Quantum automorphism groups of finite graphs

a survey

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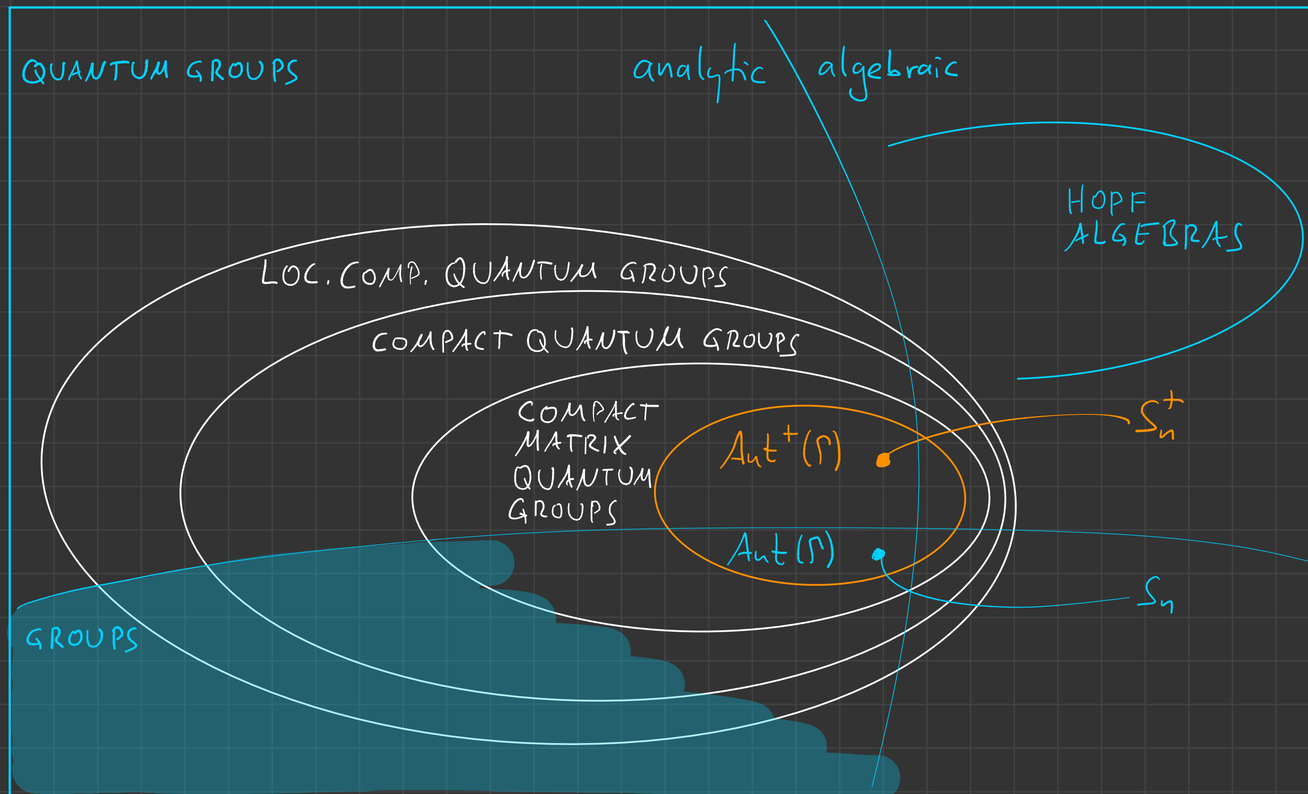
CONTEXT / BACKGROUND



CONTEXT / BACKGROUND

QUANTUM / NONCOMMUTATIVE MATHEMATICS

CLASSICAL	NONCOMMUTATIVE
TOPOLOGY	C^*-ALGEBRAS
MEASURE THEORY	VON NEUMANN ALG.
PROBABILITY TH.	FREE PROB., QU. PROB
DIFF. GEOMETRY	NONCOMM. GEOMETRY
(LOC. COMP.) GROUPS	(LOC. COMP.) QU. GROUPS
INFORMATION TH.	QU. INFORMATION TH.
COMPLEX ANALYSIS	FREE ANALYSIS

CONTEXT / BACKGROUND

Fundamental Thm (Gelfand-Naimark 1940s): A unital C^*-algebra.

