Classification of compact matrix quantum groups

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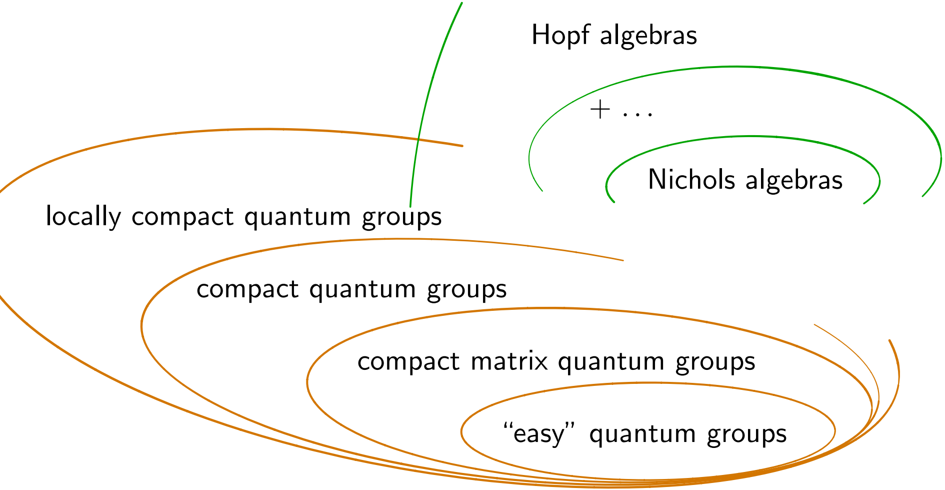

Deformation:	$xy = yx$	$\longrightarrow$	$xy = qyx,$	$q \in \mathbb{C}$
Liberation:	$xy = yx$	$\longrightarrow$	$\varnothing$	

Definition [Woronowicz 1980s]

Let $N \in \mathbb{N}$. $G = (A, u)$ is a compact matrix quantum group (CMQG) $:\Longleftrightarrow$

- *A is a unital C^*-algebra with $A = C^*(u_{ij}, 1 \mid i, j \in \{1, \dots, N\})$*
- *$u = (u_{ij})$ and $\bar{u} = (u_{ij}^*)$ invertible $N \times N$-matrices in $M_N(A)$*
- *$\Delta : A \to A \otimes A,\ u_{ij} \mapsto \sum_{k=1}^N u_{ik} \otimes u_{kj}$ *-homomorphism*

Fundamental Theorem of CMQG [Woronowicz 1980s]

$G = (A, u)$ compact matrix quantum group with $N \in \mathbb{N}$.

$$A \text{ commutative} \quad \Longleftrightarrow \quad \exists G \subseteq GL_N(\mathbb{C}) \text{ compact group} : A \cong C(G)$$

Proof.

Fundamental Theorem of CMQG [Woronowicz 1980s]

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"$\Longleftarrow$":

Let $G \subseteq GL_N(\mathbb{C})$ be a compact group.
Put $A := C(G) := \{f : G \to \mathbb{C} \mid f \text{ continuous}\}$.
Put $u_{ij}, u_{ij}^* : C(G) \to \mathbb{C}$ with $u_{ij}(g) := g_{ij}$, $u_{ij}^*(g) := \bar{g}_{ij}$.
Then (A, u) is a compact matrix quantum group:

- $A = C^*(u_{ij}, 1 \mid i, j \in \{1, \dots, N\})$ [Stone-Weierstrass]
- $u = (u_{ij}), \bar{u} = (u_{ij}^*) \in M_N(C(G))$ invertible $(u(g) = g)$
- $\Delta : C(G) \to C(G) \otimes C(G), u_{ij} \mapsto \sum_{k=1}^N u_{ik} \otimes u_{kj}$ (matrix multipl.)

Hence, compact matrix quantum groups generalize $G \subseteq GL_N(\mathbb{C})$

Fundamental Theorem of CMQG [Woronowicz 1980s]

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"$\Longrightarrow$":

Fundamental Thm in C^*-Algebras [Gelfand-Naimark 1940s]

A unital C^*-algebra.

$$A \text{ commutative} \quad \Longleftrightarrow \quad \exists X \text{ compact space} : A \cong C(X)$$

Applying it in our situation yields a compact space

$$G := \{\varphi : A \to \mathbb{C} \mid \varphi \text{ algebra homomorphism}, \varphi \neq 0\}$$

Here, $\chi : A \to C(G)$ of the form $\chi(x)(\varphi) := \varphi(x)$.
Then, G becomes a group with the help of Δ.

