Abstract. In these lectures, we aim at providing an introduction to the general theory of $C^*$-algebras (first two thirds of the lectures) as well as to the more particular area of $C^*$-dynamical systems as a tool to deal with dynamics (last third of the lectures).

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**Introduction and Motivation**

Let us briefly motivate the lectures on $C^*$-algebras and dynamics and the main results we want to learn. This introduction is meant to serve as a teaser for the lectures omitting any technical details – the mathematical background for the following will be developed throughout the upcoming lectures. So, sit back and enjoy a short overview and motivation for the future lectures.

Recall that matrices $T \in M_N(\mathbb{C})$ may be seen as linear maps $T : \mathbb{C}^N \to \mathbb{C}^N$. In functional analysis, we deal with infinite dimensional versions of these and we consider linear maps

$$T : H \to H$$

between possibly infinite dimensional Hilbert spaces $H$. In contrast to linear algebra – i.e. the finite dimensional setting – these maps do not need to be continuous (which is equivalent to being bounded), so this comes as an extra assumption making life easier. So, let us consider

$$B(H) := \{T : H \to H \mid T \text{ is linear and bounded}\},$$

where $H$ is some Hilbert space $H$. If $\dim(H) = N$, then $B(H) = M_N(\mathbb{C})$.

A main feature of bounded, linear operators on a Hilbert space is noncommutativity: We have $ST \neq TS$ in general, where $S, T \in B(H)$ and the multiplication is defined via composition of maps. We know such a feature already from the matrix multiplication in linear algebra. This noncommutativity appears in quantum physics, in linear algebra, in the representation theory of groups and in many further areas of mathematics and science.

The theory of operator algebras captures this noncommutativity turning it into a powerful tool in mathematics. The pioneers Francis Murray and John von Neumann wrote in their very first article [21] on von Neumann algebras in 1936 that

“various aspects of the quantum mechanical formalism suggest strongly the elucidation of this subject.”

In addition, they claim that their work may be viewed as part of

“attempts to generalise the theory of unitary group-representations [sic!] essentially beyond their classical frame [. . .].”

Representing groups as unitary operators in $B(H)$ has also been in the scope of Israel Gelfand and Mark Naimark, when they wrote their seminal article [13] in 1943 introducing $C^*$-algebras. In 1993, Richard Kadison commented [15] on this article

“from the vantage point of a fifty year history, it is safe to say that that paper changed the face of modern analysis. Together with the monumental ‘Rings of operators’ series [. . .] authored by F. J. Murray and J. von Neumann, it introduced ‘non-commutative analysis’, the
vast area of mathematics that provides the mathematical model for quantum physics.”

Nowadays, the following areas may be counted to such a “non-commutative analysis” or “quantum mathematics”:

<table>
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<th>Classical theory</th>
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<td>Topology</td>
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The main reason why $C^*$-algebras may be seen as a “quantum version” of topology comes from the famous Gelfand-Naimark Theorem, which we allow ourselves to call the 1st Fundamental Theorem of $C^*$-Algebras in these lectures.

**1st Fundamental Theorem of $C^*$-Algebras** (Gelfand-Naimark 1940s).

Let $A$ be a unital $C^*$-algebra. We have the following equivalence.

$$A \text{ is commutative} \iff \exists X \text{ compact} : A \cong C(X) := \{ f : X \to \mathbb{C} \text{ is continuous} \}$$

Hence, any compact topological space gives rise to a commutative unital $C^*$-algebra — on the other hand any commutative $C^*$-algebra is exactly of this form. In this sense, commutative $C^*$-algebras “correspond” to topology and we may view the theory of noncommutative $C^*$-algebras as a kind of “noncommutative topology”.

This Gelfand duality is also the basis for other quantum theories (namely von Neumann algebras, Free probability, noncommutative geometry and quantum groups).

Besides proving the above first fundamental theorem, our goal is to prove that any (abstractly defined) $C^*$-algebra may be represented concretely on a Hilbert space:

**2nd Fundamental Theorem of $C^*$-Algebras** (Gelfand-Naimark, Segal 1940s). Any $C^*$-algebra is isomorphic to a norm closed *-subalgebra of $B(H)$, for some $H$.

