ISEM24
C^*-ALGEBRAS AND DYNAMICS
LECTURE NOTES

Abstract. In these lectures, we aim at providing an introduction to the general theory of C^*-algebras (first two thirds of the lectures) as well as to the more particular area of C^*-dynamical systems as a tool to deal with dynamics (last third of the lectures).

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Introduction and Motivation

Let us briefly motivate the lectures on C^*-algebras and dynamics and the main results we want to learn. This introduction is meant to serve as a teaser for the lectures omitting any technical details – the mathematical background for the following will be developed throughout the upcoming lectures. So, sit back and enjoy a short overview and motivation for the future lectures.

Recall that matrices $T \in M_N(\mathbb{C})$ may be seen as linear maps $T : \mathbb{C}^N \to \mathbb{C}^N$. In functional analysis, we deal with infinite dimensional versions of these and we consider linear maps

$$T : H \to H$$

between possibly infinite dimensional Hilbert spaces H. In contrast to linear algebra – i.e. the finite dimensional setting – these maps do not need to be continuous (which is equivalent to being bounded), so this comes as an extra assumption making life easier. So, let us consider

$$B(H) := \{T : H \to H \mid T \text{ is linear and bounded}\},$$

where H is some Hilbert space H. If $\dim(H) = N$, then $B(H) = M_N(\mathbb{C})$.

A main feature of bounded, linear operators on a Hilbert space is noncommutativity: We have $ST \neq TS$ in general, where $S, T \in B(H)$ and the multiplication is defined via composition of maps. We know such a feature already from the matrix multiplication in linear algebra. This noncommutativity appears in quantum physics, in linear algebra, in the representation theory of groups and in many further areas of mathematics and science.

The theory of operator algebras captures this noncommutativity turning it into a powerful tool in mathematics. The pioneers Francis Murray and John von Neumann wrote in their very first article [3] on von Neumann algebras in 1936 that

> "various aspects of the quantum mechanical formalism suggest strongly the elucidation of this subject."

In addition, they claim that their work may be viewed as part of

> "attempts to generalise the theory of unitary group-representations [sic!] essentially beyond their classical frame [...]."

Representing groups as unitary operators in $B(H)$ has also been in the scope of Israel Gelfand and Mark Naimark, when they wrote their seminal article [1] in 1943 introducing C^*-algebras. In 1993, Richard Kadison commented [2] on this article

> "from the vantage point of a fifty year history, it is safe to say that that paper changed the face of modern analysis. Together with the monumental 'Rings of operators' series [...] authored by F. J. Murray and J. von Neumann, it introduced 'non-commutative analysis', the vast area of mathematics that provides the mathematical model for quantum physics."

Nowadays, the following areas may be counted to such a "non-commutative analysis" or "quantum mathematics":

Classical theory	Quantum/noncomm. version	Founders and pioneers
Topology	C^*-Algebras	Gelfand-Naimark 1940s
Measure Theory	von Neumann Algebras	Murray-vonNeumann 1930s
Probability Theory	Free Probability Theory,	Voiculescu 1980s
	Quantum Probability Theory	Accardi,
		Hudson-Parthasarathy 1970s
Differential Geometry	Noncommutative Geometry	Connes 1980s
(Compact) Groups	(Compact) Quantum Groups	Woronowicz 1980s
Information Theory	Quantum Information Theory	Feynman, Deutsch 1980s
Complex Analysis	Free Analysis	J. L. Taylor 1970s

The main reason why C^*-algebras may be seen as a "quantum version" of topology comes from the famous Gelfand-Naimark Theorem, which we allow ourselves to call the 1st Fundamental Theorem of C^*-Algebras in these lectures.

1st Fundamental Theorem of C^*-Algebras (Gelfand-Naimark 1940s)**.** *Let A be a unital C^*-algebra. We have the following equivalence.*

$$A \text{ is commutative} \quad \Longleftrightarrow \quad \exists X \text{ compact}: A \cong C(X) := \{f : X \to \mathbb{C} \text{ is continuous}\}$$

Hence, any compact topological space gives rise to a commutative unital C^*-algebra – on the other hand *any* commutative C^*-algebra is exactly of this form. In this sense, commutative C^*-algebras "correspond" to topology and we may view the theory of noncommutative C^*-algebras as a kind of "noncommutative topology".

