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ISEM 24, 2021

Part III of Talk 1 for **Project 6**

Thm (Groh, 1984*). Let $\mathcal{A}$ be C^*-Alg, $T \in \mathcal{L}(\mathcal{A})$ **Schwarz** Op with $T(1) = 1$.
Then t.f.a.e.:

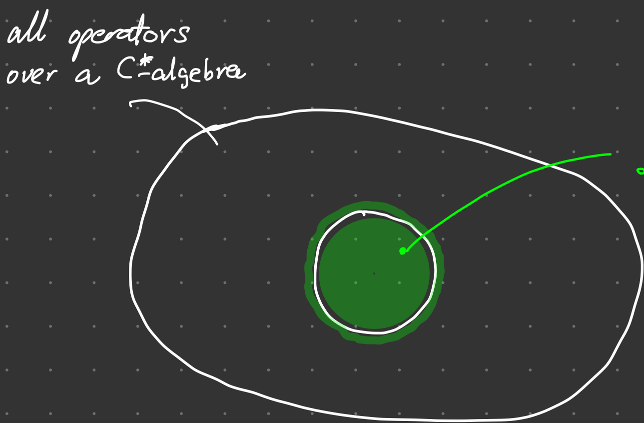
(a) T uniformly erg + $\underbrace{\dim \mathrm{Fix}(T)}_{d} < \infty$;

(b) T quasi-compact;

(c) each $\alpha \in \sigma_\pi(T)$ is pole of $\mathrm{Res}(\cdot, T)$ [of order 1] and $\dim \ker(\alpha - T)$ $[\leq d]$ $< \infty$

* [Groh, U (1984) *Uniformly Ergodic Maps on C*-Algebras*, Israel Journal of Mathematics]

POSITIVITY: *Commutative setting*



positive

$T(\mathcal{A}_+) \subseteq \mathcal{A}_+$

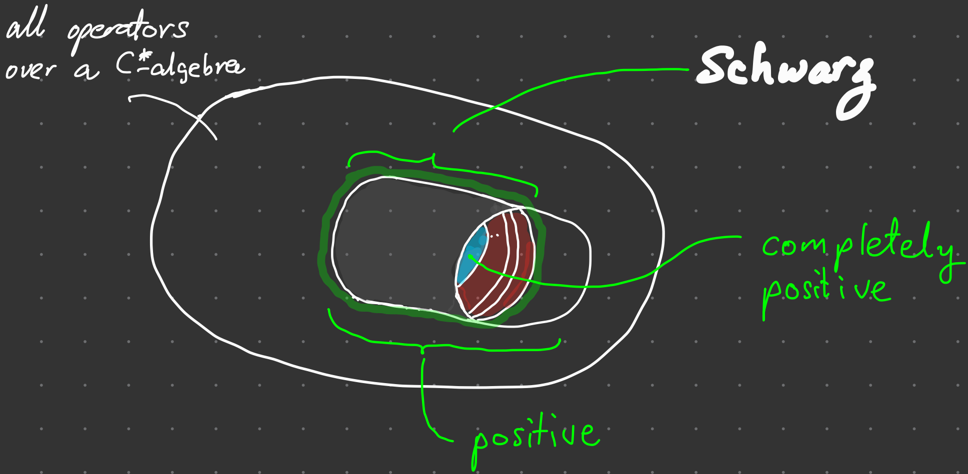
completely positive

$T^{(n)}$ pos, $n \geqslant 1$

Schwarz

$T(x)^* T(x) \leqslant \|T\| \, T(x^* x)$

POSITIVITY: Non-commutative setting



- completely positive operators occur 'naturally'.
- **Schwarz** operators: suitable generalisation.

Example

$\mathcal{H}$ ∞-dim, $u_i \in \mathcal{L}(\mathcal{H})$ isometric, $i \in \mathcal{I}$
with $u_i^* u_j = \delta_{ij} \cdot \mathbb{1}$ and $\sum_{i \in \mathcal{I}} u_i u_i^* = \mathbb{1}$.

$\mathcal{A} := \mathcal{L}(\mathcal{H})$ non-comm. unital C*-algebra

$T := \sum_{i \in \mathcal{I}} \mathrm{ad}_{u_i} \in \mathcal{L}(\mathcal{A})$

is **Schwarz** with $T(\mathbb{1}) = \mathbb{1}$.