From these fundamental theorems, we should keep in mind, that the algebra $C(X)$ of continuous functions on a compact space $X$ as well as closed (in the operator norm topology) *-subalgebras of $B(H)$ are our main examples of $C^*$-algebras.

We will spend about two thirds of the lecture (October – December 2020) in order to develop the above basic knowledge on $C^*$-algebras including also a treatment of universal $C^*$-algebras. Afterwards (January – February 2021), we turn to dynamical systems. Let us sketch some basic ideas of the latter, referring to [25] for a nice survey on dynamical systems and operator algebras.
Our starting point is a group $G$ and a compact space $X$. Assume that $G$ acts on this space, i.e. we have a map $\alpha : G \times X \to X$. This is a topological dynamical system. See [25] for a motivation how to derive this setting from more physically motivated dynamical systems or from differential equations.

Now, let us define $\alpha_g : C(X) \to C(X)$ via $\alpha_g(f)(x) := f(\alpha(g^{-1}, x))$. This induces a group homomorphism from $G$ to the automorphism group of $C(X)$ by $g \mapsto \alpha_g$. We may then construct a $C^*$-algebra $C(X) \rtimes_{\alpha} G$ containing the information of $X$, of $G$ and of the action of $G$ on $X$ (in terms of conjugation with unitaries) – hence, $C(X) \rtimes_{\alpha} G$ encodes the whole dynamical system!

Surprisingly, although $C(X)$ is commutative, the crossed product $C^*$-algebra $C(X) \rtimes_{\alpha} G$ may fail to be commutative. In fact, this is the generic situation: Unless the action is trivial, $C(X) \rtimes_{\alpha} G$ is always noncommutative (as conjugation with unitaries is trivial in commutative $C^*$-algebras). Hence, although our input $X$ and $G$ is classical data, we might want to enter the “nonclassical” or “quantum” world of noncommutative $C^*$-algebras in order to study this dynamical system. The philosophy is, that the theory of $C^*$-algebras provides a number of tools with which we may investigate $C(X) \rtimes_{\alpha} G$ – in order to learn something about the classical dynamical system.

More generally, we will treat $C^*$-dynamical systems, i.e. actions $\alpha$ of compact groups $G$ on possibly noncommutative $C^*$-algebras $A$, leading to crossed products $A \rtimes_{\alpha} G$.

We wish you a pleasant reading of the lecture notes and we hope you will enjoy the theory of $C^*$-algebras as much as we do!
1. Reminder on bounded operators on Hilbert spaces

Abstract. We recall some basic notions from Hilbert space theory, such as Hilbert spaces, Cauchy-Schwarz inequality, orthogonality, decomposition of Hilbert spaces, Riesz Representation Theorem, orthonormal bases and isomorphisms of Hilbert spaces. We then turn to bounded linear operators on Hilbert spaces, their operator norms and the existence of adjoints. We define the notion of a $C^*$-algebra and verify that $B(H)$ is a unital $C^*$-algebra. We finish this lecture with a number of algebraic reformulations of properties of operators on Hilbert spaces (such as unitaries, isometries, orthogonal projections, etc.), and we give a brief survey on compact operators. As Lecture 1 is seen as a reminder to lay the foundations for the upcoming lectures, it does not contain many complete proofs, but we give at least some ideas. You may take [3, 8, 22] as general references for Lecture 1.

1.1. Hilbert spaces. Informally speaking, Hilbert spaces are normed vector spaces equipped with a tool to measure “angles” between vectors, see also Exc. 1.5.

Definition 1.1. Let $H$ be a complex vector space. An inner product is a map $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{C}$ satisfying for all $x, y, z \in H$ and all $\lambda, \mu \in \mathbb{C}$:

1. $\langle \lambda x + \mu y, z \rangle = \lambda \langle x, z \rangle + \mu \langle y, z \rangle$
2. $\langle z, \lambda x + \mu y \rangle = \overline{\lambda} \langle z, x \rangle + \overline{\mu} \langle z, y \rangle$
3. $\langle x, y \rangle = \overline{\langle y, x \rangle}$
4. $\langle x, x \rangle \geq 0$
5. If $\langle x, x \rangle = 0$, then $x = 0$.

A space equipped with an inner product is called a pre-Hilbert space. An inner product induces a norm $\|x\| := \sqrt{\langle x, x \rangle}$. A (complex) Hilbert space is a pre-Hilbert space, which is complete with respect to the induced norm.

