$$\frac{1}{n}\sum_{k=0}^{n-1} T^k \xrightarrow[\|\cdot\|]{n\to\infty} ?$$

Uniform ergodicity on C^*-algebras

$$\frac{1}{n} \sum_{n=0}^{k} T^k \xrightarrow[\|\cdot\|]{n \to \infty} P$$

P projection onto $Fix(T) = \{x \mid Tx = x\}$

On Banach spaces ([2], [4]):

$\|T\| \leq 1$, $\dim \operatorname{Fix}(T) < \infty$

(a) T uniformly ergodic

$\Uparrow$

(b) T quasi-compact, i.e. $\exists$ compact K, $n \in \mathbb{N}$: $\|T^n - K\| < 1$

On commutative C^*-algebras ([C1]):

T positive, $\|T\| \leq 1$, $\dim \operatorname{Fix}(T) < \infty$

(a) T uniformly ergodic

$\Updownarrow$

(b) T quasi-compact

$\not\Downarrow$ in general

No positivity:

$$T : \ell^2(\mathbb{C}) \to \ell^2(\mathbb{C})$$

$$(a_n)_n \mapsto (\lambda_n a_n)_n$$

$$\lambda_1 = 1$$

$$\overline{\{\lambda_k \mid k \geq 2\}} = \overline{B_1(0) \setminus B_\varepsilon(1)}$$

→ T contraction on commutative C^*-algebra

→ $\dim \operatorname{Fix}(T) < \infty$

→ T not positive

Proposition

T contraction on a Banach space

(a) $\sigma_{\pi}(T)$: only poles of resolvent

& corresponding eigenspaces : finite dim.

$\Downarrow$

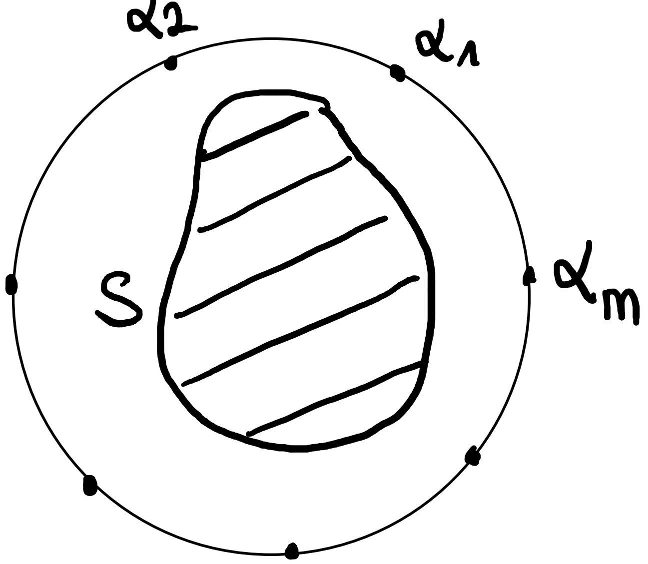
(b) T quasi-compact

$\Downarrow$ ([2],[4])

(c) T uniformly ergodic, $\dim \mathrm{Fix}(T) < \infty$

Proof (a) $\Rightarrow$ (b): Want $\|T^n - K\| < 1$, K compact

$\rightarrow$ poles



- $R := T \cdot E(S,T)$
- $r(R) < 1 \Rightarrow \|R^n\| < 1$

$\rightarrow$ finite dim.

- $E(\alpha_i, T)$ compact
- $T \cdot E(\alpha_i, T) =: P_i$ compact $\Rightarrow \sum_{i=1}^{m} P_i =: Q$ compact

$$T = Q + R$$

Q compact

$\|R^n\| < 1$

$$T^n = Q^n + R^n \Rightarrow \|T^n - Q^n\| = \|R^n\| < 1$$

$\square$

Proposition

T contraction on a Banach space

(a) $\sigma_{\mathbb{T}}(T)$: only poles of resolvent

$\Downarrow$ & corresponding eigenspaces : finite dim.

(b) T quasi-compact

$\Downarrow$ ([2],[4])

(c) T uniformly ergodic, $\dim \mathrm{Fix}(T) < \infty$

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