A commutative $\iff$ $\exists X$ compact: $A \cong C(X) := \{ f: X \to \mathbb{C} \text{ cont.} \}$

Def. (Woronowicz 1980s): $n \in \mathbb{N}$. $G = (A, u)$ compact matrix qu. group, (CMQG)

if (i) $A = C^*(1, u_{ij}, 1 \leq i,j \leq n)$

(ii) $u = (u_{ij})_{i,j=1,\dots n}$, $\bar{u} = (u_{ij}^*)_{i,j=1,\dots n} \in M_n(A)$ invertible

(iii) $\Delta: A \to A \otimes_{\min} A$, $u_{ij} \mapsto \sum_{k=1}^{n} u_{ik} \otimes u_{kj}$ $*$-hom.

Fundamental Thm (Woronowicz 1980s): (A, u) CMQG.

A commutative $\iff$ $\exists G \subseteq GL_n(\mathbb{C})$ compact group: $A \cong C(G)$

QUANTUM PERMUTATIONS

Def. (Wang 1990s): $S_n^+ := (C(S_n^+), u)$ free symmetric qu. group

$$C(S_n^+) := C^*\Big(1, u_{ij}, 1 \le i,j \le n \;\Big|\; u_{ij} = u_{ij}^* = u_{ij}^2,\ \sum_k u_{ik} = \sum_k u_{kj} = 1\Big)$$

Check: S_n^+ is a CMQG ✓

Def. (Woronowicz 1980s): $n \in \mathbb{N}$. $G = (A, u)$ compact matrix qu. group (CMQG), if

(i) $A = C^*(1, u_{ij}, 1 \le i,j \le n)$

(ii) $u = (u_{ij})_{i,j=1,\dots,n}$, $\bar{u} = (u_{ij}^*)_{i,j=1,\dots,n} \in M_n(A)$ invertible

(iii) $\Delta : A \to A \otimes_{\min} A$, $u_{ij} \mapsto \sum_{k=1}^{n} u_{ik} \otimes u_{kj}$ $*$-hom.

Check: $\sigma \in S_n \subseteq GL_n(\mathbb{C})$ permutation matrix

$$\Longrightarrow \quad \sigma_{ij} = \overline{\sigma_{ij}} = \sigma_{ij}^2 \qquad (\text{i.e. } \sigma_{ij} \in \{0,1\})$$

$$\& \quad \sum_k \sigma_{ik} = \sum_k \sigma_{kj} = 1$$

Hence: $C(S_n^+) \twoheadrightarrow C(S_n)$, $u_{ij} \longmapsto ev_{ij}$ $\rightsquigarrow$ $S_n \subseteq S_n^+$

more quantum permutations!

QUANTUM PERMUTATIONS

permutations		quantum permutations
S_4	$\subsetneq$	S_4^+
$\cup$		„ $\cup$ "

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\exists\, \pi : C(S_4^+) \to C^*(1, p, q \text{ projections})$$

$$u \mapsto \begin{pmatrix} 0 & p & 0 & 1-p \\ 0 & 1-p & 0 & p \\ q & 0 & 1-q & 0 \\ 1-q & 0 & q & 0 \end{pmatrix}$$

$\Rightarrow$ $C(S_4^+)$ noncommutative $(pq \neq qp)$

$\Rightarrow$ $C(S_4^+) \overset{\neq}{\twoheadrightarrow} C(S_4)$

$$\begin{pmatrix} 0 & \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & 0 & \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\ 0 & \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} & 0 & \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & 0 & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & 0 \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & 0 & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & 0 \end{pmatrix}$$

