1. Reminder on bounded operators on Hilbert spaces

Abstract. We recall some basic notions from Hilbert space theory, such as Hilbert spaces, Cauchy-Schwarz inequality, orthogonality, decomposition of Hilbert spaces, Riesz Representation Theorem, orthonormal bases and isomorphisms of Hilbert spaces. We then turn to bounded linear operators on Hilbert spaces, their operator norms and the existence of adjoints. We define the notion of a C^*-algebra and verify that $B(H)$ is a unital C^*-algebra. We finish this lecture with a number of algebraic reformulations of properties of operators on Hilbert spaces (such as unitaries, isometries, orthogonal projections, etc.), and we give a brief survey on compact operators. As Lecture 1 is seen as a reminder to lay the foundations for the upcoming lectures, it does not contain many complete proofs, but we give at least some ideas. You may take [1, 2, 6] as general references for Lecture 1.

1.1. **Hilbert spaces.**

Informally speaking, Hilbert spaces are vector spaces equipped with a tool to measure angles between vectors.

Definition 1.1. Let H be a complex vector space. An *inner product* is a map $\langle\cdot,\cdot\rangle : H\times H\to\mathbb{C}$ satisfying for all $x,y,z\in H$ and all $\lambda,\mu\in\mathbb{C}$:

(1) $\langle\lambda x+\mu y,z\rangle=\lambda\langle x,z\rangle+\mu\langle y,z\rangle$
(2) $\langle z,\lambda x+\mu y\rangle=\bar\lambda\langle z,x\rangle+\bar\mu\langle z,y\rangle$
(3) $\overline{\langle x,y\rangle}=\langle y,x\rangle$
(4) $\langle x,x\rangle\geq 0$
(5) If $\langle x,x\rangle=0$, then $x=0$.

A space equipped with an inner product is called a *pre-Hilbert space.* An inner product induces a norm $\|x\|:=\sqrt{\langle x,x\rangle}$. A *(complex) Hilbert space* is a pre-Hilbert space, which is complete with respect to the induced norm.

Example 1.2. The following spaces are examples of Hilbert spaces.

(a) Given $n\in\mathbb{N}$, the vector space $\mathbb{C}^n$ endowed with $\langle x,y\rangle:=\sum_{i=1}^n x_i\bar y_i$, $x,y\in\mathbb{C}^n$ is a Hilbert space. The induced norm is the well-known Euclidean norm.
(b) The space $\ell^2(\mathbb{N})$ of complex-valued sequences $(a_n)_{n\in\mathbb{N}}$ with $\sum_{n\in\mathbb{N}}|a_n|^2<\infty$ endowed with $\langle(a_n)_{n\in\mathbb{N}},(b_n)_{n\in\mathbb{N}}\rangle:=\sum_{n\in\mathbb{N}}a_n\bar b_n$, $(a_n)_{n\in\mathbb{N}},(b_n)_{n\in\mathbb{N}}\in\ell^2(\mathbb{N})$ is a Hilbert space.
(c) More generally, recall that we may define $L^2(X,\mu)$ where (X,μ) is a measure space. The inner product is then given by:

$$\langle f,g\rangle:=\int_X f(x)\bar g(x)\ \mathrm{d}\mu(x),\qquad f,g\in L^2(X,\mu)$$

Note that for $X=[0,1]$ the unit interval and $\mu=\lambda$ the Lebesgue measure, this defines an inner product on the space $\mathcal{C}([0,1])$ of continuous complex-valued functions. However, $\mathcal{C}([0,1])$ is not complete with respect to the induced norm (which is the so called L^2-norm), i.e. it is only a pre-Hilbert space but no Hilbert space.

Choosing $X = I$ a set and $\mu = \zeta$ the counting measure, we obtain $\ell^2(I)$, with the above examples $\ell^2(\mathbb{N})$ and $\mathbb{C}^n$ as special cases.

(d) Any closed subspace of a Hilbert space is a Hilbert space (closed with respect to the norm topology, subspace in the sense of a linear subspace).

The most important inequality for inner products is the following one.

Proposition 1.3 (Cauchy-Schwarz inequality). *If H is a Hilbert space (or a pre-Hilbert space), we have for all $x, y \in H$:*

$$|\langle x, y\rangle| \leq \|x\|\|y\|$$

Here, equality holds if and only if x and y are linearly dependent.

