W*-dynamical systems

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Definition

$\mathfrak{A}$ is a W^*-algebra if

- $\mathfrak{A}$ is a C^*-algebra,
- there exists a Banach space $\mathfrak{A}_*$ such that $(\mathfrak{A}_*)^* \simeq \mathfrak{A}$.

Definition

A linear functional $\varphi\colon \mathfrak{A} \to \mathbb{C}$ is normal if it is w^*-continuous.
$\mathfrak{A}_*$ is the space of normal functionals.

Example

- $\mathfrak{A} = \mathrm{L}^\infty(X, \mu)$, $\varphi(f) = \int_X f \,\mathrm{d}\mu$, where $\mu \geqslant 0$ finite measure.
- $\mathfrak{A} = \mathrm{B}(\mathcal{H})$, $\varphi(a) = \mathrm{Tr}(Ta)$, where $T \geqslant 0$ trace class.

Definition

$(\mathfrak{A}, T, \varphi)$ is a W^*-dynamical system if

- $\mathfrak{A}$ is a W^*-algebra,
- φ is a faithful normal state on $\mathfrak{A}$,
- $T\colon \mathfrak{A} \to \mathfrak{A}$ is completely positive s.t. $T(\mathbb{1}) = \mathbb{1}$ and $T^*(\varphi) = \varphi$.

Remark

One can show that T is normal, hence the preadjoint $T_*\colon \mathfrak{A}_* \to \mathfrak{A}_*$ exists.

Definition

A W^*-dynamical system $(\mathfrak{A}, T, \varphi)$ is irreducible if the fixed point space of T equals $\mathbb{C}\mathbb{1}$.

Theorem

Let $(\mathfrak{A}, T, \varphi)$ be an irreducible W^*-dynamical system.

1. The sets of peripheral eigenvalues of T and T_* are equal. It is a subgroup of $\mathbb{T}$.
2. There exists a faithful normal conditional expectation Q

$$Q \colon \mathfrak{A} \to \mathfrak{M} = \overline{\operatorname{span}}^{w^*}\{x \in \mathfrak{A} \,|\, T(x) = \alpha x,\ |\alpha| = 1\}.$$

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3. $\mathfrak{M} \subseteq \mathfrak{A}$ is a W^*-subalgebra and φ is a trace on $\mathfrak{M}$.

- There exists a faithful normal conditional expectation Q

$$Q\colon \mathfrak{A} \to \mathfrak{M} = \overline{\mathrm{span}}^{w^*}\{x \in \mathfrak{A} \,|\, T(x) = \alpha x,\ |\alpha| = 1\}.$$

- Existence of Q uses Jacobs-DeLeeuw-Glicksberg theory and theory of compact semigroups (see "Decomposition of operator semigroups on W* algebras" – Bátkai, Groh, Kunszenti-Kovács, Schreiber).

Theorem

Let $(\mathrm{B}(\mathcal{H}), T, \varphi)$ be an irreducible W^*-dynamical system. Then the set of peripheral eigenvalues of T equals Γ_h, the group of all h-th roots of unity for some $h \geqslant 1$.

Sketch of proof:

- We have a normal faithful conditional expectation $Q\colon \mathrm{B}(\mathcal{H}) \to \mathfrak{M}$.
- (Tomiyama) This means that $\mathfrak{M}$ is purely atomic, i.e. there is a set $\{p_i\}$ of minimal projections with $\sum_i p_i = \mathbb{1}$.
- Consequently $\mathfrak{M}$ is of type I and finite (φ is a faithful trace on $\mathfrak{M}$).
- It follows that $\mathcal{Z}(\mathfrak{M})$ is also purely atomic, i.e. $\mathcal{Z}(\mathfrak{M}) \simeq \ell^\infty(S)$ for some set $S \neq \emptyset$.

- Assume by contradiction that $\mathcal{Z}(\mathfrak{M})$ is infinite dimensional, $\mathcal{Z}(\mathfrak{M}) \simeq \ell^\infty(\mathbb{N})$.
- One can show that $T|_{\mathfrak{M}}$ is a $\star$-automorphism, hence $T|_{\mathcal{Z}(\mathfrak{M})}$ is also a $\star$-automorphism induced by some bijection $\tau \colon \mathbb{N} \to \mathbb{N}$.
- $\varphi|_{\mathcal{Z}(\mathfrak{A})} \in \ell^\infty(\mathbb{N})^+_* = \ell^1(\mathbb{N})^+$ is given by $\varphi(f) = \sum_{n=1}^\infty \varphi_n f(n)$ for some $\varphi_n > 0$. $T_*(\varphi) = \varphi$ gives

$$\varphi_n = \varphi_{\tau(n)} \quad (n \in \mathbb{N}).$$

- If the orbit $\{1, \tau(1), \tau^2(1), \dots\}$ is finite then $\mathbb{1} \neq \sum_{k=0}^{k_0} \delta_{\tau^k(1)}$ is invariant under T, contradiction. Otherwise $\varphi(\mathbb{1}) = +\infty$, also contradiction.

- It follows that $\dim \mathcal{Z}(\mathfrak{M}) < +\infty$.
- Recall that $\mathfrak{M}$ is of type I and finite, hence

$$\mathfrak{M} \simeq \mathrm{M}_{n_1} \oplus \cdots \oplus \mathrm{M}_{n_k}$$

  for some $n_1, \ldots, n_k \in \mathbb{N}$.
- Consequently $\dim \mathfrak{M} < +\infty$ and since

$$\mathfrak{M} = \overline{\mathrm{span}}^{w^*}\{x \in \mathfrak{A} \,|\, T(x) = \alpha x,\ |\alpha| = 1\}$$

  the peripheral point spectrum of T is finite. It is a subgroup of $\mathbb{T}$, consequently is equal to Γ_h for some root of unity h.

Bibliography

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Thank you!