QUANTUM PERMUTATIONS

S_n^+ as the quantum symmetry group of n points:

n points $\rightsquigarrow$ $X_n = \{1, \dots, n\}$

$\rightsquigarrow$ $C(X_n) = \mathbb{C}^n = C^*(p_1, \dots p_n \text{ proj.} \mid \sum_k p_k = 1)$

$S_n \curvearrowright X_n$: $\alpha : S_n \times X_n \longrightarrow X_n$, $(\sigma, i) \longmapsto \sigma(i)$

$S_n^+ \curvearrowright X_n$: $\alpha : C(X_n) \longrightarrow C(S_n^+) \otimes C(X_n)$, $p_i \longmapsto \underbrace{\sum_k u_{ik} \otimes p_k}$

α $*$-hom.: $p_i'^2 = \sum_{k,l} u_{ik} u_{il} \otimes \underbrace{p_k p_l}_{= \delta_{kl} p_k} = \sum_k \underbrace{u_{ik}^2}_{= u_{ik}} \otimes p_k = p_i'$ $\qquad =: p_i'$

S_n^+ maximal with this action!

OPEN: $\exists G$ CMQG: $S_n \subsetneq G \subsetneq S_n^+$?

$$\boxed{\begin{array}{l} C(S_n^+) \overset{\not\cong}{\twoheadrightarrow} B \overset{\not\cong}{\twoheadrightarrow} C(S_n) \\ \quad u_{ij} \longmapsto w_{ij} \longmapsto ev_{ij} \\ \& \quad w_{ij} \longmapsto \sum_k w_{ik} \otimes w_{kj} \ \text{*-hom.} \end{array}}$$

QUANTUM SYMMETRIES OF GRAPHS

$\Gamma = (\{1, \dots, n\}, E)$ finite graph with adj. matrix $\varepsilon \in M_n(\{0,1\})$

$\mathrm{Aut}(\Gamma) := \{\sigma \in S_n \mid \sigma\varepsilon = \varepsilon\sigma\} \subseteq S_n$ automorphism group (symmetries of Γ)

<u>Def. (Banica 2005):</u> $\mathrm{Aut}^+(\Gamma)$ quantum automorphism group

$$C(\mathrm{Aut}^+(\Gamma)) := C^*\Big(u_{ij} \;\Big|\; u_{ij} = u_{ij}^* = u_{ij}^2,\ \sum_k u_{ik} = \sum_k u_{kj} = 1,\ u\varepsilon = \varepsilon u\Big)$$

$$\begin{array}{ccc} \mathrm{Aut}^+(\Gamma) & \subseteq & S_n^+ \\ \cup\!\!\mid & & \cup\!\!\mid \\ \mathrm{Aut}(\Gamma) & \subseteq & S_n \end{array} \qquad\qquad \begin{array}{ccc} C(\mathrm{Aut}^+(\Gamma)) & \twoheadleftarrow & C(S_n^+) \\ \downarrow & & \downarrow \\ C(\mathrm{Aut}(\Gamma)) & \twoheadleftarrow & C(S_n) \end{array}$$

Γ has quantum symmetries $:\Longleftrightarrow$ $\mathrm{Aut}(\Gamma) \subsetneq \mathrm{Aut}^+(\Gamma)$ ($C(\mathrm{Aut}^+(\Gamma))$ noncomm.)

