Dynamical systems and crossed products of C^*-algebras

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Context: „Quantum Mathematics“

Classical	Quantum
Topology	C^*-Algebras [Gelfand-Naimark 1940s]
Measure Theory	Von Neumann Algebras [Murray-von Neumann 1940s]
Probability Theory	Free Probability Theory [Voiculescu 1980s]
	& Quantum Probability [Accardi, Hudson-Parthasarathy 1970s]
Differential Geometry	Noncommutative Geometry [Connes 1980s]
(Compact) Groups	Compact Quantum Groups [Woronowicz 1980s]
Information Theory	Quantum Information Theory [Feynmann, Deutsch 1980s]
Complex Analysis	Free Analysis [J.L.Taylor 1970s]

1st Fundamental Theorem of C^*-Algebras (Gelfand-Naimark 1940s)

A unital C^*-algebra.

$$A \text{ commutative} \iff \exists X \text{ compact} : A \cong C(X) := \{f : X \to \mathbb{C} \text{ cont.}\}$$

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Philosophy behind Quantum Mathematics:

commutative algebras $\Longleftrightarrow$ classical situation

noncommutative algebras $\Longleftrightarrow$ quantum situation

Example (Algebra of functions)

Let X be a compact Hausdorff space.

$$C(X) := \{f : X \to \mathbb{C} \mid f \text{ continuous}\}$$

has nice structure:

- *algebra (ptwise oper.): multiplication* $(fg)(x) := f(x)g(x)$
  addition $(f+g)(x) := f(x) + g(x)$
  scalar multiplication $(\lambda f)(x) := \lambda f(x)$, $\lambda \in \mathbb{C}$
- *unital:* $1(x) := 1$ *for all* $x \in X$ *(constant map)*
- $f^*(x) := \overline{f(x)}$ *complex conjugation*
- *supremum norm* $\|f\|_\infty := \sup_{x\in X}|f(x)|$
- *complete with respect to this norm*

Example (Bounded linear operators)

Let H be a Hilbert space.

$$B(H) := \{T : H \to H \mid T \text{ linear and bounded (aka continuous)}\}$$

has nice structure:

- *algebra:* *multiplication* $(ST)(x) := S(Tx)$ *(composition)*
  addition $(S+T)(x) := Sx + Tx$
  scalar multiplication $(\lambda S)(x) := \lambda Sx$, $\lambda \in \mathbb{C}$
- *unital:* $1(x) := x$ *for all* $x \in X$ *(identity map)*
- T^* *adjoint:* $\langle Tx, y\rangle = \langle x, T^* y\rangle$ *for all* $x, y \in H$
- *operator norm* $\|T\|_\infty := \sup_{x \in H, \|x\|=1} \|Tx\|$
- *complete with respect to this norm*

Definition (Gelfand-Naimark, Segal 1940s)

A C^*-algebra A is

- *an algebra over* $\mathbb{C}$
- *which may or may not be unital (today: always unital)*
- *with an involution* $^* : A \to A$
  i.e. $(\lambda a + \mu b)^* = \bar{\lambda} a^* + \bar{\mu} b^*$ *for* $\lambda, \mu \in \mathbb{C}$, $(ab)^* = b^* a^*$, $(a^*)^* = a$
- *and a norm satisfying* $\|ab\| \leq \|a\| \|b\|$ *and* $\|a^* a\| = \|a\|^2$
- *complete with respect to this norm (i.e.: A Banach algebra)*

Example

- $C(X)$ *with* $\|\cdot\|_{\sup}$
- $B(H)$ *or* $M_N(\mathbb{C})$
- *every closed (unital)* **-subalgebra* $A \subseteq B(H)$

Example (rotation algebra/noncommutative torus)

H separable Hilbert space, ONB $(e_n)_{n\in\mathbb{Z}}$, $\vartheta \in \mathbb{R}$.

$$U e_n := e^{2\pi i n\vartheta} e_n = \lambda^n e_n \qquad \lambda := e^{2\pi i\vartheta} \in \mathbb{C}$$

$$V e_n := e_{n+1} \qquad \textit{bilateral shift}$$

$$A_\vartheta := C^*(U,V) \subseteq B(H) \qquad \textit{smallest closed } {}^*\textit{-subalgebra}$$

Check (easy):

- $UV = \lambda VU$
- *U and V are unitaries* $:\Leftrightarrow UU^* = U^*U = 1$
  $\Leftrightarrow$ *U surj. and isometric ($\|U\| = 1$), i.e. Hilbert space isomorphism*