Proof (idea): Use $0 \leq \langle x + \lambda y, x + \lambda y\rangle = \langle x, x\rangle - \frac{|\langle x,y\rangle|^2}{\langle y,y\rangle}$ with $\lambda = -\frac{\langle x,y\rangle}{\langle y,y\rangle}$. $\square$

Actually, one needs the Cauchy-Schwarz inequality for proving that the norm in Def. 1.1 is a norm indeed. There are two further important properties of the inner product and its induced norm.

Proposition 1.4. *Let H be a Hilbert space (or a pre-Hilbert space) and let $x, y \in H$.*

(a) *The* parallelogram identity *holds:* $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$.

(b) *The* polarisation identity *holds:* $\langle x, y\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \|x + i^k y\|^2$.

Proof (idea): (a): Direct computation. (b): Use $\|x + i^k y\|^2 = \langle x + i^k y, x + i^k y\rangle$. $\square$

The first of the above identities characterizes pre-Hilbert spaces: A normed space is a pre-Hilbert space if and only if the parallelogram identity holds. The second identity shows that the inner product is completely determined by its induced norm.

1.2. **Orthogonality and decomposition of Hilbert spaces.** As mentioned before, an inner product is the abstract information of an angle between vectors, see also Exc. 1.5. The notion of orthogonality plays the role of right angles.

Definition 1.5. Let H be a Hilbert space and $K, K_1, K_2 \subseteq H$ be subsets.

(a) Two vectors $x, y \in H$ are *orthogonal* ($x \perp y$), if $\langle x, y\rangle = 0$.

(b) We write $K_1 \perp K_2$, if $x \perp y$ for all $x \in K_1$ and $y \in K_2$.

(c) The *orthogonal complement* of K is $K^\perp := \{x \in H \mid x \perp y \text{ for all } y \in K\}$.

Even when K is just a subset without any further structure, its orthogonal complement will be of a nice form.

Lemma 1.6. *Given a subset $K \subseteq H$, its orthogonal complement $K^\perp \subseteq H$ is a closed subspace of H and we have $(\bar{K})^\perp = K^\perp$, where $\bar{K}$ is the closure of K.*

Proof (idea): Due to the continuity of the inner product (Exc. 1.4). $\square$

The following is a version of the antique Greek theorem by Pythagoras verifying that orthogonality corresponds to right angles indeed, see also Exc. 1.5.

Proposition 1.7 (Pythagoras' Theorem)**.** *If H is a Hilbert space and $x, y \in H$ are orthogonal, then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.*

Proof. Direct computation. □

One of the most important features of Hilbert spaces is that we may decompose them into direct sums.

Definition 1.8. Let $K_1, K_2 \subseteq H$ be two closed subspaces of a Hilbert space, such that $K_1 \perp K_2$. We then write $K_1 \oplus K_2 \subseteq H$ for the subspace given by elements $x + y \in H$, where $x \in K_1$ and $y \in K_2$.

Proposition 1.9. *Given a closed subspace $K \subseteq H$, we may decompose the Hilbert space H as a direct sum:*

$$H = K \oplus K^\perp$$

Then, every vector $x \in H$ has a unique decomposition $x = x_1 + x_2$ with $x_1 \in K$ and $x_2 \in K^\perp$.

Proof (idea): By Lemma 1.6, $K^\perp$ is closed. Trivially, $K \perp K^\perp$. We need to do some hard work to show that given $x \in H$, there is a unique "best approximation" $x_1 \in K$ such that $\|x - x_1\| = \inf\{\|x - y\| \mid y \in K\}$. With some further efforts, we then show $x_2 := x - x_1 \in K^\perp$. That this decomposition of x is unique easily follows from $K \cap K^\perp = \{0\}$. □

Corollary 1.10. *Given a subspace $K \subseteq H$, the double complement $(K^\perp)^\perp$ coincides with the closure $\bar{K}$ of K.*

Proof. By the previous proposition and using Lemma 1.6, we may decompose H in two ways, $H = \bar{K} \oplus K^\perp$ and $H = (K^\perp)^\perp \oplus K^\perp$, which shows $\bar{K} = (K^\perp)^\perp$. □

1.3. **Dual space and the Representation Theorem of Riesz.** Another nice feature of Hilbert spaces is that they have nice dual spaces – themselves! Given $y \in H$, we denote by $f_y : H \to \mathbb{C}$ the linear map given by $f_y(x) := \langle x, y\rangle$. In Exc. 1.4, it is shown that f_y is linear and continuous.