QUANTUM SYMMETRIES OF GRAPHS

$\Gamma = (\{1,\dots,n\}, E)$ finite graph with adj. matrix $\varepsilon \in M_n(\{0,1\})$

$\mathrm{Aut}(\Gamma) := \{\sigma \in S_n \mid \sigma\varepsilon = \varepsilon\sigma\} \subseteq S_n$ automorphism group

(symmetries of Γ)

<u>Def. (Banica 2005):</u> $\mathrm{Aut}^+(\Gamma)$ quantum automorphism group

$$C(\mathrm{Aut}^+(\Gamma)) := C^*\left(u_{ij} \mid u_{ij} = u_{ij}^* = u_{ij}^2,\ \sum_k u_{ik} = \sum_k u_{kj} = 1,\ u\varepsilon = \varepsilon u\right)$$

$$u\varepsilon = \varepsilon u \iff \left(u_{ik}u_{jl} = 0 \quad \text{if} \quad \varepsilon_{ij} \neq \varepsilon_{kl}\right)$$

By the way: $C(\mathrm{Aut}^*(\Gamma)) := C^*(u_{ij} \mid \dots\ \&\ u_{ik}u_{jl} = u_{jl}u_{ik}$ if $\varepsilon_{ij} = \varepsilon_{kl} = 1)$

$\mathrm{Aut}(\Gamma) \subseteq \mathrm{Aut}^*(\Gamma) \subseteq \mathrm{Aut}^+(\Gamma)$ [Bichon 2003]

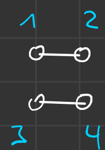
QUANTUM SYMMETRIES OF GRAPHS – $\boxed{\text{HAS QSYM}}$

$\Gamma = (\{1,\dots,n\}, E)$ finite graph with adj. matrix $\varepsilon \in M_n(\{0,1\})$

$C(\mathrm{Aut}^+(\Gamma)) := C^*\left(u_{ij} \mid u_{ij} = u_{ij}^* = u_{ij}^2,\ \sum_k u_{ik} = \sum_k u_{kj} = 1,\ u\varepsilon = \varepsilon u\right)$

a) $\Gamma =$  n points : $\mathrm{Aut}^+(\Gamma) = S_n^+ \neq S_n$, $n \geq 4$

$\rightsquigarrow$ has qsym

b) $\Gamma =$ 

$\varepsilon = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

$C(\mathrm{Aut}^+(\Gamma)) \longrightarrow C^*(p, q \text{ proj.})$

$u \longmapsto \begin{pmatrix} p & 1-p & 0 & 0 \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & q \end{pmatrix}$

$\Rightarrow C(\mathrm{Aut}^+(\Gamma))$ noncomm. $\rightsquigarrow$ has qsym

QUANTUM SYMMETRIES OF GRAPHS – HAS QSYM

$\Gamma = (\{1,\dots,n\}, E)$ finite graph with adj. matrix $\varepsilon \in M_n(\{0,1\})$

$$C(\mathrm{Aut}^+(\Gamma)) := C^*\Big(u_{ij} \;\Big|\; u_{ij} = u_{ij}^* = u_{ij}^2,\ \sum_k u_{ik} = \sum_k u_{kj} = 1,\ u\varepsilon = \varepsilon u\Big)$$

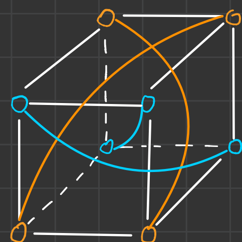
Thm. (Schmidt 2020): Γ has two disjoint automorphisms $\Rightarrow$ has qsym

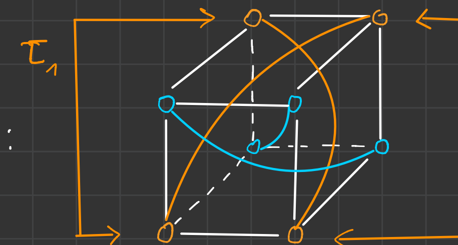
$$\tau_1, \tau_2 \in \mathrm{Aut}(\Gamma) \text{ disjoint} :\Leftrightarrow \forall i \in V: \begin{pmatrix} \tau_1(i) \neq i \Rightarrow \tau_2(i) = i \\ \tau_2(i) \neq i \Rightarrow \tau_1(i) = i \end{pmatrix}$$