Check (hard): Given any $u, v \in A$ with $uv = \lambda vu$, u, v unitaries, we have:

$$A_\vartheta \cong C^*(u,v) \subseteq A$$

1st Fundamental Theorem of C^*-Algebras (GN 1940s)

A unital C^*-algebra.

$$A \text{ commutative} \iff \exists X \text{ compact} : A \cong C(X) := \{f : X \to \mathbb{C} \text{ cont.}\}$$

Proof (rough sketch for $\Rightarrow$):

- $X := \mathrm{Spec}(A) := \{\varphi : A \to \mathbb{C} \mid \varphi \text{ algebra hom.}, \varphi \neq 0\}$ compact H.dorff
- Define $\chi : A \to C(X)$ by $\chi(x)(\varphi) := \varphi(x)$
- check (easy): χ algebra hom.
- check (less easy): any φ is *-preserving and so is χ
- check (hard): χ isometric (in particular injective)
- use Stone-Weierstraß: $\chi(A)$ is a closed unital *-subalgebra of $C(X)$ separating the points $\Longrightarrow$ $\chi(A) = C(X)$

□

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A unital C^*-algebra.

$$A \text{ commutative} \iff \exists X \text{ compact} : A \cong C(X) := \{f : X \to \mathbb{C} \text{ cont.}\}$$

Getting back to the rotation algebra

$$A_\vartheta \cong C^*(u, v \mid u, v \text{ unitaries}, uv = e^{2\pi i\vartheta} vu)$$

What happens for $\vartheta = 0$? Then:

$$uv = vu$$

Thus, by the 1st Fundamental Theorem:

$$\exists X : A_\vartheta \cong C(X)$$

What is X in this case? It is $\mathbb{T}^2$, the torus. Hence for $\vartheta \in \mathbb{R}\backslash\{0\}$:

$$\text{“}A_\vartheta = C(\mathbb{T}^2_\vartheta)\text{''} \qquad \text{noncommutative torus}$$

1st Fundamental Theorem of C^*-Algebras (GN 1940s)

A unital C^*-algebra.

$$A \text{ commutative} \iff \exists X \text{ compact} : A \cong C(X) := \{f : X \to \mathbb{C} \text{ cont.}\}$$

Continuous functional calculus for free:
A unital C^*-algebra, $x \in A$ with $x^*x = xx^*$ ("normal").
Then $C^*(x,1) \subseteq A$ commutative (smallest C^*-subalg. containing x and 1)
Thus, there is an X (in fact, the "spectrum of x") and an isomorphism:

$$\Phi : C(X) \to C^*(x,1) \subseteq A$$

Hence, we can "apply" continuous functions to x, for instance $\sqrt{x}$ or $\log x$ (if the spectrum is nice), simply by putting $f(x) := \Phi(f) \in A$ for $f \in C(X)$.

Ex.: $$\sqrt{\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

1st Fundamental Theorem of C^*-Algebras (GN 1940s)

A unital C^*-algebra.

$$A \text{ commutative} \iff \exists X \text{ compact} : A \cong C(X) := \{f : X \to \mathbb{C} \text{ cont.}\}$$

2nd Fundamental Theorem of C^*-Algebras (GN+Segal 1940s)

A unital C^*-algebra. There is a Hilbert space H and an injective *-homomorphism $\pi : A \to B(H)$. Hence, we have:

$$A \cong \pi(A) \subseteq B(H)$$

Definition

A C^*-dynamical system is a tripel (A, G, α) where A is a C^*-algebra, G is a compact group and $\alpha : G \to \mathrm{Aut}(A)$ is a cont. group hom.

Dynamical system for topological spaces:

- X compact space, $\varphi : X \to X$ homeomorphism (bij., cont., φ^{-1} cont.)
- Then: $C(X) \to C(X)$, $f \mapsto f \circ \varphi$ automorphism (bij. *-algebra hom.)
- Assume φ depends on $t \in \mathbb{Z}$ (or $t \in \mathbb{R}$), i.e. $\varphi_t : X \to X$; put $\varphi_0 := \mathrm{id}$
- Assume $\varphi_{s+t} = \varphi_s \circ \varphi_t$
- Then: $A = C(X)$, $G = \mathbb{Z}$, $\alpha : \mathbb{Z} \to \mathrm{Aut}(C(X))$, $\alpha_t(f) := f \circ \varphi_t$
  (Note: $\mathrm{Aut}(A)$ is a group with respect to the composition)

Example

$\vartheta \in \mathbb{R}$, $X = S^1 := \{z \in \mathbb{C} \mid |z| = 1\} \subseteq \mathbb{C}$ *sphere*, $\varphi_t : S^1 \to S^1$, $z \mapsto e^{2\pi i t \vartheta} z$

Definition

A C^*-dynamical system is a tripel (A, G, α) where A is a C^*-algebra, G is a (locally) compact group and $\alpha : G \to \mathrm{Aut}(A)$ is a cont. group hom.