Example 1.2. The following spaces are examples of Hilbert spaces.

(a) Given $n \in \mathbb{N}$, the vector space $\mathbb{C}^n$ endowed with $\langle x, y \rangle := \sum_{i=1}^{n} x_i y_i$, $x, y \in \mathbb{C}^n$ is a Hilbert space. The induced norm is the well-known Euclidean norm.

(b) The space $\ell^2(\mathbb{N})$ of complex-valued sequences $(a_n)_{n \in \mathbb{N}}$ with $\sum_{n \in \mathbb{N}} |a_n|^2 < \infty$ endowed with $\langle (a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}} \rangle := \sum_{n \in \mathbb{N}} a_n \overline{b_n}$, $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}} \in \ell^2(\mathbb{N})$ is a Hilbert space.

(c) More generally, recall that we may define $L^2(X, \mu)$ where $(X, \mu)$ is a measure space. The inner product is then given by:

$$\langle f, g \rangle := \int_X f(x) \overline{g(x)} \, d\mu(x), \quad f, g \in L^2(X, \mu)$$

Note that for $X = [0, 1]$ the unit interval and $\mu = \lambda$ the Lebesgue measure, this defines an inner product on the space $C([0, 1])$ of continuous complex-valued functions. However, $C([0, 1])$ is not complete with respect to the induced norm (which is the so called $L^2$-norm), i.e. it is only a pre-Hilbert space but no Hilbert space.
Choosing \( X = I \) a set and \( \mu = \zeta \) the counting measure, we obtain \( \ell^2(I) \), with the above examples \( \ell^2(\mathbb{N}) \) and \( \mathbb{C}^n \) as special cases.

(d) Any closed subspace of a Hilbert space is a Hilbert space (closed with respect to the norm topology, subspace in the sense of a linear subspace).

The most important inequality for inner products is the following one.

**Proposition 1.3** (Cauchy-Schwarz inequality). If \( H \) is a Hilbert space (or a pre-Hilbert space), we have for all \( x, y \in H \):

\[
|\langle x, y \rangle| \leq \|x\| \|y\|
\]

Here, equality holds if and only if \( x \) and \( y \) are linearly dependent.

**Proof (idea):** Use

\[
0 \leq \langle x + \lambda y, x + \lambda y \rangle = \langle x, x \rangle - \frac{\langle x, y \rangle^2}{\langle y, y \rangle} \quad \text{with} \quad \lambda = -\frac{\langle x, y \rangle}{\langle y, y \rangle}.
\]

Actually, one needs the Cauchy-Schwarz inequality for proving that the norm in Def. 1.1 is a norm indeed; moreover we may derive continuity of the inner product. There are two further important properties of the inner product and its induced norm.

**Proposition 1.4.** Let \( H \) be a Hilbert space (or a pre-Hilbert space) and let \( x, y \in H \).

(a) The parallelogram identity holds: \( \|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2) \).

(b) The polarisation identity holds: \( \langle x, y \rangle = \frac{1}{4} \sum_{k=0}^{3} i^k \|x + i^k y\|^2 \).

**Proof (idea):** (a): Direct computation. (b): Use \( \|x + i^k y\|^2 = \langle x + i^k y, x + i^k y \rangle \).

The first of the above identities characterizes pre-Hilbert spaces: A normed space is a pre-Hilbert space if and only if the parallelogram identity holds. The second identity shows that the inner product is completely determined by its induced norm.

1.2. **Orthogonality and decomposition of Hilbert spaces.** As mentioned before, an inner product is the abstract information of an angle between vectors, see also Exc. 1.5. The notion of orthogonality plays the role of right angles.

**Definition 1.5.** Let \( H \) be a Hilbert space and \( K, K_1, K_2 \subseteq H \) be subsets.

(a) Two vectors \( x, y \in H \) are **orthogonal** (\( x \perp y \)), if \( \langle x, y \rangle = 0 \).

(b) We write \( K_1 \perp K_2 \), if \( x \perp y \) for all \( x \in K_1 \) and \( y \in K_2 \).

(c) The **orthogonal complement** of \( K \) is \( K^\perp := \{ x \in H \mid x \perp y \text{ for all } y \in K \} \).

Even when \( K \) is just a subset without any further structure, its orthogonal complement will be of a nice form.