Proof: Idea:

$$\begin{pmatrix} \tau_1 & 0 & 0 \\ 0 & \tau_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightsquigarrow u \longmapsto \begin{pmatrix} \begin{matrix} p & 1-p \\ 1-p & p \end{matrix} & 0 & 0 \\ 0 & \begin{matrix} q & 1-q \\ 1-q & q \end{matrix} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

QUANTUM SYMMETRIES OF GRAPHS – $\boxed{\text{HAS QSYM}}$

c) $FQ_n =$  folded cube, disj. autom:



$\rightsquigarrow$ has qsym, i.e. $Aut(FQ_n) \neq Aut^+(FQ_n) = ?$

<u>Thm. (Schmidt 2020):</u> $Aut^+(FQ_n) = SO_n^{-1}$, n odd

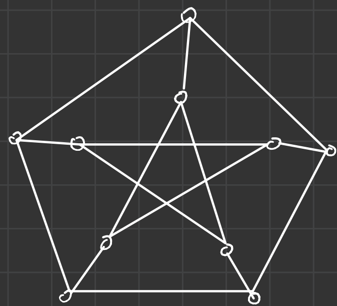
$$C(SO_n^{-1}) := C^*\left(u_{ij}=u_{ij}^* \;\middle|\; u^t u = u u^t = 1,\ u_{ij}u_{kl} = \begin{cases} u_{kl}u_{ij} & i\neq k \text{ and } j\neq l \\ -u_{kl}u_{ij} & i=k \underset{\uparrow\,\text{strict}}{\text{ or }} j=l\end{cases} \;\&\; \sum_{\sigma\in S_n} u_{\sigma(1)1}\cdots u_{\sigma(n)n} = 1\right)$$

<u>Proof:</u> Consider $\alpha: C^*(p_1,\dots p_N) \longrightarrow C(G)\otimes C^*(p_1,\dots p_N)$, $p_i \mapsto \sum_k u_{ki}\otimes p_k$

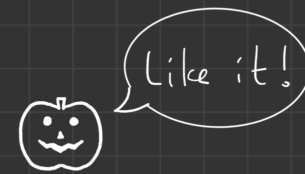
u_{ij} satisfy SO_n^{-1} relations $\Longleftrightarrow$ α exists, $u\varepsilon = \varepsilon u$ $\quad N = |\text{vertices of } FQ_n|$

Hence $G = SO_n^{-1}$ maximal with this action

$\Gamma =$



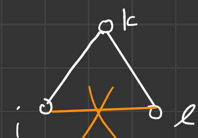
Petersen graph



Banica–Bichon 2007: has qsym?

Thm. (Schmidt 2018): The Petersen graph has no qsym.

Proof: graph properties:



(1) given $\wedge$ $\Rightarrow$ $\not-$

(2) given $\not-$ $\Rightarrow$ $\exists!$ $\wedge$

$u_{ii'} u_{ll'} = u_{ll'} u_{ii'}$ in the case $i \circ\!\!\not-\!\!\circ\, l$ $\quad i' \circ\!\!\not-\!\!\circ\, l'$ since:

$$u_{ii'} u_{ll'} = u_{ii'} \Big(\sum_s u_{ks}\Big) u_{ll'} = u_{ii'} u_{kk'} u_{ll'} \underset{[\dots]}{=} u_{ii'} u_{kk'} u_{ll'} u_{ii'} = u_{ii'} u_{ll'} u_{ii'}$$

selfadjoint!

SUMMARY ON THE EXISTENCE OF QUANTUM SYMMETRIES:

$\boxed{\Gamma \text{ finite graph}}$

QSYM?

show that u_{ij} commute
(Schmidt: tools)

no

e.g. has disj. autom.

yes

find a quantum group G
acting on Γ
& any such action implies
relations of G

what is $Aut^+(\Gamma)$?

Criteria for existence of qsym?