Proposition 1.11 (Riesz Representation Theorem)**.** *Let H be a Hilbert space and denote by H' its dual space, i.e. the space consisting in all linear, continuous maps $f : H \to \mathbb{C}$. The map $j : H \to H'$ given by $j(y) := f_y$ is an antilinear isometric isomorphism.*

Proof (idea): By Exc. 1.4, j is well defined and isometric (and hence injective); antilinearity follows from Def. 1.1(2). As for surjectivity, let $f \in H'$ be non-zero and decompose $H = K \oplus K^\perp$, where $K := \ker f$. You will find out that $K^\perp$ is one-dimensional and $j(f) = f_y$ for some $y \in K^\perp$. □

This has some nice consequences when working with Hilbert spaces. For instance, given a linear, continuous functional $f : L^2(X, \mu) \to \mathbb{C}$, then it must come from a function $g \in L^2(X, \mu)$, i.e. $f(h) = \int_X h\bar{g}\, \mathrm{d}\mu$ for all $h \in L^2(X, \mu)$.

1.4. **Orthonormal basis for a Hilbert space.** In finite dimensions, we usually understand vector spaces with respect to certain coordinates. We may transport this concept to the infinite-dimensional setting within the framework of Hilbert spaces.

Lemma 1.12. *Let H be a Hilbert space and let $(e_i)_{i\in I}$ be an* orthonormal system*, i.e. $\langle e_i, e_j\rangle = \delta_{ij}$. The following are equivalent:*

(1) $\|x\|^2 = \sum_{i\in I} |\langle x, e_i\rangle|^2$ *for all $x \in H$*
(2) $x = \sum_{i\in I} \langle x, e_i\rangle e_i$ *for all $x \in H$*
(3) $\operatorname{span}\{e_i \mid i \in I\} \subseteq H$ *is dense.*
(4) *If $z \in H$ is orthogonal to all e_i, $i \in I$, then $z = 0$.*
(5) *$(e_i)_{i\in I}$ is a maximal orthonormal family (with respect to inclusion).*

Proof (idea): The easy parts are the equivalences of (2) and (3) (just a reformulation), of (4) and (5) (just a reformulation) as well as of (2) and (4) (use $z := x - \sum_{i\in I}\langle x, e_i\rangle e_i$). The hard part is the equivalence of (1) and (2), where we use Pythagoras (Prop. 1.7) on finite subsets $F \subseteq I$ proving that $\|x - \sum_{i\in F}\langle x, e_i\rangle e_i\|^2$ tends to zero. The key words are Bessel's Inequality and Parseval's Identity. □

Definition 1.13. An orthonormal system is called an *orthonormal basis* of a Hilbert space, if one of the equivalent conditions in Lemma 1.12 is satisfied.

We should not be misled by the word "basis" here: The elements of an orthonormal basis are linearly independent, but they do *not* necessarily form a basis in the sense of linear algebra (Hamel basis) – we may *not* represent any vector in H by a *finite* linear combination of the e_i. However, passing to *infinite* linear combination, we may do so. This is the content of Lemma 1.12(2) – and we even know the coefficients thanks to our inner product. See also Schauder bases for the general Banach space setting.

Example 1.14. For $\mathbb{C}^n$, the vectors e_i having 1 at the i-th entry and zero otherwise form an orthonormal basis – in fact, in finite dimensions any orthonormal basis is also a (Hamel) basis.

More generally, for $\ell^2(I)$, the sequence having 1 at the i-th entry and zero otherwise form an orthonormal basis. If I is infinite, then this is not a basis.

Lemma 1.15. *Any Hilbert space possesses an orthonormal basis $(e_i)_{i\in I}$ and the cardinality of I is independent of the choice of the vectors.*

Proof (idea): Use Zorn's Lemma for the existence and Cantor-Schröder-Bernstein for the uniqueness of the cardinality. □

Definition 1.16. Given a Hilbert space H with orthonormal basis $(e_i)_{i\in I}$, its *(Hilbert space) dimension* is defined as the cardinality of I. If I is countable, we call H *separable.*

Thanks to the above lemma, the dimension is well-defined.

1.5. **Isomorphisms of Hilbert spaces.** Let us think about isomorphisms of Hilbert spaces – which structure are they supposed to preserve? Well, the vector space and the inner product!

Definition 1.17. Let H and K be Hilbert spaces. An *isomorphism* between H and K is a surjective linear map $U : H \to K$ which is *isometric*, i.e. it satisfies $\langle Ux, Uy\rangle_K = \langle x, y\rangle_H$ for all $x, y \in H$.

The preservation of the inner product implies that U is injective, which means that it is an isomorphism of the level of vector spaces, in particular. One can show that Hilbert spaces are isomorphic if and only if they have the same Hilbert space dimension in the sense of Def. 1.16. Hence, any Hilbert space is isomorphic to some $\ell^2(I)$. In particular, $\ell^2(\mathbb{N})$ is *the* separable Hilbert space.