**Lemma 1.6.** Given a subset \( K \subseteq H \), its orthogonal complement \( K^\perp \subseteq H \) is a closed subspace of \( H \) and we have \( (K^\perp)^\perp = K^\perp \), where \( K^\perp \) is the closure of \( K \).

**Proof (idea):** Due to the continuity of the inner product (Exc. 1.4).

The following is a version of the antique Greek theorem by Pythagoras verifying that orthogonality corresponds to right angles indeed, see also Exc. 1.5.
Proposition 1.7 (Pythagoras’ Theorem). If $H$ is a Hilbert space and $x, y \in H$ are orthogonal, then $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.

Proof. Direct computation. \qed

One of the most important features of Hilbert spaces is that we may decompose them into direct sums.

Definition 1.8. Let $K_1, K_2 \subseteq H$ be two closed subspaces of a Hilbert space, such that $K_1 \perp K_2$. We then write $K_1 \oplus K_2 \subseteq H$ for the subspace given by elements $x + y \in H$, where $x \in K_1$ and $y \in K_2$.

Proposition 1.9. Given a closed subspace $K \subseteq H$, we may decompose the Hilbert space $H$ as a direct sum:

$$H = K \oplus K^\perp$$

Then, every vector $x \in H$ has a unique decomposition $x = x_1 + x_2$ with $x_1 \in K$ and $x_2 \in K^\perp$.

Proof (idea): By Lemma 1.6, $K^\perp$ is closed. Trivially, $K \perp K^\perp$. We need to do some hard work to show that given $x \in H$, there is a unique “best approximation” $x_1 \in K$ such that $\|x - x_1\| = \inf\{\|x - y\| : y \in K\}$. With some further efforts, we then show $x_2 := x - x_1 \in K^\perp$. That this decomposition of $x$ is unique easily follows from $K \cap K^\perp = \{0\}$. \qed

Corollary 1.10. Given a subspace $K \subseteq H$, the double complement $(K^\perp)^\perp$ coincides with the closure $\overline{K}$ of $K$.

Proof. By the previous proposition and using Lemma 1.6, we may decompose $H$ in two ways, $H = \overline{K} \oplus K^\perp$ and $H = (K^\perp)^\perp \oplus K^\perp$, which shows $\overline{K} = (K^\perp)^\perp$. \qed

1.3. Dual space and the Representation Theorem of Riesz. Another nice feature of Hilbert spaces is that they have nice dual spaces - themselves! Given $y \in H$, we denote by $f_y : H \to \mathbb{C}$ the linear map given by $f_y(x) := \langle x, y \rangle$. In Exc. 1.4, it is shown that $f_y$ is linear and continuous.

Proposition 1.11 (Riesz Representation Theorem). Let $H$ be a Hilbert space and denote by $H'$ its dual space, i.e. the space consisting in all linear, continuous maps $f : H \to \mathbb{C}$. The map $j : H \to H'$ given by $j(y) := f_y$ is an antilinear isometric isomorphism.

Proof (idea): By Exc. 1.4, $j$ maps to $H'$ and it is isometric, i.e. $\|f_y\| = \|y\|$ (and hence injective); antilinearity follows from Def. 1.1(2). As for surjectivity, let $f \in H'$ be non-zero and decompose $H = K \oplus K^\perp$, where $K := \ker f$. You will find out that $K^\perp$ is one-dimensional and $j(y) = f_y = f$ for some $y \in K^\perp$. \qed

This has some nice consequences when working with Hilbert spaces. For instance, given a linear, continuous functional $f : L^2(X, \mu) \to \mathbb{C}$, then it must come from a function $g \in L^2(X, \mu)$, i.e. $f(h) = \int_X h \overline{g} \, d\mu$ for all $h \in L^2(X, \mu)$.
1.4. **Orthonormal basis for a Hilbert space.** In finite dimensions, we usually understand vector spaces with respect to certain coordinates. We may transport this concept to the infinite-dimensional setting within the framework of Hilbert spaces.