Goal: Want to have a C^*-algebra $A \rtimes_\alpha G$ containing the whole information on the C^*-dynamical system.

Quick look at $\mathrm{Aut}(B(H))$: Let $U \in B(H)$ be unitary $(UU^* = U^*U = 1)$.

$$T \mapsto UTU^*$$

is an automorphism of $B(H)$ (with inverse $T \mapsto U^*TU$)
Can show: *All* automorphisms of $B(H)$ are of this form ("inner")

Strategy: Add unitaries u_t to A to make all $\alpha_t \in \mathrm{Aut}(A)$ inner

Definition

A C^*-dynamical system is a tripel (A, G, α) where A is a C^*-algebra, G is a (locally) compact group and $\alpha : G \to \mathrm{Aut}(A)$ is a cont. group hom.

Definition (informal)

Assume that G is discrete (for instance: $G = \mathbb{Z}$). The crossed product $A \rtimes_\alpha G$ is given by adding elements u_t, $t \in G$ to A such that:

- *the u_t are unitaries: $u_t u_t^* = u_t^* u_t = 1$*
- *the unitaries respect the group G: $u_s u_t = u_{s+t}$, $u_t^* = u_{t^{-1}}$*
- *the unitaries make the α_t inner: $\alpha_t(a) = u_t a u_t^*$*

Example

Consider $\vartheta \in \mathbb{R}$, $X = S^1 \subseteq \mathbb{C}$ *sphere,* $\varphi_t : S^1 \to S^1$, $z \mapsto e^{2\pi i t\vartheta} z$.
C^**-dyn. syst.:* $A = C(S^1)$, $G = \mathbb{Z}$, $\alpha : \mathbb{Z} \to \mathrm{Aut}(C(S^1))$, $\alpha_t(f) = f \circ \varphi_t$
Note: $\alpha_t = \alpha_1 \circ \ldots \alpha_1$ *for* $t > 0$ *and* $\alpha_t = \alpha_1^{-1} \circ \ldots \alpha_1^{-1}$ *for* $t < 0$

Add a unitary u_1 *to* $C(S^1)$ *with* $\alpha_1(a) = u_1 a u_1^*$ *and put* $u_t := u_1^t$
Then: u_t *are unitaries,* $u_s u_t = u_{s+t}$, $u_t^* = u_{t^{-1}}$, $\alpha_t(a) = u_t a u_t^*$

Check that $v := \mathrm{id} : S^1 \to S^1$ *is a unitary element in* $C(S^1)$ *with* $\alpha_1(v) = v \circ \varphi_1 = \mathrm{id} \circ \varphi_1 = e^{2\pi i \vartheta}\,\mathrm{id} = e^{2\pi i\vartheta} v$. *Hence*

$$e^{2\pi i\vartheta} v = \alpha_1(v) = u_1 v u_1^* \quad \Longleftrightarrow \quad u_1 v = e^{2\pi i \vartheta} v u_1$$

from which we infer:

$$A_\vartheta \cong C(S^1) \rtimes_\alpha \mathbb{Z}$$

Dynamical Systems and Crossed Products

Some facts about crossed products:

- Note that $C(X) \rtimes G$ may be noncommutative (see A_ϑ)
- Takai duality: $(A \rtimes G) \rtimes \hat{G} \cong A \otimes \mathcal{K}(H)$
  where G is an abelian group, $\hat{G} := \{\psi : G \to \mathbb{C} \text{ group hom.}\}$ is the dual group and $\mathcal{K}(H)$ compact operators on some Hilbert space
  (Compare with Pontryagin duality: $\hat{\hat{G}} \cong G$ for abelian groups)
- Gootman-Rosenberg-Sauvageot Theorem: In the classical situation of a compact group G acting on a compact space X, if things are nice (G amenable, second countable, G acts "freely" on X), we have:

$$C(X) \rtimes_\alpha G \text{ simple (i.e. has no ideals)} \iff G \text{ acts minimal}$$

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