- disjoint autom. 1810.11284
- Sinkhorn type algorithm 1911.04912
- computer algebra tools 1906.12097
- 👻 2503.14159

SUMMARY ON THE EXISTENCE OF QUANTUM SYMMETRIES:

no qsym: [Schmidt] Petersen, odd graphs O_k, Hamming $H(n,3)$, Johnson $J(n,2)$, Kneser $K(n,2)$, Moore (diameter 2), cubic distance-transitive (order ≥ 10), P_9, P_{13}, P_{17}, Shrikhande, ...

OPEN: $J(6,3)$, $J(n,k)$ $(k \geq 3)$, Paley P_k $(k > 17)$, Tutte 12-cage, ...

have qsym: [Schmidt, Banica-Bichon, ...] complete graphs, complete bipartite, cycles, cube, folded cube FQ_n (n odd), Clebsch, 4x4 rook's, crown graphs, Hamming $H(n,k)$ $(k > 3)$, Higman-Sims, ...

OPEN: $Aut^+(FQ_n) = ?$ (n even), $Aut^+(4\times 4 \text{ rook's}) = ?$, Higman-Sims, ...

OPEN: What is A_n^+ "quantum alternating group"? $\exists \Gamma: Aut(\Gamma) = A_n$, has qsym?

FURTHER DIRECTIONS

PROBABILISTIC STATEMENTS

	have symmetries	have quantum symmetries
graphs	$\mathbb{P} \to 0$ as $N \to \infty$ Erdös – Renyi 1963	$\mathbb{P} \to 0$ as $N \to \infty$ Lupini – Mancinska – Roberson 2017
trees	$\mathbb{P} \to 1$ as $N \to \infty$ Erdös – Renyi 1963	$\mathbb{P} \to 1$ as $N \to \infty$ Junk-Schmidt-W. 2019

clearly $\mathrm{Aut}^+(\Gamma) = \{e\} \implies \mathrm{Aut}(\Gamma) = \{e\}$

OPEN: $\Leftarrow$? $\exists \Gamma : \mathrm{Aut}(\Gamma) = \{e\}, \ \mathrm{Aut}^+(\Gamma) \neq \{e\}$?

FURTHER DIRECTIONS

QUANTUM ISOMORPHISMS OF GRAPHS

Def. (Atserias–Mančinska–Roberson–Šámal–Severini–Varvitsiotis 2019): $\Gamma_i = (V_i, E_i)$, $|V_i| = n$, $i = 1, 2$

$\Gamma_1 \cong_q \Gamma_2 \;:\Longleftrightarrow\; \exists\, \pi : C(S_n^+) \to A$, A some C^*-algebra: $\pi(u)\,\varepsilon_1 = \varepsilon_2\,\pi(u)$

e.g.: $v \in M_n(M_m(\mathbb{C}))$ with

$$v_{ij} = v_{ij}^* = v_{ij}^2, \quad \sum_k v_{ik} = \sum_k v_{kj} = 1, \quad v\varepsilon_1 = \varepsilon_2 v$$

if $m = 1$: v permutation matrix

$\Gamma_1 \cong \Gamma_2 \;\underset{\not\Longleftarrow}{\Longrightarrow}\; \Gamma_1 \cong_q \Gamma_2$ There are graphs which are quantum isomorphic but not isomorphic!