**Lemma 1.12.** Let \( H \) be a Hilbert space and let \((e_i)_{i \in I}\) be an orthonormal system, i.e. \( \langle e_i, e_j \rangle = \delta_{ij} \). The following are equivalent:

1. \( \|x\|^2 = \sum_{i \in I} |\langle x, e_i \rangle|^2 \) for all \( x \in H \)
2. \( x = \sum_{i \in I} \langle x, e_i \rangle e_i \) for all \( x \in H \)
3. span\{\(e_i \mid i \in I\}\} \subseteq H \) is dense.
4. If \( z \in H \) is orthogonal to all \( e_i, i \in I \), then \( z = 0 \).
5. \((e_i)_{i \in I}\) is a maximal orthonormal family (with respect to inclusion).

**Proof (idea):** First note that the sums over a possibly uncountable index set \( I \) are defined as the limits of nets of finite sums; in particular, only countably many summands are non-zero. As for the proof of the lemma, the easy parts are the equivalences of (2) and (3) (just a reformulation), of (4) and (5) (just a reformulation) as well as of (2) and (4) (use \( z := x - \sum_{i \in F} \langle x, e_i \rangle e_i \)). The hard part is the equivalence of (1) and (2), where we use Pythagoras (Prop. 1.7) on finite subsets \( F \subseteq I \) proving that \( \|x - \sum_{i \in F} \langle x, e_i \rangle e_i \| \) tends to zero. The key words are Bessel’s Inequality and Parseval’s Identity. □

**Definition 1.13.** An orthonormal system is called an **orthonormal basis** of a Hilbert space, if one of the equivalent conditions in Lemma 1.12 is satisfied.

We should not be misled by the word “basis” here: The elements of an orthonormal basis are linearly independent, but they do not necessarily form a basis in the sense of linear algebra (Hamel basis) – we may not represent any vector in \( H \) by a finite linear combination of the \( e_i \). However, passing to infinite linear combination, we may do so. This is the content of Lemma 1.12(2) – and we even know the coefficients thanks to our inner product. See also Schauder bases for the general Banach space setting.

**Example 1.14.** For \( \mathbb{C}^n \), the vectors \( e_i \) having 1 at the \( i \)-th entry and zero otherwise form an orthonormal basis – in fact, in finite dimensions any orthonormal basis is also a (Hamel) basis.

More generally, for \( \ell^2(I) \), the sequence having 1 at the \( i \)-th entry and zero otherwise form an orthonormal basis. If \( I \) is infinite, then this is not a basis.

**Lemma 1.15.** Any Hilbert space possesses an orthonormal basis \((e_i)_{i \in I}\) and the cardinality of \( I \) is independent of the choice of the vectors.

**Proof (idea):** Use Zorn’s Lemma for the existence and Cantor-Schröder-Bernstein for the uniqueness of the cardinality. □

**Definition 1.16.** Given a Hilbert space \( H \) with orthonormal basis \((e_i)_{i \in I}\), its **(Hilbert space) dimension** is defined as the cardinality of \( I \). If \( I \) is countable, we call \( H \) separable.

Thanks to the above lemma, the dimension is well-defined.
1.5. **Isomorphisms of Hilbert spaces.** Let us think about isomorphisms of Hilbert spaces – which structure are they supposed to preserve? Well, the vector space and the inner product!

**Definition 1.17.** Let $H$ and $K$ be Hilbert spaces. An **isomorphism** between $H$ and $K$ is a surjective linear map $U : H \to K$ which is isometric (or preserves the inner product), i.e. it satisfies $\langle Ux, Uy \rangle_K = \langle x, y \rangle_H$ for all $x, y \in H$.

The preservation of the inner product implies that $U$ is injective, which means that it is an isomorphism of the level of vector spaces, in particular. One can show that Hilbert spaces are isomorphic if and only if they have the same Hilbert space dimension in the sense of Def. 1.16. Hence, any Hilbert space is isomorphic to some $\ell^2(I)$. In particular, $\ell^2(\mathbb{N})$ is the separable Hilbert space.

1.6. **Bounded linear operators on Hilbert spaces.** In the subsequent lectures, we are not so much interested in the theory of Hilbert spaces as such but rather in the theory of bounded linear operators on Hilbert spaces. Let us first prove that “bounded” and “continuous” means the same for linear operators.

**Lemma 1.18.** Let $H, K$ be Hilbert spaces and let $T : H \to K$ be linear. The following are equivalent:

(a) $T$ is continuous everywhere.

(b) $T$ is continuous in zero.

