 \cong \Gamma_2 \;\underset{\not\Longleftarrow}{\Longrightarrow}\; \Gamma_1 \cong_q \Gamma_2$



[Lupini-Mancinska-Roberson 2017]

nonlocal game [AMRSSV19]: referee $\underset{y_A, y_B \in V_1 \dot\cup V_2}{\overset{x_A, x_B \in V_1 \dot\cup V_2}{\rightleftarrows}}$ Alice & Bob

classical strategy: win $\big((x_1, y_1) \in E_1 \Longleftrightarrow (x_2, y_2) \in E_2\big)$ with $\mathbb{P} = 1 \Longleftrightarrow \Gamma_1 \cong \Gamma_2$

quantum strategy: win with $\mathbb{P} = 1 \Longleftrightarrow \Gamma_1 \cong_q \Gamma_2$

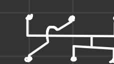
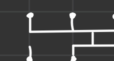
[+Musto, Reutter, Verdon, Brannan, Paulsen, Ganesan, Harris, Eifler, Soltan, Schmidt, ...]

QUANTUM LOVASZ THEOREM

graph homomorphism: $\varphi : \Gamma' \longrightarrow \Gamma$ hom. $:\Longleftrightarrow \left[(x,y) \in E' \Longrightarrow (\varphi(x), \varphi(y)) \in E \right]$

[Lovasz 1967]: $\Gamma_1 \cong \Gamma_2 \Longleftrightarrow \forall \Gamma'$ graph: $|\{\varphi : \Gamma' \to \Gamma_1 \text{ hom.}\}| = |\{\varphi : \Gamma' \to \Gamma_2 \text{ hom.}\}|$

[Mancinska-Roberson 2019]: $\Gamma_1 \cong_q \Gamma_2 \Longleftrightarrow \forall \Gamma'$ planar graph: $|\{\varphi : \Gamma' \to \Gamma_1 \text{ hom.}\}| = |\{\varphi : \Gamma' \to \Gamma_2 \text{ hom.}\}|$

(quantum) group	rep. category	diagrams
S_N	all partitions	
S_N^+	planar part.	
$Aut^+(\Gamma)$ [Mancinska-Roberson 2019]	$\lvert\{\varphi : \Gamma' \to \Gamma \text{ hom.}\}\rvert$	bilabelled planar graphs Γ'

QIT (graph isom. game) $\longleftrightarrow$ QG (rep. th. $Aut^+(\Gamma)$) $\longleftrightarrow$ Alg. Comb. (LovaszThm)

QUANTUM SYMMETRIES OF (QUANTUM) GRAPH C^*-ALGEBRAS

graph C^*-algebras [Cuntz–Krieger 1980s]: $\Gamma = (V, E)$, $s(e) \xrightarrow{e} r(e)$

$$C^*(\Gamma) := C^*\left(p_v, v \in V, s_e, e \in E \mid p_v = p_v^2 = p_v^*,\ s_e^* s_e = p_{r(e)},\ \sum_{s(e)=v} s_e s_e^* = p_v\right)$$

[Schmidt–W. 2018]: $QSym(C^*(\Gamma)) = Aut^+(\Gamma)$ [+ Banica–Skalski 2013, Joardar–Mandal 2018]: NCG...

quantum graphs [Weaver 2012, Duan–Severini–Winter 2013, Musto–Reutter–Verdon 2019, Brannan–Chirvasitu–Eifler–Harris–Paulsen–Su–Wasilewski 2020]:

graph $\Gamma = (\{1, \dots, N\}, \varepsilon)$: $\mathbb{C}^N \xrightarrow{\varepsilon} \mathbb{C}^N$

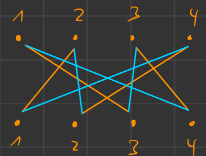
quantum graph $\Gamma_q = \left(\left(\bigoplus_{a=1}^N M_{N_a}(\mathbb{C}), \psi\right), A_\varepsilon\right)$: $\bigoplus_{a=1}^N M_{N_a}(\mathbb{C}) \xrightarrow{A_\varepsilon} \bigoplus_{a=1}^N M_{N_a}(\mathbb{C})$

quantum graph C^*-algebras [Brannan–Eifler–Voigt–W. 2020]: $C^*(\Gamma_q) := \dots$

$$QSym(C^*(\Gamma_q)) \supseteq Aut^+(\Gamma_q)$$

SUMMARY

quantum permutation



$$\begin{pmatrix} 0 & \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & 0 & \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\ 0 & \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} & 0 & \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & 0 & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & 0 \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & 0 & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & 0 \end{pmatrix}$$

S_N^+

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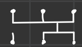
Philosophy: commutative $\longleftrightarrow$ classical
$\cap$ $\cap$
noncommutative $\longleftrightarrow$ quantum

$S_N^+ = QSym(N \text{ pts}) \supseteq S_N$

$S_N^+ = QSym(\text{free independence})$ (planar)

$Aut^+(\Gamma) = QSym(\Gamma) \supseteq Aut(\Gamma)$ $(\neq ?)$

$Aut^+(\Gamma) = QSym(C^*(\Gamma))$

$Rep(S_N^+) \longleftrightarrow$  (planar)

$\Gamma_1 \cong \Gamma_2 \;\overset{\not\Leftarrow}{\Longrightarrow}\; \Gamma_1 \cong_q \Gamma_2$ (QIT)

$Rep(Aut^+(\Gamma)) \longleftrightarrow$ hom. counts (planar)

quantum Lovasz Thm: $\Gamma_1 \cong_q \Gamma_2$ (planar)

T H A N K S

(ref. $\rightarrow$)

books on quantum groups:

Neshveyev-Tuset, Compact quantum groups and their rep. categ., 2013
Timmermann, An invitation to quantum groups and duality, 2008

S_N^+ & "easy" qu. groups:

Sh. Wang, Quantum symmetry groups of finite spaces, 1998
Banica, Speicher, Liberation of orthogonal Lie groups, 2009
1308.6390, 1311.7630, 1312.3857, 1512.00195, 1901.03266, 2003.00569 (U_n^+)

$Aut^+(\Gamma)$:

Bichon, Quantum automorphism groups of finite graphs, 2003
Banica, Quantum automorphism groups of homogeneous graphs, 2005
1706.08833, 1801.02942, 1810.11284, 1906.06537 (Schmidt)
math/0605257, math/0601758, math/0107029 (Banica-Bichon$^+$)
1911.04912, 1906.12097 (U_n^+: Sinkhorn, comp. alg.)
1504.05671, 1904.00455 (Chassaniol)
1712.01820, 1911.02952, 2011.14149 (probabilistic)

QIT:

1611.09837, 1712.01820, 1910.06958, 2012.13328 (Mančinska-Roberson$^+$)
1609.07775, 1711.07945, 1801.09705 (Musto-Reutter-Verdon$^+$)
1903.12369, 1703.00960, 2009.07229, 1908.03842, 2011.03867 (games)

$C^*(\Gamma)$:

1109.6184, 1706.08833, 1711.04253, 1811.08735 (qsym C^*(graphs))
1005.0354, 1002.2514, 1711.07945, 1812.11474, 2009.09466 (qu.-graphs)