Definition [Gelfand-Naimark 1940s]

A C^*-algebra A is

- *an (associative) algebra over $\mathbb{C}$*
- *which may or may not be unital (for CMQG: always unital)*
- *with an involution $^* : A \to A$*
  i.e. $(\lambda a + \mu b)^* = \bar{\lambda} a^* + \bar{\mu} b^*$ for $\lambda, \mu \in \mathbb{C}$, $(ab)^* = b^* a^*$, $(a^*)^* = a$
- *and a norm satisfying $\|ab\| \leq \|a\| \|b\|$ and $\|a^* a\| = \|a\|^2$*
- *complete with respect to this norm (i.e.: A Banach algebra)*

□

Definition [Woronowicz 1980s]

Let $N \in \mathbb{N}$. $G = (A, u)$ is a compact matrix quantum group $:\Longleftrightarrow$

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Fundamental Theorem of CMQG [Woronowicz 1980s]

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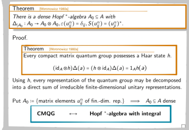
Theorem [Woronowicz 1980s]

There is a dense Hopf *-algebra $A_0 \subseteq A$ with
$\Delta_{|A_0} : A_0 \to A_0 \otimes A_0, \varepsilon(u_{ij}^\alpha) = \delta_{ij}, S(u_{ij}^\alpha) = (u_{ji}^\alpha)^*$.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Proof.

Theorem [Woronowicz 1980s]

Every compact matrix quantum group possesses a Haar state h.

$$(\mathrm{id}_A \otimes h)\Delta(a) = (h \otimes \mathrm{id}_A)\Delta(a) = 1_A h(a)$$

Using h, every representation of the quantum group may be decomposed into a direct sum of irreducible finite-dimensional unitary representations.

Put $A_0 := \{\text{matrix elements } u_{ij}^\alpha \text{ of fin.-dim. rep.}\} \implies A_0 \subseteq A$ dense

CMQG $\longleftrightarrow$ Hopf *-algebra with integral

Example (Symmetric quantum group) [Sh. Wang 1990s]

$S_N^+ := (A_S(N), u)$ *compact matrix quantum group with*

$$A_S(N) := C^*\left(u_{ij}, 1 \mid u_{ij} = u_{ij}^2 = u_{ij}^*, \sum_k u_{ik} = \sum_k u_{kj} = 1\right) \quad \twoheadrightarrow \quad C(S_N)$$

$$S_N \quad \subseteq \quad S_N^+ \quad \textit{quantum permutations}$$

$$\rotatebox{90}{$\in$} \qquad \text{“}\rotatebox{90}{$\in$}\text{”}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & 0 & \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\ 0 & \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} & 0 & \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & 0 & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & 0 \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & 0 & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & 0 \end{pmatrix}$$

Example (Symmetric quantum group) [Sh. Wang 1990s]

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$$A_S(N) := C^*(u_{ij}, 1 \mid u_{ij} = u_{ij}^2 = u_{ij}^*, \sum_k u_{ik} = \sum_k u_{kj} = 1) \quad \twoheadrightarrow \quad C(S_N)$$

Example (Free orthogonal quantum group) [Sh. Wang 1990s]

$O_N^+ := (A_O(N), u)$ *compact matrix quantum group with*

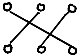
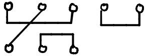
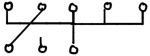
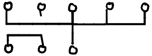
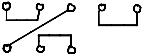
$$A_O(N) := C^*(u_{ij}, 1 \mid u_{ij} = u_{ij}^*, \sum_k u_{ik} u_{jk} = \sum_k u_{ki} u_{kj} = \delta_{ij}) \quad \twoheadrightarrow \quad C(O_N)$$

algebraic relations	**solutions in $\mathbb{C}$:**	**group**
	solutions in $M_N(\mathbb{C})$:	**quantum group**

CMQG: some aspects

- Have quantum versions $S_N \subseteq S_N^+$, $O_N \subseteq O_N^+$, $U_N \subseteq U_N^+, \dots$
- Associated "reduced" C^*-algebras $C_{\text{red}}(G)$ and von Neumann algebras $L(G)$ are interesting [Banica, Vaes, Vergnioux, Brannan, Freslon,...]
- S_N^+, O_N^+, U_N^+ yield quantum symmetries for free probability or Connes's noncommutative geometry [Köstler, Speicher, Curran, Banica, Goswami,...]
- S_N^+ is a Calabi-Yau algebra of dimension 3 [Bichon, Franz, Gerhold,...]
- (Hochschild) cohomological dimensions of S_N^+, O_N^+, U_N^+ are 3 [Thom, Bichon, Franz, Gerhold, Das, Kula, Skalski,...]
- L^2-Betti numbers of S_N^+, O_N^+ and U_N^+ known: $\beta_p^{(2)} = 0$ except $\beta_1^{(2)}(U_N^+) = 1$ [Vergnioux, Collins, Härtl, Thom, Bichon, Raum, Kyed, Vaes, Valvekens,...]

