ISEM2021

– Project 6 –
Positive Operators on $C^\star$-algebras

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The setting

- A a unital C*-algebra, $A_h := \{x \in A : x^* = x\}$ the selfadjoint part
- $A_+ := \{x^*x : x \in A\}$ the positive cone
- A^* the dual space, $A^*_+ := \{\varphi \in A^* : \varphi(x^*x) \geqslant 0\}$
- $S(A) := \{\varphi \in A^*_+ : \varphi(1) = 1\}$
- $\sigma_\pi := \{\lambda \in \sigma(T) : |\lambda| = r(T)\}$

Different types of positivity: Positive operators

Definition

We call a linear operator $T : A \to A$ positive if $T(x^*x) \geqslant 0$.

- Such an operator is automatically continuous with $\|T\| = \|T(1)\|$.
- Kadison inequality: For every positive operator it holds

$$T(x)^2 \leqslant \|T\| T(x^2) \quad \forall x \in A_h$$

- If T and T^{-1} are positive contractions we even get $T(x)^2 = T(x^2)$.

Different types of positivity: Schwarz operators

- Jordan multiplication: $x \circ y := \frac{1}{2}(xy + yx)$
- For T, T^{-1} positive contractions it holds that $T(x \circ y) = T(x) \circ T(y)$

Definition

We call a linear operator $T : A \to A$ Schwarz operator if $T(x)^* T(x) \leqslant \|T\| T(x^* x)$.

- Every Schwarz operator is positive
- If T and T^{-1} are Schwarz operators and contractions then $T(x)^* T(y) = T(x^* y)\ \forall x, y \in A$

$\to$ "nice" Schwarz operators are morphisms of the C^* structure

Different types of positivity: n-positivity

- If A is a unital C*-algebra then the Matrix algebra $M_n(A)$ is a unital C*-algebra as well.

Definition

$T^{(n)} : M_n(A) \to M_n(A)$, $T^{(n)}((a_{i,j}) = (T(a_{i,j}))$

- T is called n-positive if $T^{(n)}$ is positive.
- T is called completely positive if $T^{(n)}$ is positive for all $n \in \mathbb{N}$.

- If A is not commutative then:
  n-positive $\implies$ 2-positive $\implies$ Schwarz $\implies$ positive.
- If A is a commutative C*-algebra then these notions are all equivalent.
- Positive is equivalent to completely positive if and only if A is commutative (M. Choi)

Spectral properties of positive operators

Theorem

$T : A \to A$ positive

(i) $r(T) \in \sigma(T)$

(ii) *If r(T) is a pole of $R(\cdot, T)$ then its order is maximal among the poles in $\sigma_\pi(T)$*

(iii) *there exists a state $\varphi \in S(A)$ such that $T'\varphi = r(T)\varphi$*

Remark: (iii) is no longer true if A is without unit: Take $A = c_0$ the C*-algebra of all sequences convergent to zero, and T the left shift on A.

Proof of (i)

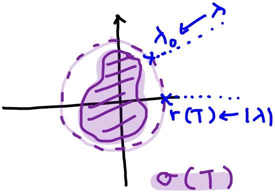
- $\|R(\lambda, T)\| \to \infty$ *iff* $\lambda \to \lambda_0 \in \sigma(T)$
- $|\varphi(R(\lambda, T)x)| \leqslant \varphi(R(|\lambda|, T)x)$ for all $x \in A_+$ and $\varphi \in A_+^*$

Choose $\lambda_0 \in \sigma_\pi(T)$

then $\|R(\lambda, T)\| \to \infty$ for $\lambda \to \lambda_0$, $|\lambda| \geq r(T)$;

$\Rightarrow$ $\exists\, \varphi \in S(A)$ $\exists\, x \in A_+$, $\|x\| = 1$

s.t. $|\varphi(R(\lambda, T)x)| \to \infty$



$\Rightarrow$ $|\varphi(R(\lambda, T)x)| \leq \varphi(R(|\lambda|, T)x) \to \infty$

$\Rightarrow$ $\|R(|\lambda|, T)\| \to \infty$ for $|\lambda| \to |\lambda_0| = r(T)$

$\Rightarrow$ $\boxed{r(T) \in \sigma(T)}$