This Gelfand duality is also the basis for other quantum theories (namely von Neumann algebras, Free probability, noncommutative geometry and quantum groups).

Besides proving the above first fundamental theorem, our goal is to prove that any (abstractly defined) C^*-algebra may be represented concretely on a Hilbert space:

2nd Fundamental Theorem of C^*-Algebras (Gelfand-Naimark, Segal 1940s)**.** *Any C^*-algebra is isomorphic to a norm closed *-subalgebra of $B(H)$, for some H.*

From these fundamental theorems, we should keep in mind, that the algebra $C(X)$ of continuous functions on a compact space X as well as closed (in the operator norm topology) *-subalgebras of $B(H)$ are our main examples of C^*-algebras.

We will spend about two thirds of the lecture (October – December 2020) in order to develop the above basic knowledge on C^*-algebras including also a treatment of universal C^*-algebras. Afterwards (January – February 2021), we turn to dynamical systems. Let us sketch some basic ideas of the latter, referring to [4] for a nice survey on dynamical systems and operator algebras.

Our starting point is a group G and a compact space X. Assume that G acts on this space, i.e. we have a map $\alpha : G \times X \to X$. This is a topological dynamical system. See [4] for a motivation how to derive this setting from more physically motivated dynamical systems or from differential equations.

Now, let us define $\alpha_g : C(X) \to C(X)$ via $\alpha_g(f)(x) := f(\alpha(g,x))$. This induces a group homomorphism from G to the automorphism group of $C(X)$ by $g \mapsto \alpha_g$. We may then construct a C^*-algebra $C(X) \rtimes_\alpha G$ containing the information of X, of G and of the action of G on X (in terms of conjugation with unitaries) – hence, $C(X) \rtimes_\alpha G$ encodes the whole dynamical system!

Surprisingly, although $C(X)$ is commutative, the crossed product C^*-algebra $C(X) \rtimes_\alpha G$ may fail to be commutative. In fact, this is the generic situation: Unless the action is trivial, $C(X) \rtimes_\alpha G$ is always noncommutative (as conjugation with unitaries is trivial in commutative C^*-algebras). Hence, although our input X and G is classical data, we might want to enter the "nonclassical" or "quantum" world of noncommutative C^*-algebras in order to study this dynamical system. The philosophy is, that the theory of C^*-algebras provides a number of tools whith which we may investigate $C(X) \rtimes_\alpha G$ – in order to learn something about the classical dynamical system.

More generally, we will treat C^*-dynamical systems, i.e. actions α of compact groups G on possibly noncommutative C^*-algebras A, leading to crossed products $A \rtimes_\alpha G$.

We wish you a pleasant reading of the lecture notes and we hope you will enjoy the theory of C^*-algebras as much as we do!

References

[1] I. Gel'fand and M. Neumark. On the imbedding of normed rings into the ring of operators in Hilbert space. In *C^*-algebras: 1943–1993 (San Antonio, TX, 1993)*, volume 167 of *Contemp. Math.*, pages 2–19. Amer. Math. Soc., Providence, RI, 1994. Corrected reprint of the 1943 original.

[2] R. V. Kadison. Notes on the Gel'fand-Neumark theorem. In *C^*-algebras: 1943–1993 (San Antonio, TX, 1993)*, volume 167 of *Contemp. Math.*, pages 20–53. Amer. Math. Soc., Providence, RI, 1994.

[3] F. J. Murray and J. Von Neumann. On rings of operators. *Ann. of Math. (2)*, 37(1):116–229, 1936.

[4] I. Raeburn. Dynamical systems and operator algebras. In *National Symposium on Functional Analysis, Optimization and Applications*, pages 109–119, Canberra AUS, 1999. Centre for Mathematics and its Applications, Mathematical Sciences Institute, The Australian National University.