Positivity

&

Lyapunov's stability condition

ISem 24 Workshop

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What is the problem?

stability
on
C^*-algebras

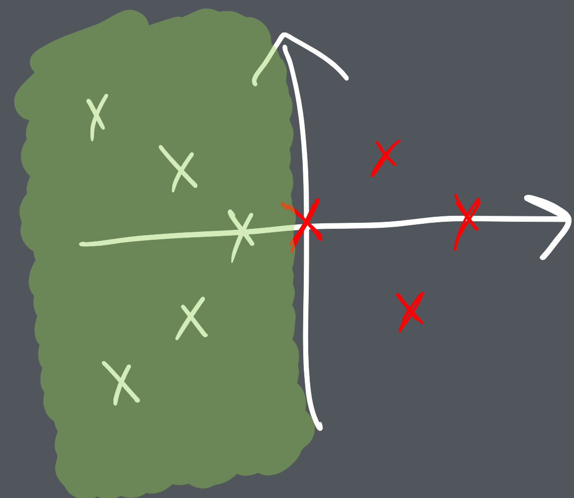
system
$$\begin{cases} \text{Hilbert } H = \mathbb{C}^n \\ A \in \mathcal{L}(H) \quad C^*\text{-algebra} \\ \dot{u}(t) = A\,u(t) \\ T(t) := e^{tA} := \sum_{k=0}^{\infty} \frac{(tA)^k}{k!} \end{cases}$$

system stable $:\Longleftrightarrow \lim_{t\to\infty} \|T(t)\| = 0$

$\Longleftrightarrow \forall \lambda \in \sigma(A): \ \operatorname{Re}\lambda < 0$

$\Longleftrightarrow s(A) := \sup\{\operatorname{Re}\lambda \mid \lambda \in \sigma(A)\} < 0$

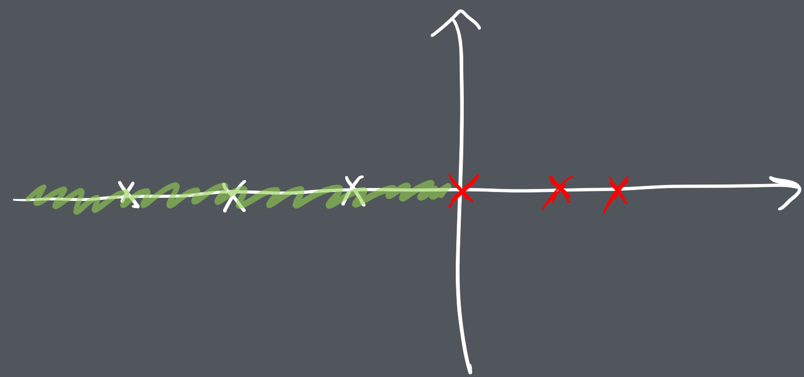
↳ spectral bound



Solution #1: Perron-Frobenius theory

$(\mathbb{R}^n, \leq)$ Banach lattice, in particular ordered Banach space (oBs)

Thm. (Perron-Frobenius): $\boxed{e^{tA} \geq 0 \quad \forall t \geq 0 \quad \Rightarrow \quad s(A) \in \sigma(A)}$



$\Rightarrow$ POSITIVITY HELPS

? "Make our problem positive"

Solution #2: Def. implemented semigroup:

$\mathcal{T}(t) : \mathcal{L}(H) \to \mathcal{L}(H)$, $\mathcal{T}(t)B := T(t)^* B\, T(t) = e^{tA^*} B\, e^{tA}$

- on unital C^*-a. $\mathcal{L}(H)$ with $\mathbb{1}$
- $\mathcal{L}(H)_{sa}$ oBs with $0 \le B = C^*C$
- $\boxed{\mathcal{T}(t) \geq 0}$

Let $B \in \mathcal{L}(H)$, $B = C^*C \geq 0$.
$\mathcal{T}(t)B = T(t)^* C^* C\, T(t) = (C\,T(t))^* C\,T(t) \geq 0$ $\Big\} \Rightarrow \mathcal{T}(t) \geq 0.$ $\square$

- $\mathcal{T}(t)$ C_0-semigroup with generator $\mathcal{A}B := A^*B + BA$
- $\boxed{s(\mathcal{A}) \in \sigma(\mathcal{A})}$
- $\|\mathcal{T}(t)\| = \|T(t)\|^2 \Rightarrow$ stability is preserved

Lyapunov stability for the implemented semigroup

TFAE:


[5]

Alexandr Mikhailovich Lyapunov (1857–1918)

(a) $s(A) < 0$ (Lyapunov criterion)

(b) $s(\mathcal{A}) < 0$

(c) $\forall \lambda \geq 0 \; \exists R(\lambda, \mathcal{A}) = \int_0^\infty e^{-\lambda s} \mathcal{T}(s)\, ds \geq 0$

(d) $\exists R(0, \mathcal{A}) = (-\mathcal{A})^{-1} \geq 0$, i.e. $\forall C \geq 0 \; \exists B \geq 0 : \mathcal{A} B = -C$

(e) $\exists B \geq 0 : \mathcal{A} B = -\mathbb{1} = A^* B + B A$ (Lyapunov equation)

- numerical solution: Hammarling / ADI algorithm [6]
- similarly
  - on von Neumann algebras ($\dim H = \infty$)
  - for non-autonomous Cauchy problems [1]

Sources:

[1] R. Nagel, A. Rhandi, Positivity and Liapunov's stability conditions for linear systems, Adv. Math. Sci. Appl. 3 (1993/94), Special Issue, 33-41

[2] U. Groh, F. Neubrander, Stabilität starkstetiger, positiver Operatorhalbgruppen auf C^*-Algebren, Math. Ann. 256 (1981), 509-516

[3] R. Nagel (ed.), One-parameter semigroups of Positive Operators, Lecture Notes in Mathematics, vol. 1184, Springer-Verlag, Berlin (1986), 403-406

[4] T. Eisner, Stability of Operators and Operator Semigroups, Birkhäuser, Basel (2010), 35-38, 75-77

[5] P.C. Parks, Lyapunov's stability theory – 100 years on, IMA Journal of Mathematical Control and Information, Volume 9, Issue 4 (1992), 275-303

[6] S.J. Hammarling, Numerical solution of the stable, non-negative definite Lyapunov equation, IMA J. Numer. Anal (1982), 303-323.

Thank you

for your

attention!