"Easy" QG: Schur-Weyl/Tannaka-Krein first

(quantum) group	representation category	diagrams
U_N	permutations (Schur-Weyl)	
O_N	pair partitions (Brauer diagrams)	
S_N	all partitions of sets	
S_N^+	noncrossing partitions	
O_N^+	noncrossing pair partitions	

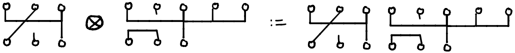
[Woronowicz 1980s]

Denote by $P(k,l)$ the set of partitions on k upper and l lower points.

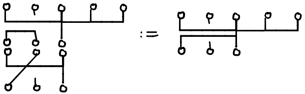
Definition [Banica-Speicher 2009]

A category of partitions is a set $\mathcal{C} \subseteq \bigcup_{k,l\in\mathbb{N}_0} P(k,l)$ which is closed under

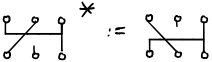
tensor products



composition



involution



and containing  *and* 

Example

(a) all partitions, *(b) pair partitions,* *(c) noncrossing partitions (NC)*
(d) noncrossing pair partitions, *(e)* $\{p \in NC \mid$ blocks of size 1 or 2$\}$

“Easy” QG: Definition

Definition [Banica-Speicher 2009]

A category of partitions is a set $\mathcal{C} \subseteq \bigcup_{k,l\in\mathbb{N}_0} P(k,l)$ *which is closed under tensor products, composition and involution, containing also*  *and* 

Definition [Banica-Speicher 2009]

A CMQG $G = (A, u)$ *with* $S_N \subseteq G \subseteq O_N^+$ *is „easy“, if its representation theory is given (via Tannaka-Krein) by a category of partitions* $\mathcal{C}$*:*

$$\mathrm{Mor}(u^{\otimes k}, u^{\otimes l}) = \mathrm{span}\{T_p \mid p \in \mathcal{C} \cap P(k,l)\}$$

Here, $\mathrm{Mor}(u^{\otimes k}, u^{\otimes l}) := \{T : (\mathbb{C}^N)^{\otimes k} \to (\mathbb{C}^N)^{\otimes l} \text{ linear} \mid Tu^{\otimes k} = u^{\otimes l} T\}$.

“easy” QG $\longleftrightarrow$ **categories of partitions**

“Easy” QG: The map T_p

Definition of $T_p : (\mathbb{C}^N)^{\otimes k} \to (\mathbb{C}^N)^{\otimes l}$ as follows.

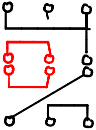
$$T_p(e_{i_1} \otimes \ldots \otimes e_{i_k}) := \sum_{j_1,\ldots,j_l} \delta_p(i_1,\ldots,i_k; j_1,\ldots,j_l) e_{j_1} \otimes \ldots \otimes e_{j_l}$$

Then: $T_p u^{\otimes k} = u^{\otimes l} T_p \quad \longrightarrow \quad$ relations on the u_{ij}

crossings $\longleftrightarrow$ **commutativity relations**

"Easy" QG: Side remark (Deligne categories)

Idea [Deligne 2007]: Weight for loops $t \in \mathbb{N} \longrightarrow t \in \mathbb{C}$



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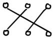
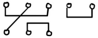
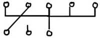
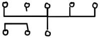
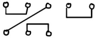
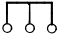
U_N	permutations (Schur-Weyl)	
O_N	pair partitions (Brauer diagrams)	
S_N	all partitions of sets	
S_N^+	noncrossing partitions	
O_N^+	noncrossing pair partitions	

Categories of partitions $\longrightarrow$ many new interpolating categories

[Flake-Maaßen 2020]

"easy" QG $\longleftrightarrow$ categories of partitions

Flavour: There are exactly two categories of noncrossing partitions containing

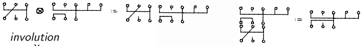
1. Case: $\mathcal{C} \subseteq NC$,  $\in \mathcal{C}$,  $\notin \mathcal{C}$
Then: $\mathcal{C} = \{p \in NC \mid$ blocks of size 1 or 2$\}$

Denote by $P(k,l)$ the set of partitions on k upper and l lower points.