OPEN: smaller examples? $(|V| \geq 16)$

$\Gamma = (\{1, \dots, n\}, E)$ finite graph with adj. matrix $\varepsilon \in M_n(\{0,1\})$

$\mathrm{Aut}(\Gamma) := \{\sigma \in S_n \mid \sigma\varepsilon = \varepsilon\sigma\} \subseteq S_n$ automorphism group (symmetries of Γ)

Def. (Banica 2005): $\mathrm{Aut}^+(\Gamma)$ quantum automorphism group

$$C(\mathrm{Aut}^+(\Gamma)) := C^*\Big(u_{ij} \;\Big|\; u_{ij} = u_{ij}^* = u_{ij}^2,\ \sum_k u_{ik} = \sum_k u_{kj} = 1,\ u\varepsilon = \varepsilon u\Big)$$

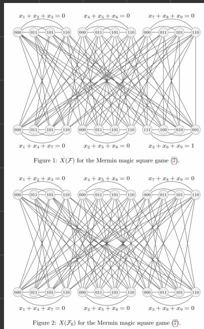


Figure 1: $X(\mathcal{F})$ for the Mermin magic square game [7].

Figure 2: $X(\mathcal{F}_0)$ for the Mermin magic square game [7].

1712.01820

FURTHER DIRECTIONS

QUANTUM ISOMORPHISMS OF GRAPHS

Def. (Atserias–Mančinska–Roberson–Šámal–Severini–Varvitsiotis 2019): $\Gamma_i = (V_i, E_i)$, $|V_i| = n$, $i = 1,2$

$\Gamma_1 \cong_q \Gamma_2 \;:\Longleftrightarrow\; \exists\, \pi : C(S_n^+) \to A$, A some C^*-algebra: $\pi(u)\,\varepsilon_1 = \varepsilon_2\,\pi(u)$

nonlocal game:
- given $\Gamma_1 = (V_1, E_1)$, $\Gamma_2 = (V_2, E_2)$, $|V_1| = |V_2|$
- referee gives $v_A, v_B \in V_1 \dot\cup V_2$ to Alice & Bob
- Alice & Bob reply with $w_A, w_B \in V_1 \dot\cup V_2$
- win, if $|\{v_A, v_B, w_A, w_B\} \cap V_i| = 2$ & linked in $\Gamma_1 \Longleftrightarrow$ linked in Γ_2

Thm. (AMRSSV 2019): Alice & Bob win with quantum strategy $\Longleftrightarrow \Gamma_1 \cong_q \Gamma_2$

note: Alice & Bob win classically $\Longleftrightarrow \Gamma_1 \cong \Gamma_2$

FURTHER DIRECTIONS

A QUANTUM LOVASZ THEOREM & INTERTWINER SPACES

Thm. (Lovasz 1967): $\Gamma_1 \cong \Gamma_2 \iff \forall H$ graph: $|\{\varphi: H \to \Gamma_1 \text{ hom.}\}| = |\{\varphi: H \to \Gamma_2 \text{ hom.}\}|$

(recall: $\varphi: H \to \Gamma$ graph hom. $\iff (i \sim j \Rightarrow \varphi(i) \sim \varphi(j))$)

Thm. (Mancinska-Roberson 2019): $\Gamma_1 \cong_q \Gamma_2 \iff \forall H$ planar graph: $|\{\varphi: H \to \Gamma_1 \text{ hom.}\}| = |\{\varphi: H \to \Gamma_2 \text{ hom.}\}|$

This answers an old question on Lovasz's theorem: are planar graphs enough? no!

Besides, Mancinska & Roberson describe the representation theory (aka intertwiner spaces):

$$\mathrm{Mor}_{\mathrm{Aut}^+(\Gamma)}(u^{\otimes k}, u^{\otimes \ell}) = \mathrm{span}\{T_\varphi \mid \varphi \text{ homomorphisms from planar graphs to } \Gamma\}$$