1.6. **Bounded linear operators on Hilbert spaces.** In the subsequent lectures, we are not so much interested in the theory of Hilbert spaces as such but rather in the theory of bounded linear operators on Hilbert spaces. Let us first prove that "bounded" and "continuous" means the same for linear operators.

Lemma 1.18. *Let H, K be Hilbert spaces and let $T : H \to K$ be linear. The following are equivalent:*

(a) *T is continuous everywhere.*
(b) *T is continuous in zero.*
(c) *T is bounded, i.e. there is a $C > 0$ such that $\|Tx\| \leq C\|x\|$ for all $x \in H$.*

Proof (idea): The step from (a) to (b) is trivial. Assuming (b) with $\varepsilon = 1$, there is a $\delta > 0$ such that $\|x\| \leq \delta$ implies $\|Tx\| \leq 1$; put $C := \delta^{-1}$ to derive (c). Passing from (c) to (a) is straightforward. ☐

Definition 1.19. Given a Hilbert space H, we denote by $B(H)$ the space of all bounded, linear operators $T : H \to H$.

Example 1.20. If $\dim(H) = N$, i.e. if $H = \mathbb{C}^N$, then $B(H) = M_N(\mathbb{C})$, the algebra of $N \times N$ matrices with complex entries. Indeed, in this case, *any* linear map is automatically bounded.

Definition 1.21. Given $T \in B(H)$, we denote by

$$\|T\| := \inf\{C > 0 \mid \|Tx\| \leq C\|x\| \text{ for all } x \in H\}$$

the *operator norm* of T.

One can check that the operator norm is a norm indeed.

Lemma 1.22. *Given $T \in B(H)$, we have $\|Tx\| \leq \|T\|\|x\|$ for all $x \in H$.*

Proof. Choosing $C_n > \|T\|$ with $C_n \to \|T\|$ yields $\|Tx\| \leq C_n\|x\| \to \|T\|\|x\|$. ☐

Let us express the operator norm in an alternative way.

Lemma 1.23. *The norm* $\|T\|$ *may be written as*

$$\|T\| = \sup\{\|Tx\| \mid \|x\| = 1\}.$$

You may replace $\|x\| = 1$ *by* $\|x\| \leq 1$*, if you prefer.*

Proof. By Lemma 1.22, we have $\|Tx\| \leq \|T\|$, if $\|x\| \leq 1$. Thus the supremum s over all $\|Tx\|$ with $\|x\| = 1$ is less or equal to $\|T\|$. Conversely,

$$\|Tx\| = \|T\left(\frac{x}{\|x\|}\right)\|\|x\| \leq s\|x\|$$

whenever $x \neq 0$, so $\|T\| \leq s$ by Def. 1.21, which yields $\|T\| = s$ in total. The same proof works if s is the supremum over $\|Tx\|$ with $\|x\| \leq 1$. $\square$

1.7. **Existence of adjoints.** How does a bounded, linear operator T behave with respect to evaluations under the inner product? Here, the existence of adjoints is a useful fact.

Proposition 1.24. *Let* H *be a Hilbert space and* $T \in B(H)$*. There exists a unique operator* $T^* \in B(H)$ *(the* adjoint *of* T*) such that*

$$\langle Tx, y\rangle = \langle x, T^*y\rangle$$

for all $x, y \in H$*.*

Proof (idea): Let $y \in H$. We define $g^y : H \to \mathbb{C}$ by $g^y(x) := \langle Tx, y\rangle$. Then $g^y \in H'$ and by the Riesz Representation Theorem 1.11, there is a $z \in H$ such that $g^y = f_z$. Thus $\langle Tx, y\rangle = \langle x, z\rangle$ and we put $T^*y := z$. Check $T^* \in B(H)$. $\square$

Example 1.25. If $H = \mathbb{C}^N$ and $T \in B(H) = M_N(\mathbb{C})$, we may express T by $Te_i = \sum_j t_{ji}e_j$ for the canonical basis $e_1, \ldots, e_N$ of $\mathbb{C}^N$. Thus, $T \in M_N(\mathbb{C})$ has coefficients t_{ij} and $T^* \in M_N(\mathbb{C})$ has coefficients $\bar{t}_{ji}$.

Some operators coincide with their adjoints; they will play a special role.

Definition 1.26. An operator $T \in B(H)$ is called *selfadjoint* (or *Hermitian*), if $T = T^*$.