Thm (Groh, 1984). Let $\mathcal{A}$ be unital C^*-Alg, $T \in \mathcal{L}(\mathcal{A})$ **Schwarz** Op with $T(1) = 1$.
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Proof

(b) ⇒ (a):

Holds for T contractive over Banach sp.

Cf. Cor VIII.8.4 **[**Dunford-Schwartz (1958) Linear Operators**]**.

(c) ⇒ (b):

Holds for T contractive over Banach sp.

Cf. previous slides.

APPROACH (a) $\Rightarrow$ (c)

$\mathcal{A}$ unital C*-alg;

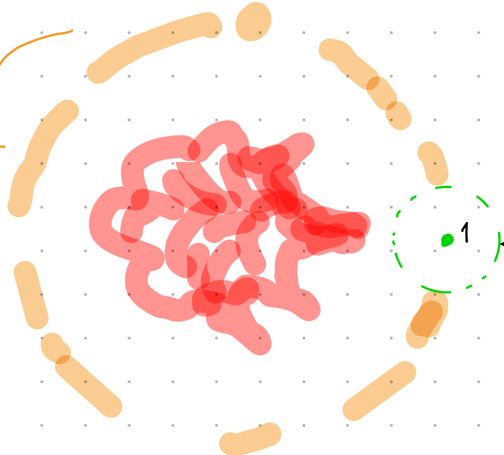
$T \in \mathcal{L}(\mathcal{A})$ **Schwarz**, $T1 = 1$, T uniform ergodic.

To show

each $\alpha \in$

are poles,

$\dim(\alpha - T) \leq$ **d**



dim Eigensp = **d** $< \infty$.

FIG 1. Spectrum of T.

STEP 1. apply Ultrafilters + move to W*-setting;

STEP 2. obtain domination result in W*-setting;

STEP 3. transfer back to **T**.

Proof of Thm: STEP 1.

Space	Properties	Operator	Properties.
$\mathcal{A}$	C^*-Alg with unit	T	$1 \mapsto 1$, **Schwarz**, u-erg. $d := \dim \mathrm{Fix}(T) < \infty$.
$\hat{\mathcal{A}}$ Ultraproduct	C^*-Alg with unit	$\hat{T}$	//
$\hat{\mathcal{A}}^{**}$ double dual	$\boxed{W^*\text{-Alg}}$ (univ. enveloping algebra)	$\hat{T}^{**}$	//

ULTRAPRODUCT

$$\sigma(\hat{T}) = \sigma(T) \quad \text{and} \quad \overbrace{\sigma_\pi(\hat{T}) \subseteq \sigma_{ap}(\hat{T})}^{\text{in general}} \underbrace{\subseteq \sigma_p(\hat{T})}_{}$$

Ultrafilter turns approx seq. into Eigenvectors.

Proof of Thm ctd...

Sep$^{\underline{n}}$ Lemma. Let
- $\mathcal{E}$ be Banach sp.;
- $S \in \mathcal{L}(\mathcal{E})$ contraction;
- $\alpha \in \sigma_\pi(S)$ Eigenvalue.

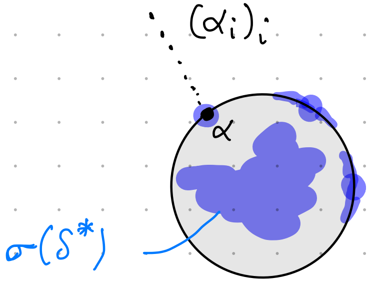
Then $\ker(\alpha - S^*)$ separates $\ker(\alpha - S)$.
In particular: $\dim \ker(\alpha - S) \leq \dim \ker(\alpha - S^*)$.

Pf. Let $V \subseteq \ker(\alpha - S)$ cl. subsp., $x_0 \in \ker(\alpha - S) \setminus V$.

To find: $\tilde{\psi} \in S^*$ s.t. $\tilde{\psi} \in \ker(\alpha - S^*)$, ✱

$$\tilde{\psi}|_V \equiv 0, \quad \tilde{\psi}(x_0) \neq 0. \quad ✱✱$$

By Hahn-Banach, $\exists\, \psi \in S^*$ s.t. $\|\psi\| = 1$, satisfying ✱✱.



Set $\psi_i := \underbrace{(\alpha_i - \alpha)\mathrm{Res}(\alpha_i, S^*)}_{\text{in unit ball of } \mathcal{E}^*}\psi$.

