Transition

- **So far:** Discrete Semigroup Representation:

$$\mathcal{T} : \mathbb{N} \to \mathcal{L}(C), \quad n \mapsto T^n$$

for T positive\Schwarz-Operator on C^*-Algebra C

- **Now:** Strongly Continuous Semigroup Representation

$$T : \mathbb{R}_{\geq 0} \to \mathcal{L}(C), \quad t \mapsto T(t)$$

with $T(t)$ positive on commutative, unital C^*-Algebra C

- Short Notation: $((T(t))_{t \geq 0}$ is C_0-Semigroup on C

- By Gelfand-Naimark: $\exists K$ compact Hausdorff-space s.t.: $C \cong C(K)$

Characterisation of positive C_0-(Semi)groups on $C(K)$

ISem 24, Project 6

10.06.2021

Remark

- Characterisation in terms of:
  - Expression for each $T(t)$
  - Property of generator A

- **Reminder:** Generator of C_0-Semigroup $(T(t))_{t\geq 0}$

$$D(A) := \{x \in C \mid \lim_{t \searrow 0} \frac{T(t)x - x}{t} \quad \text{exists}\}$$

$$Ax := \lim_{t \searrow 0} \frac{T(t)x - x}{t}$$

Definition 1.1

K compact Hausdorff space. $\varphi : [0, \infty) \to C(K, K), t \mapsto \varphi_t$ is called **continuous semiflow on** K if:

1. $\varphi_0 = \mathrm{id}_K$
2. $\varphi_{s+t} = \varphi_s \circ \varphi_t \;\; \forall s, t \geq 0$
3. $[0, \infty) \times K \ni (t, x) \mapsto \varphi_t(x)$ is continuous

Derivations

Definition 1.2

C C^*-Algebra, $(A, D(A))$ linear operator on C. A is called **derivation** if:

1. $D(A)$ is *-Algebra
2. A is involutive
3. $A(fg) = (Af)g + f(Ag) \ \ \forall f, g \in D(A)$

Characterisation of C_0-Semigroup of *-Homomorphisms

Theorem 1.3

K compact Hausdorff space, $(T(t))_{t\geq 0}$ C_0-Semigroup on $C(K)$ with generator A. Equivalent:

1. *$T(t)$ *-Homomorphism* $\forall t \geq 0$
2. *A is a derivation*
3. *$(T(t))_{t\geq 0}$ is a* Koopman-semigroup[a], *i.e.* $\exists \varphi$ *continuous semiflow on K s.t.:* $T(t)f = f \circ \varphi_t \; \forall t \geq 0$

[a] Bernard Osgood Koopman (1900–1981), French-American Mathematician

- We have seen:

$(C(K), \mathbb{R}, T)$ C^*-dynamical system	$\Longleftrightarrow$	$(K, \mathbb{R}, \varphi)$ topological dynamical system: $T(t)f = f \circ \varphi_t$
$\Updownarrow$		$\Updownarrow$
$(T(t))_{t\in\mathbb{R}}$ C_0-group of *-Homomorphisms on $C(K)$		$(\varphi_t)_{t\in\mathbb{R}}$ continuous flow on K, i.e. $(T(t))_{t\in\mathbb{R}}$ Koopman-group

Theorem 1.4

K compact Hausdorff space, $(T(t))_{t\geq 0}$ C_0-Semigroup on $C(K)$ with generator A. Equivalent:

1. *$T(t)$ *-Homomorphism* $\forall t \geq 0$
2. *A is a derivation*
3. *$(T(t))_{t\geq 0}$ is a* Koopman-semigroup[a], *i.e.* $\exists \varphi$ *continuous semiflow on K s.t.: $T(t)f = f \circ \varphi_t$* $\forall t \geq 0$

[a] Bernard Osgood Koopman (1900–1981), French-American Mathematician

(1) $\Leftrightarrow$ (2): Sketch of Proof

- (1) $\Rightarrow$ (2) :

$$A(fg) = \lim_{t \searrow 0} \frac{T(t)(fg) - fg}{t} = \lim_{t \searrow 0} \frac{(T(t)f)(T(t)g) - fg}{t}$$

  - Use Product Rule

Definition 1.5

C C^*-Algebra, $(A, D(A))$ linear operator on C. A is called **derivation** if:

1. $D(A)$ is *-Algebra
2. A is involutive
3. $A(fg) = (Af)g + f(Ag) \ \ \forall f, g \in D(A)$

$(1) \Leftrightarrow (2)$: Sketch of Proof

- $\underline{(2) \Rightarrow (1)}$:
  - Multiplicativity:

$$\eta : [0,t] \to C(K),\ \eta(s) := T(t-s)((T(s)f)(T(s)g)) \quad f,g \in D(A)$$

$$\left.\begin{aligned} \eta(0) &= T(t)(fg) \\ \eta(t) &= (T(t)f)(T(t)g) \end{aligned}\right\} \Rightarrow \text{Suffices to show} : \eta(0) = \eta(t)$$

- To that end:
  - verify $\eta'(s) = 0\ \forall s \in [0,t]$ + Hahn-Banach + Mean Value Theorem
- $T(t)$ involutive works similarly

Take-Home Message

- (1) and (2) in Theorem 1.3 are also equivalent in non-commutative C^*-Algebras
- New object to characterize dynamics on C^*-Algebras: Generators of C_0-(Semi)groups
- **Question:** When is a derivation on a C^*-Algebra a generator?[1]

[1]See: D. Evans. Quantum dynamical semigroups, symmetry groups, and locality.
Acta Applicandae Mathematicae, 2:333–352, 1984

Final Remarks

1. There more characterisation theorems, for instance
   - For arbitrary positive C_0-Group on $C(K)$:
     - $A = V\delta V^{-1} + h$ where $h, V \in C(K)$, $V > 0$ and δ derivation and generator of C_0-Group on $C(K)$
     - $(T(t)f)(x) = g_t(x)f(\varphi_t(x))$ where $(\varphi_t)_{t\in\mathbb{R}}$ continuous semiflow and $(g_t)_t \in \mathbb{R}$ continuous cocycle
   - For $K = [a, b]$: One can show: δ first order differential operator
   - All previous theorems also work on $C_0(X)$ for X locally compact

References

- More characterisations can be found in Part B.II and D.II in [1]

[1] W. Arendt, A. Grabosch, G. Greiner, U. Moustakas, R. Nagel, U. Schlotterbeck, U. Groh, H. Lotz, and F. Neubrander. One-parameter semigroups of positive operators. Springer-Verlag, 1986.

[2] D. Evans. Quantum dynamical semigroups, symmetry groups, and locality. *Acta Applicandae Mathematicae*, 2:333–352, 1984.