There is a useful formula relating the kernel of T with the image of its adjoint. We denote by $\ker T$ the space of all $x \in H$ such that $Tx = 0$, whereas $\operatorname{ran} T$ denotes the set of all Tx, where $x \in H$.

Lemma 1.27. *For* $T \in B(H)$*, we have* $\ker T = (\operatorname{ran} T^*)^\perp$ *and* $(\ker T)^\perp = \overline{\operatorname{ran} T^*}$*.*

Proof. A vector x is in $(\operatorname{ran} T^*)^\perp$ if and only if $\langle Tx, y\rangle = \langle x, T^*y\rangle = 0$ for all y, i.e. if and only if x is in the kernel of T. Use Lemma 1.6 for the second part. $\square$

Implicitely, we used the following lemma in the proof above.

Lemma 1.28. *Let* $T \in B(H)$*. If* $\langle Tx, y\rangle = 0$ *for all* $y \in H$*, then* $Tx = 0$*. In particular,* $\langle Tx, y\rangle = \langle Sx, y\rangle$ *for all* $x, y \in H$ *implies* $S = T$*.*

Proof. Put $y = Tx$ for the first part and use the first part for the second. $\square$

1.8. **Algebraic structure of $B(H)$ and C^*-algebras.** Let us now turn to the main structure of these lectures: to C^*-algebras. It turns out that it describes the algebraic structure of $B(H)$ pretty well.

Definition 1.29. We define the following algebraic notions.

(a) An *algebra* A over $\mathbb{C}$ is a complex vector space equipped with a bilinear associative multiplication $\cdot : A \times A \to A$ satisfying $\lambda(xy) = (\lambda x)y = x(\lambda y)$ for $x, y \in A$ and $\lambda \in \mathbb{C}$. The algebra is *unital*, if it contains a unit 1 with respect to the multiplication, i.e. $1x = x1 = x$ for all $x \in A$.

(b) A *normed algebra* A is an algebra which is also a normed vector space and whose norm is *submultiplicative*: It satisfies $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in A$.

(c) A *Banach algebra* is a normed algebra which is complete.

(d) An *involution* on an algebra A is an antilinear map $^* : A \to A$ such that $(x^*)^* = x$ and $(xy)^* = y^*x^*$ for all $x, y \in A$.

(e) A *C^*-algebra* is a Banach algebra A with an involution satisfying the *C^*-identity* $\|x^*x\| = \|x\|^2$ for all $x \in A$.

We conclude that a C^*-algebra combines algebraic structures (algebra with involution) with topological ones (norm and completion). The most important link between these two worlds is the C^*-identity, which turns C^*-algebras into a very special subclass of Banach algebras. We will see later how this identity comes into play. Also, we will discuss basic properties of the above definition in the next lecture. For now, let us be patient and let us only check that $B(H)$ is a C^*-algebra.

Proposition 1.30. *Given a Hilbert space H, the adjoint operators $T \mapsto T^*$ give rise to an involution and the composition of maps gives rise to a multiplication. Together with the operator norm, this turns $B(H)$ into a unital C^*-algebra.*

Proof. Using Lemma 1.28, we may directly check that we have an involution on $B(H)$ given by the adjoints. For instance:

$$\langle (T^*)^*x, y\rangle = \langle x, T^*y\rangle = \langle Tx, y\rangle$$

By Lemma 1.28 this yields $(T^*)^* = T$. Submultiplicativity of the norm follows from Lemma 1.23 when taking the supremum over $\|S(Tx)\| \leq \|S\|\|Tx\| \leq \|S\|\|T\|\|x\|$.

Let us now check that the involution is isometric (a fact that holds in general in C^*-algebras). Using Cauchy-Schwarz (Prop. 1.3), we have:

$$\|T^*x\|^2 = \langle T^*x, T^*x\rangle = \langle TT^*x, x\rangle \leq \|TT^*x\|\|x\| \leq \|T\|\|T^*x\|\|x\|$$

This implies $\|T^*x\| \leq \|T\|\|x\|$ and taking the supremum, we obtain $\|T^*\| \leq \|T\|$, by Lemma 1.23. On the other hand, $\|T\| = \|(T^*)^*\| \leq \|T^*\|$ which proves that the involution satisfies $\|T^*\| = \|T\|$.

We may now check the C^*-identity. Again, Cauchy-Schwarz yields

$$\|Tx\|^2 \leq \|T^*T\|\|x\|^2 = \|T^*T\|,$$

in case $\|x\| = 1$. Taking the supremum and using that the involution is isometric, we obtain:

$$\|T\|^2 \leq \|T^*T\| \leq \|T^*\|\|T\| = \|T\|^2$$

Hence, we have equality in the above computation.