By w^*-compactness $(\psi_i)_i$ has a cluster pt. $\tilde{\psi}$.
Since $(\alpha_i - \alpha)x = (\alpha_i - S)x$ for $x \in \ker(\alpha - S)$,

$$\tilde{\psi}|_{\ker(\alpha - S)} \equiv \psi|_{\ker(\alpha - S)}.$$

$\Rightarrow$ easy to see $\tilde{\psi}$ satisfies ✱ + ✱✱.

$\square$

Proof of Th$^{\underline{m}}$ ctd...

Sep$^{\underline{n}}$ Lemma. Let
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Then $\ker(\alpha - S^*)$ separates $\ker(\alpha - S)$.
In particular: $\dim \ker(\alpha - S) \leq \dim \ker(\alpha - S^*)$.

Application 1 (for Th$^{\underline{m}}$)

- $\mathcal{E} := \hat{\mathcal{A}}^*$; $\quad S := \hat{T}^*$; $\quad \underbrace{\alpha := 1}_{\text{Eigenvalue (cf. 1st part of talk)}}$.
- Then $\mathcal{E}^* = \hat{\mathcal{A}}^{**}$, $\quad S^* = \hat{T}^{**}$.

$$\Rightarrow \dim \mathrm{Fix}(\hat{T}^*) \leq \dim \mathrm{Fix}(\hat{T}^{**}) = d$$

Proof of Th^m: STEP 2.

Lemma (Geom. Dominance). Let

- $\mathcal{M}$ be **W^*-algebra**;
- $S \in \mathcal{L}(\mathcal{M})$ be **Schwarz** with $S1 = 1$.

Then $\dim \operatorname{Fix} S_* < \infty$

$\Rightarrow \forall \alpha \in \mathbb{T}: \quad \dim \ker(\alpha - S_*) \leq \dim \operatorname{Fix}(S_*)$.

IDEA: Transform $\psi \in \ker(\alpha - S_*)$ into $|\psi| \in \operatorname{Fix}(S_*)$.

Note: Here we need **Schwarz** + W^*-alg!

PROPERTIES OF SCHWARZ OPs

Let $\mathcal{M}$ be a W^*-algebra.

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vN-Theory: - $\mathcal{M}_*$ = 'Predual' $\begin{cases} \text{el} \equiv \text{functionals}, \\ \text{pos. el} \equiv \text{states.} \end{cases}$

- For $\varphi \in \mathcal{M}_*$, $a \in \mathcal{M}$ denote $L_a\varphi = \varphi(a^* \cdot)$, $R_a\varphi = \varphi(\cdot\, a)$,
  $\varphi^*(a) = (\varphi(a^*))^*$
- $\psi \in \mathcal{M}_*^+ \Rightarrow \exists$ min. Proj., $s_\psi := p \in \mathcal{M}$ s.t. $\psi(p) = \psi(1)$
  MOREOVER: $L_p\psi = \psi = R_p\psi$.
- $\varphi \in \mathcal{M}_* \Rightarrow \exists$ unique **polar decomp**
  $u \in \mathcal{M}$, $|\varphi| \in \mathcal{M}_*^+$ with $u^*u = s_{|\varphi|}$ & $\varphi = R_u|\varphi|$.

Let $S \in \mathcal{L}(\mathcal{M})$ be **Schwarz** with $T1 = 1$.

I) map $B: \mathcal{M}\times\mathcal{M} \to \mathcal{M}$; $(x,y) \longmapsto S(y^*x) - S(y)^*S(x)$
is sesquilinear form ($\because$ **Schwarz** inequality).

II) For $\alpha \in \sigma_\pi(S_*)$ Eigenvalue of S_*.
Assume $\mathrm{Fix}(S_*)^+$ faithful for $\mathcal{M}$.

- $\mathrm{Fix}(S)$ a C^*-subalgebra (containing 1).
  (use I + faithfulness to prove this).

  $x \in \mathrm{Fix}(S)$
  $S(x^*x) \geq x^*x$

- $\varphi \in \ker(\alpha - S_*) \Rightarrow |\varphi|, |\varphi^*| \in \mathrm{Fix}(S_*)$.
- $''$ + $p\mathcal{M}$ S-invar $\Rightarrow S\big|_{p\mathcal{M}p} \equiv \alpha L_{u^*} S L_u$.
  where $p = s_{|\varphi|}$

$\leadsto$ TOOLS to partition $\mathrm{Fix}(S)$ & compute $\dim\ker(\alpha - S)$.
+ vN-Theory Via **Sep^n Lemma** get bound * for $\dim\ker(\alpha - S_*)$.