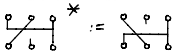
Definition [Banica-Speicher 2009]

A category of partitions is a set $\mathcal{C} \subseteq \bigcup_{k,l \in \mathbb{N}_0} P(k,l)$ which is closed under

tensor products ... *composition*



involution



and containing *and* 

Example

(a) all partitions, *(b) pair partitions,* *(c) noncrossing partitions (NC)*
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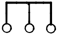
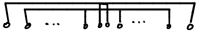
2. Case: $\mathcal{C} \subseteq NC$,  $\in \mathcal{C}$,  $\in \mathcal{C}$
Then: $\mathcal{C} = NC$

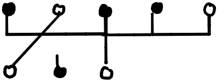
Theorem [Banica-Speicher 2009, Banica-Curran-Speicher 2010, W. 2013, Raum-W. 2016]

Orthogonal "easy" QG (i.e. $S_N \subseteq G \subseteq O_N^+$) are completely classified:

	Categories of partitions	*Quantum groups*
	$\{$*all partitions*$\}$, $\{\lvert b\rvert = 2\}$, $\{\lvert b\rvert = 1 \text{ or } 2\}$, $\{\lvert b\rvert$ *even*$\}$, $\{\lvert p\rvert$ *even*$\}$, $\{\lvert p\rvert$ *even*, $\lvert b\rvert = 1 \text{ or } 2\}$	$S_N, O_N, B_N,$ $\mathbb{Z}_2 \wr S_N, S_N \times \mathbb{Z}_2,$ $B_N \times \mathbb{Z}_2$
	$\{NC\}$, $\{NC, \lvert b\rvert = 2\}$, $\{NC, \lvert b\rvert = 1 \text{ or } 2\}$, $\{NC, \lvert b\rvert$ *even*$\}$, $\{NC, \lvert p\rvert$ *even*$\}$, $\{NC, \lvert p\rvert$ *even*, $\lvert b\rvert = 1, 2\}$, $\{+$ *leg dist. even*$\}$	$S_N^+, O_N^+, B_N^+,$ $\mathbb{Z}_2 \wr_* S_N^+, S_N^+ \times \mathbb{Z}_2,$ $B_N^+ \times \mathbb{Z}_2, B_N^+ * \mathbb{Z}_2$
	$\{\lvert b\rvert = 2$, *leg dist. even*$\}$, ...	$O_N^*, B_N^*, H_N^*, \ldots$
	*use* $(\mathbb{Z}_2)^{*N} \twoheadrightarrow \Gamma$	$\hat{\Gamma} \bowtie S_N$
	use 	

"EASY" QG: UNITARY CASE

How about U_N?
Non-Hermitian: $u := (u_{ij}) \neq (u_{ij}^*) =: \bar{u}$



Definition [Tarrago-W. 2016]

A category of two-colored partitions is a set $\mathcal{C} \subseteq \bigcup_{k,l \in \mathbb{N}_0} P^{\circ\bullet}(k,l)$ *closed under tensor products, composition, involution, containing*  *and* 

Definition [Tarrago-W. 2016]

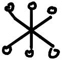
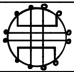
A CMQG $G = (A, u)$ *with* $S_N \subseteq G \subseteq U_N^+$ *is „easy", if its representation theory is given (via Tannaka-Krein) by a category of two-col. partitions* $\mathcal{C}$*:*

$$\mathrm{Mor}(u \otimes \bar{u} \ldots \otimes u, u \otimes \ldots \otimes \bar{u}) = \mathrm{span}\{T_p \mid p \in \mathcal{C} \cap P^{\circ\bullet}(k,l) \text{ colored } \}$$

unitary "easy" QG $\longleftrightarrow$ categories of two-colored partitions

Theorem [Tarrago-W. 2018, Gromada 2018, Mang-W. 2019, 2020]

Unitary “easy” QG (i.e. $S_N \subseteq G \subseteq U_N^+$) are partially classified:

	Categories of partitions	Quantum groups
	$\{$all two-col. partitions$\}$, $\{+\lvert b\rvert = 2$ and  $\}$, rules on block sizes and colorings	S_N, U_N, $S_N \times \mathbb{Z}_k, \ldots,$
	way more noncrossing ones	S_N^+, U_N^+, $S_N^+ \times \mathbb{Z}_k, S_N^+ * \mathbb{Z}_k$, $(S_N^+ * \mathbb{Z}_d) \times \mathbb{Z}_k, \ldots$
	use $D \subseteq (\mathbb{N}_0, +)$  	many U_N^* versions
	rules on block sizes and colorings and crossings	?

Classification of "Easy" QG: Summary

way richer! →

Cases	orthogonal (complete) $u_{ij}=u_{ij}^*$		unitary (ongoing) $u_{ij}\neq u_{ij}^*$	
groups $(ab=ba)$	$S_N, O_N, S_N\times\mathbb{Z}_2 \dots$	✓	$+\ U_N, S_N\times\mathbb{Z}_k, \dots$	✓
free case "noncrossing"	$S_N^+, O_N^+, S_N^+\times\mathbb{Z}_2 \dots$	✓	$+\ U_N^+, S_N^+ * \mathbb{Z}_k, \dots$	✓
half-liberated $(abc=cba)$	$O_N^*, \dots$	✓	many! $(\mathcal{D}\subseteq(\mathbb{N}_0,+))$	✓
other cases	$\hat{\Gamma}\bowtie S_N, \ \mathbb{Z}_2^{*N}\twoheadrightarrow\Gamma$	✓	...	?