As for the completeness of $B(H)$ with respect to the operator norm, this is a general fact on Banach spaces, which we omit here.

The unit on $B(H)$ is the identity map $x \mapsto x$, denoted by 1. $\square$

We now have a good example of a C^*-algebra at hand: it is $B(H)$, or $M_N(\mathbb{C})$, if you prefer the finite-dimensional setting. We may easily obtain further examples.

Example 1.31. Any closed *-subalgebra of $B(H)$ is a C^*-algebra. More precisely, let $A \subseteq B(H)$ be a linear subspace, which is closed under taking products and adjoints (i.e. it is a *-subalgebra), and which is also closed in the operator norm topology. Then, A is a C^*-algebra.

Finally, let us remark that $B(H)$ is also closed under taking inverses with respect to the composition, i.e. the inverse as a map is also the inverse with respect to the multiplication.

Proposition 1.32. *Let $T \in B(H)$ be a bijective map. Then also $T^{-1} \in B(H)$ and $(T^{-1})^* = (T^*)^{-1}$.*

Proof (idea): It is easy to see that T^{-1} is linear, but we need the Open Mapping Theorem for boundedness. The second assertion follows from Lemma 1.28. $\square$

1.9. **Algebraic formulations of Hilbert space features.** Being aware of the algebraic structure of $B(H)$ has some advantages: We may express certain properties of operators by purely algebraic means.

Definition 1.33. Let A be a unital C^*-algebra. Let $U, V, P \in A$.

(a) U is called *unitary*, if $U^*U = UU^* = 1$.
(b) V is called *isometry*, if $V^*V = 1$.
(c) P is called *(orthogonal) projection*, if $P = P^* = P^2$.

Let us take a look at the above definition in the special case $A = B(H)$ and see how the naming is motivated. Recall that $1 \in B(H)$ denotes the identity map.

Proposition 1.34. *Let $U, V, P \in B(H)$.*

(a) *U is a unitary if and only if it is a Hilbert space isomorphism of H.*
(b) *V is an isometry if and only if $\langle Vx, Vy\rangle = \langle x, y\rangle$ for all $x, y \in H$.*
(c) *P is a projection if and only if there is a closed subspace $K \subseteq H$ such that $P(x + y) = x$ for $x + y \in K \oplus K^\perp = H$, i.e. $\operatorname{ran} P = K$.*

Proof. Item (b) is an easy consequence of Lemma 1.28. As for (a), assume that U is a unitary. By (b), it is isometric, and from $UU^* = 1$ follows surjectivity. Hence, it is a Hilbert space isomorphism in the sense of Def. 1.17. Conversely, if U is a

Hilbert space isomorphism, we use (b) to deduce $U^*U = 1$. We prove $UU^* = 1$ as follows, making use of Lemma 1.28. Given $x, y \in H$ there is $x_0 \in H$ with $Ux_0 = x$ and hence:

$$\langle UU^*x, y\rangle = \langle UU^*Ux_0, y\rangle = \langle Ux_0, y\rangle = \langle x, y\rangle$$

Showing (c), let us first assume that P is a projection. Put $K := \operatorname{ran} P$, the range of P. Then K is a linear subspace of H. Moreover, any $x \in \operatorname{ran} P$ satisfies $Px = x$, since $P^2 = P$. Thus, for any sequence $x_n \to x$ with $x_n \in \operatorname{ran} P$, we have $x_n = Px_n \to Px$ by continuity of P. As the limit is unique, we have $x = Px \in \operatorname{ran} P$, which means that K is closed. We may hence decompose $H = K \oplus K^\perp$ and we observe that $K^\perp = \ker P$ using Lemma 1.27 and $P = P^*$. Thus, $P(x + y) = x$ for $x \in K$ and $y \in K^\perp$.

Conversely, let $K \subseteq H$ be a closed subspace and $P(x+y) = x$ as in the assertion. Then $P^2 = P$. Moreover, $P^* = P$ holds, since for $x, x' \in K$ and $y, y' \in K^\perp$:

$$\begin{aligned}\langle P^*(x + y), x' + y'\rangle &= \langle x + y, P(x' + y')\rangle = \langle x + y, x'\rangle = \langle x, x'\rangle = \langle x, x' + y'\rangle \\ &= \langle P(x + y), x' + y'\rangle\end{aligned}$$

We then use Lemma 1.28 to finish the proof. □

We conclude, that even in an abstract C^*-algebra A in the sense of Def. 1.29, we may define unitaries, isometries and projections as in Def. 1.33 – and this will allow us to deal abstractly with Hilbert space isomorphisms, the preservation of inner products and closed subspaces even if there is no underlying Hilbert space at hand!