Proof of Th$^{\underline{m}}$: STEP 2.

Lemma (Geom. Dominance). Let
- $\mathcal{M}$ be **W*-algebra**;
- $S \in \mathcal{L}(\mathcal{M})$ be **Schwarz** with $S1 = 1$.

Then $\dim \operatorname{Fix} S_* < \infty$

$\Rightarrow \forall \alpha \in \mathbb{T}: \quad \dim \ker(\alpha - S_*) \leq \dim \operatorname{Fix}(S_*)$.

Application 2 (for Th$^{\underline{m}}$)

- $\mathcal{M} := \hat{\mathcal{A}}^{**}$; $\quad S := \hat{T}^{**}$.
- Then $S_* = \hat{T}^*$ over $\mathcal{M}_* = \hat{\mathcal{A}}^*$.
- By **Appl. 1**, $\dim \operatorname{Fix}(S_*) \leq d < \infty$.

  So $\forall \alpha \in \mathbb{T}: \quad \dim \ker(\alpha - \hat{T}^*) \leq d$.

$\Rightarrow$ Since <u>all</u> $\alpha \in \sigma_\pi(\hat{T})$ are Eigenvalues, **Sep$^{\underline{n}}$ Lemma** yields

$$\dim \ker(\alpha - \hat{T}) \leq \dim \ker(\alpha - \hat{T}^*) \leq d.$$

Proof of Thm 1: STEP 3.

Transfer Lemma. Let
- $\mathcal{E}$ Banach sp.;
- $S \in \mathcal{L}(\mathcal{E})$ contraction;
- $\alpha \in \sigma_\pi(\hat{S})$ with $\dim \ker(\alpha - \hat{S}) < \infty$.

Then (i) α Eigenvalue of S;
(ii) $\dim \ker(\alpha - S) = \dim \ker(\alpha - \hat{S})$.

In fact:
(iii) α pole of $\mathrm{Res}(\cdot, S)$ of order 1.

Pf. (i) Since $\sigma_\pi(\hat{S}) \subseteq \sigma_p(\hat{S}) = \sigma_{ap}(\hat{S}) = \sigma_{ap}(S)$,

$\exists (x_n)_n \subseteq \mathcal{E}$, normalised: $(S-\alpha)x_n \longrightarrow 0$.

Case 1: ex. conv. subseq. ✓ $\underbrace{}_{\Delta}$

Case 2: w.l.o.g. $\exists \delta > 0 : \forall n \neq n' \text{ in } \mathbb{N} : \|x_n - x_{n'}\| \geq \delta$

Shifts: $\hat{x}_i := [(x_{n+i})_{n\in\mathbb{N}}]_{\mathcal{F}} \in \hat{\mathcal{E}}$.

$\Rightarrow (\hat{S} - \alpha)\hat{x}_i = 0$

Since $\ker(\hat{S} - \alpha)$ finite dim, ex. conv. subseq.
Use this + Ultrafilter to contradict Δ.

(ii) By arguments in (i), each $\hat{x} \in \ker(\alpha - \hat{S})$ corr. to (exactly) one $x \in \ker(\alpha - S)$.

ctd...

Proof of Thm: STEP 3.

Transfer Lemma. Let
- $\mathcal{E}$ Banach sp.;
- $S \in \mathcal{L}(\mathcal{E})$ contraction;
- $\alpha \in \sigma_\pi(\hat{S})$ with $\dim \ker(\alpha - \hat{S}) < \infty$.

Then (i) α Eigenvalue of S;
(ii) $\dim \ker(\alpha - S) = \dim \ker(\alpha - \hat{S})$.

In fact:
(iii) α pole of $\mathrm{Res}(\cdot, S)$ of order 1.

Pf. (iii) Let $\{x_1, x_2, \dots, x_n\}$ basis for $\ker(\alpha - S)$. Eigenvalue, by (i)

(**Sepⁿ Lemma**) $\exists\, \varphi_1, \varphi_2, \dots, \varphi_n \in \ker(\alpha - S^*)$:

each φ_i sep x_i from $\{x_j\}_{j \neq i}$.