$\rightsquigarrow$ $\mathrm{Mor}_{S_n^+}$ is well-known: $\mathrm{span}\{T_p \mid p \text{ noncrossing/planar partition of sets}\}$

more systematically: "easy" quantum groups with Mor_G coming from partitions

FURTHER DIRECTIONS

LINKS WITH GRAPH C^*-ALGEBRAS, QUANTUM GRAPHS

Thm. (Schmidt-W. 2018): $\mathrm{QSym}(C^*(\Gamma)) = \mathrm{Aut}^+(\Gamma)$

hence $\Gamma \longmapsto C^*(\Gamma)$ respects the quantum symmetries of Γ

$$C^*(\Gamma) := C^*\Big(p_v \text{ proj.}, v \in V;\ s_e \text{ partial isom.}, e \in E \ \Big|\ s_e^* s_e = p_{r(e)},\ \sum_{s(e)=v} s_e s_e^* = p_v \ (\text{if } s^{-1}(v) \neq \emptyset)\Big)$$

where $r, s: E \longrightarrow V$ are range and source map

Proof: 1.) $C^*(\Gamma) \longrightarrow C(\mathrm{Aut}^+(\Gamma)) \otimes C^*(\Gamma)$

$$p_v \longmapsto \sum_{k \in V} u_{vk} \otimes p_k, \qquad s_e \longmapsto \sum_{f \in E} u_{s(e)s(f)} u_{r(e)r(f)} \otimes s_f$$

is a (left) action (and we also have one from the right

2.) $C^*(\Gamma) \to C(G) \otimes C^*(\Gamma)$ left & right action as above

$\implies u_{ij} \in C(G)$ satisfy relations of $C(\mathrm{Aut}^+(\Gamma))$

Thm. (Schmidt 2020): $\mathrm{Aut}^+(FQ_n) = SO_n^{-1}$, n odd

$$C(SO_n^{-1}) := C^*\left(u_{ij} = u_{ij}^* \ \middle|\ u^t u = u u^t = 1,\ u_{ij} u_{kl} = \begin{cases} u_{kl} u_{ij} & i \neq k \text{ and } j \neq l \\ -u_{kl} u_{ij} & i = k \text{ or } j = l \end{cases} \ \& \sum_{\sigma \in S_n} u_{\sigma(1)1} \cdots u_{\sigma(n)n} = 1\right)$$

strict

Proof: Consider $\alpha: C^*(p_1, \dots p_N) \longrightarrow C(G) \otimes C^*(p_1, \dots p_N)$, $p_i \mapsto \sum_k u_{ki} \otimes p_k$

u_{ij} satisfy SO_n^{-1} relations $\iff$ α exists, $u\varepsilon = \varepsilon u$ $N = |\text{vertices of } FQ_n|$

Hence $G = SO_n^{-1}$ maximal with this action

FURTHER DIRECTIONS

LINKS WITH GRAPH C*-ALGEBRAS, QUANTUM GRAPHS

Thm. (Schmidt-W. 2018): $QSym(C^*(\Gamma)) = Aut^+(\Gamma)$

hence $\Gamma \mapsto C^*(\Gamma)$ respects the quantum symmetries of Γ

$C^*(\Gamma) := C^*\left(p_v \text{ proj.}, v \in V;\ s_e \text{ partial isom.}, e \in E \mid s_e^* s_e = p_{r(e)}, \sum_{s(e)=v} s_e s_e^* = p_v\right)$ (if $s^{-1}(v) \neq \emptyset$)

where $r, s: E \to V$ are range and source map

Def. (Brannan-Eifler-Voigt-W. 2020): $\Gamma \mapsto C^*(\Gamma)$ extension to quantum graphs

graph $\Gamma = (V, E)$, $|V| = n$ $\rightsquigarrow$ $\mathbb{C}^n \xrightarrow{\varepsilon} \mathbb{C}^n$ adj. matrix

qu. graph Γ : $\bigoplus_{i=1}^{k} M_{n_i}(\mathbb{C}) \xrightarrow{\varepsilon} \bigoplus_{i=1}^{k} M_{n_i}(\mathbb{C})$

have $Aut^+(\Gamma)$ for qu. graphs, have slightly weaker statement $QSym(C^*\Gamma) \supseteq Aut^+(\Gamma)$

FURTHER DIRECTIONS

 OPEN: $Aut^{+}($  $) = ?$

THANKS