Example 1.35. Let us briefly look at some examples of unitaries and isometries.

(a) In the finite dimensional setting, any isometry is automatically unitary. Indeed, by Prop. 1.34 we know that any isometry $V \in M_N(\mathbb{C})$ is injective: $Vx = 0$ implies $\langle x, x\rangle = \langle Vx, Vx\rangle = 0$. In finite dimensions, injectivity implies surjectivity, thus V is a unitary.

(b) In the infinite dimensional setting, these two notions may differ. Consider the Hilbert space $\ell^2(\mathbb{N})$ with an orthonormal basis e_n, $n \in \mathbb{N}$, see Exm. 1.14 for instance. The *unilateral shift* $S \in B(\ell^2(\mathbb{N}))$ is defined by $Se_n := e_{n+1}$, for all $n \in \mathbb{N}$. It is easy to see that $S^*e_n = e_{n-1}$ for $n \geq 2$ and $S^*e_1 = 0$. So, $S^*S = 1$, but $SS^* \neq 1$. See also Exc. 1.7.

1.10. **Compact operators.** We have seen that $B(H)$ is a unital C^*-algebra. Let us come to another important example of a C^*-algebra, in fact a non-unital one.

Definition 1.36. An operator $T \in B(H)$ is *compact* if one of the following equivalent conditions is satisfied:

(a) For any bounded set $M \subseteq H$, the closed set $\overline{TM}$ is compact.

(b) The closed image $\overline{TB(0, 1)}$ of the unit ball $B(0, 1) := \{x \in H \mid \|x\| \leq 1\}$ is compact.

(c) For any bounded sequence $(x_n)_{n\in\mathbb{N}}$ in H, the sequence $(Tx_n)_{n\in\mathbb{N}}$ contains a convergent subsequence.

We denote by $\mathcal{K}(H) \subseteq B(H)$ the set of all compact operators.

Example 1.37. (a) In $M_N(\mathbb{C})$, any operator is compact (Heine-Borel).

(b) Let H be infinite dimensional and assume that $T \in B(H)$ has *finite rank*, i.e. its image $\operatorname{ran} T$ is finite dimensional. Then T is compact. This follows again from some Heine-Borel argument, since $\overline{TB(0,1)}$ is contained in $\overline{\{y \in \operatorname{ran} T \mid \|y\| \leq C\}}$ with $C = \|T\|$, by Lemma 1.23.

(c) Let H be infinite dimensional. The operator $1 \in B(H)$ (i.e. the identity map) is *not* compact, since the closed unit ball is not compact. In fact, any normed vector space is finite-dimensional if and only if the closed unit ball is compact. We infer $\mathcal{K}(H) \subsetneq B(H)$ in infinite dimensions.

So, compact operators seem to be close to the finite dimensional setting – that is indeed the case: they may be approximated by finite rank operators as we will see in the next proposition. Thus, compact operators play the role of "small" operators.

Proposition 1.38. *The compact operators have the following properties.*

(a) *$\mathcal{K}(H)$ is a closed two-sided ideal of $B(H)$, i.e. it is a closed linear subspace satisfying $ST, TS \in \mathcal{K}(H)$ for all $S \in \mathcal{K}(H)$ and $T \in B(H)$.*

(b) *Given $T \in \mathcal{K}(H)$, we may find a sequence $T_n \in B(H)$ of finite rank operators approximating T in the operator norm.*

(c) *$\mathcal{K}(H)$ is closed under taking adjoints.*

(d) *$\mathcal{K}(H)$ is a C^*-algebra. It is non-unital, if and only if H is infinite dimensional.*

Proof (idea): The proof of (a) is no fun. That $\mathcal{K}(H)$ is a linear subspace follows easily from the continuity of the addition. Also, the ideal property is doable. But showing that $\mathcal{K}(H)$ is closed requires some tedious arguments with a diagonal sequence (but no magic).

In order to show (b), let us restrict to the case when H is separable with orthonormal basis e_n, $n \in \mathbb{N}$. We denote by $E(H)$ the set of finite rank operators. By (a) and Exm. 1.37, we know $\overline{E(H)} \subseteq \mathcal{K}(H)$. For the converse inclusion, denote by P_n the projection onto $\operatorname{span}\{e_1, \ldots, e_n\}$. Then $T_n := P_n T$ is of finite rank. One can then directly show that $T_n x \to Tx$, using Lemma 1.12. But this is only pointwise convergence! In order to show convergence in the operator norm, we need to use that T is compact.