Set $\mathcal{E}_1 := \ker(\alpha - S)$
$\mathcal{E}_2 := \bigcap_{i=1}^{n} \ker \varphi_i$
S-invariant closed subspaces, $\mathcal{E}_1 \oplus \mathcal{E}_2 = \mathcal{E}$

✓ $\mathrm{Res}(\cdot, S|_{\mathcal{E}_1})$ pole order 1 in α.
✓ $\alpha \in \rho(S|_{\mathcal{E}_2})$

$\Rightarrow$ α pole order 1 for $\mathrm{Res}(\cdot, \underbrace{S|_{\mathcal{E}_1} \oplus S|_{\mathcal{E}_2}}_{S})$.

$\square$

Proof of Thm: STEP 3.

Transfer Lemma. Let
- $\mathcal{E}$ Banach sp.;
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- $\alpha \in \sigma_\pi(\hat{S})$ with $\dim \ker(\alpha - \hat{S}) < \infty$.

Then (i) α Eigenvalue of S;
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In fact:
(iii) α pole of $\mathrm{Res}(\cdot, S)$ of order 1.

Application 3 (for Thm)

- $\mathcal{E} := \mathcal{A}$; $S := T$; $\alpha \in \sigma_\pi(T)$ arbitrary.
- Then $\alpha \in \sigma_{ap}(T)$, $|\alpha| = r(T) = 1$.

Ultraproduct: $\underbrace{\alpha \in \sigma_p(\hat{T}) \text{ and } |\alpha| = r(T) = r(\hat{T})}_{\Rightarrow \alpha \in \sigma_\pi(\hat{T})}$

$\Rightarrow$
- α pole of $\mathrm{Res}(\cdot, T)$ of order 1.
- $\dim \ker(\alpha - T) = \dim \ker(\alpha - \hat{T}) \underbrace{\leq d}_{\text{by Appl. 2}}$

RECAP OF APPROACH (a) ⇒ (c)

$\mathcal{A}$ unital C^*-alg;
$T \in \mathcal{L}(\mathcal{A})$ **Schwarz**, $T1 = 1$, T uniform ergodic.

To show

each $\alpha \in$
are poles,
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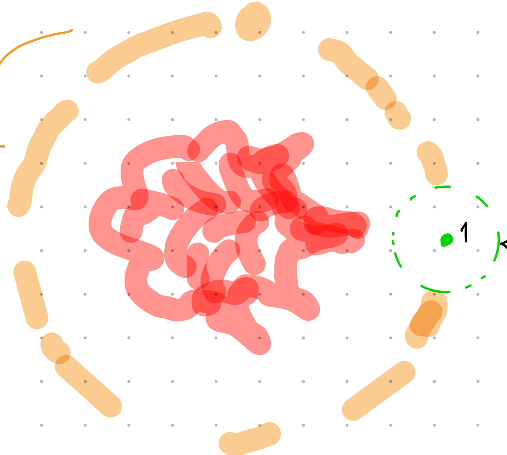

dim Eigensp = **d** $< \infty$.

FIG 1. Spectrum of T.

STEP 1. apply Ultrafilters + move to W^*-setting
- Ultaproduct turns per. spec. values into Eigenvalues.

STEP 2. obtain properties in W^*-setting:
- relied on **Schwarz** to get geom. dominance.

STEP 3. transfer back to T.
- obtain poles in the process.

Thm (**Groh, 1984**). Let $\mathcal{A}$ be unital C^*-Alg, $T \in \mathcal{L}(\mathcal{A})$ **Schwarz** Op with $T(1) = 1$. Then t.f.a.e.:

(a) T uniformly erg $+$ $\underbrace{\dim \operatorname{Fix}(T)}_{d} < \infty$;

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and $\dim \ker(\alpha - T)$ $[\leq d]$ $< \infty$

Summary

$\Leftarrow$ $\Leftarrow$ Conds. SUFFICIENT in general.

$\Rightarrow$ To show NECESSITY

- **Schwarz** was crucial to obtain geometric dominance.

- finite dimensionality cannot be left out:

E.g. $\mathcal{A} = \ell^\infty_{\mathbb{C}^2}(\mathbb{N})$, $\quad T(x_n)_n = \left(\begin{pmatrix} 0 & 1 \\ 1-n^{-1} & n^{-1} \end{pmatrix} x_n\right)_n$.

Then T unif ergodic, $\quad \dim \operatorname{Fix}(T) = \aleph_0$,

(c) fails: $-1 \in \sigma_\pi(T)$ not isolated.

Part $\overline{\underline{IV}}$ of Talk 1 follows...