Part (c) is known as Schauder's Theorem (which holds for general Banach spaces H). In our case, it follows easily from (b) (but (b) is not true for general Banach spaces H): Let $T \in \mathcal{K}(H)$ and pick a sequence T_n of finite rank operators approximating T. Then T_n^* is also of finite rank, since $P_m T_n = T_n$ for some $m \in \mathbb{N}$ and $T_n^* = T_n^* P_m$, i.e. T_n^* acts only on a finite dimensional subspace. Now, the involution is isometric and hence continuous, i.e. $T_n^* \to T^*$ and T^* is compact by (b).

(d) We conclude that $\mathcal{K}(H)$ is a closed *-subalgebra and hence it is a C^*-algebra by Exm. 1.31. Why isn't it unital in the infinite dimensional case? Let $(e_i)_{i \in I}$ be an orthonormal basis of H. Assume $P \in \mathcal{K}(H)$ was a unit for $\mathcal{K}(H)$, i.e. $PT = TP = T$ for all $T \in \mathcal{K}(H)$. Then also $PQ_i = Q_i P = Q_i$ where Q_i is the projection onto $\mathbb{C}e_i$, the one-dimensional subspace spanned by the i-th basis vector. This implies $Pe_i = e_i$ for all $i \in I$, and hence $P = 1$. But $1 \notin \mathcal{K}(H)$ by Exm. 1.37. $\square$

1.11. **Exercises.**

Exercise 1.1.
(a) Check that property (2) of Def. 1.1 may be derived from (1) and (3).
(b) Check that $\|x + y\|^2 = \|x\|^2 + 2\mathrm{Re}\langle x, y\rangle + \|y\|^2$ holds, where Re is the real part of a complex number.

Exercise 1.2. Prove the Cauchy-Schwarz inequality (Prop. 1.3) and show that equality holds if and only if the vectors are linearly dependent.

Exercise 1.3. Show that the norm induced by an inner product is a norm indeed. Use Cauchy-Schwarz and Exc. 1.1(b). Show that the mapping $x \mapsto \|x\|$ is continuous. This turns a Hilbert space into a topological vector space.

Exercise 1.4. Show that $f_y(x) := \langle x, y\rangle$ is linear and bounded with norm $\|f_y\| = \|y\|$. Thus, f_y is an element in the dual space of a Hilbert space H and the inner product is continuous in the sense that $x \mapsto \langle x, y\rangle$ is continuous.

Exercise 1.5. In Def. 1.1, we defined Hilbert spaces only for complex vector spaces, but the definition of real Hilbert spaces is completely analogous. Let us consider $\mathbb{R}^2$ with the inner product $\langle x, y\rangle = \sum_{i=1}^{2} x_i y_i$.
(a) Describe all unit vectors (i.e. vectors with norm 1) with the help of sine and cosine.
(b) Describe all vectors that are orthogonal to a given vector $x = (x_1, x_2) \in \mathbb{R}^2$.
(c) Show that $\frac{\langle x,y\rangle}{\|x\|\|y\|} = \cos\varphi$, where φ is the angle between x and y.
(d) Convince yourself that Prop. 1.7 is really Pythagoras Theorem for $H = \mathbb{R}^2$.

Exercise 1.6. Prove Lemma 1.18.

Exercise 1.7. Consider the unilateral shift $S \in B(\ell^2(\mathbb{N}))$ from Exm. 1.35.
(a) Verify $S^* e_n = e_{n-1}$ for $n \geq 2$ and $S^* e_1 = 0$. Verify that S is an isometry but no unitary.
(b) Now, consider the bilateral shift $\tilde{S} \in B(\ell^2(\mathbb{Z}))$ given by $Se_n = e_{n+1}$, where e_n, $n \in \mathbb{Z}$ is an orthonormal basis. How about this one, is it an isometry, is it a unitary?
(c) Which matrix is a reasonable analogue of $\tilde{S}$ in $M_N(\mathbb{C})$?

Exercise 1.8. An operator $V \in B(H)$ is called a *partial isometry*, if $VV^*V = V$.
(a) Show that V is a partial isometry if and only if V^*V is a projection (if and only if VV^* is a projection) in the sense of Def. 1.33.

(b) Show that V is a partial isometry if and only if there is a closed subspace $K \subseteq H$ such that $\langle Vx, Vy \rangle = \langle x, y \rangle$ for all $x, y \in K$ and $Vx = 0$ for $x \in K^{\perp}$. Compare with Prop. 1.34.

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