(c) $T$ is bounded, i.e. there is a $C > 0$ such that $\|Tx\| \leq C\|x\|$ for all $x \in H$.

**Proof (idea):** The step from (a) to (b) is trivial. Assuming (b) with $\varepsilon = 1$, there is a $\delta > 0$ such that $\|x\| \leq \delta$ implies $\|Tx\| \leq 1$; put $C := \delta^{-1}$ to derive (c). Passing from (c) to (a) is straightforward.

**Definition 1.19.** Given a Hilbert space $H$, we denote by $B(H)$ the space of all bounded, linear operators $T : H \to H$.

**Example 1.20.** If $\dim(H) = N$, i.e. if $H = \mathbb{C}^N$, then $B(H) = M_N(\mathbb{C})$, the algebra of $N \times N$ matrices with complex entries. Indeed, in this case, any linear map is automatically bounded.

**Definition 1.21.** Given $T \in B(H)$, we denote by

$$\|T\| := \inf\{C > 0 \mid \|Tx\| \leq C\|x\| \text{ for all } x \in H\}$$

the **operator norm** of $T$.

One can check that the operator norm is a norm indeed.

**Lemma 1.22.** Given $T \in B(H)$, we have $\|Tx\| \leq \|T\|\|x\|$ for all $x \in H$.

**Proof.** Choosing $C_n > \|T\|$ with $C_n \to \|T\|$ yields $\|Tx\| \leq C_n\|x\| \to \|T\|\|x\|$. □

Let us express the operator norm in an alternative way.
Lemma 1.23. The norm $\|T\|$ may be written as
$$\|T\| = \sup\{\|Tx\| \mid \|x\| = 1\}.$$ You may replace $\|x\| = 1$ by $\|x\| \leq 1$, if you prefer.

Proof. By Lemma 1.22, we have $\|Tx\| \leq \|T\|$, if $\|x\| \leq 1$. Thus the supremum $s$ over all $\|Tx\|$ with $\|x\| = 1$ is less or equal to $\|T\|$. Conversely, $\|Tx\| = \|T\left(\frac{x}{\|x\|}\right)\|\leq s\|x\|$ whenever $x \neq 0$, so $\|T\| \leq s$ by Def. 1.21, which yields $\|T\| = s$ in total. The same proof works if $s$ is the supremum over $\|Tx\|$ with $\|x\| \leq 1$. □

1.7. Existence of adjoints. How does a bounded, linear operator $T$ behave with respect to evaluations under the inner product? Here, the existence of adjoints is a useful fact.

Proposition 1.24. Let $H$ be a Hilbert space and $T \in B(H)$. There exists a unique operator $T^* \in B(H)$ (the adjoint of $T$) such that
$$\langle Tx, y \rangle = \langle x, T^* y \rangle$$
for all $x, y \in H$.

Proof (idea): Let $y \in H$. We define $g^y : H \to \mathbb{C}$ by $g^y(x) := \langle Tx, y \rangle$. Then $g^y \in H'$ and by the Riesz Representation Theorem 1.11 there is a $z \in H$ such that $g^y = f_z$. Thus $\langle Tx, y \rangle = \langle x, z \rangle$ and we put $T^* y := z$. Check $T^* \in B(H)$. □

Example 1.25. If $H = \mathbb{C}^N$ and $T \in B(H) = M_N(\mathbb{C})$, we may express $T$ by $Te_i = \sum_j t_{ij} e_j$ for the canonical basis $e_1, \ldots, e_N$ of $\mathbb{C}^N$. Thus, $T \in M_N(\mathbb{C})$ has coefficients $t_{ij}$ and $T^* \in M_N(\mathbb{C})$ has coefficients $\overline{t_{ji}}$.

Some operators coincide with their adjoints; they will play a special role.

Definition 1.26. An operator $T \in B(H)$ is called selfadjoint (or Hermitian), if $T = T^*$.

There is a useful formula relating the kernel of $T$ with the image of its adjoint. We denote by ker $T$ the space of all $x \in H$ such that $Tx = 0$, whereas ran $T$ denotes the set of all $Tx$, where $x \in H$.

Lemma 1.27. For $T \in B(H)$, we have ker $T = (\text{ran } T^*)^\perp$ and $\overline{(\text{ker } T)^\perp} = \text{ran } T^*$.

Proof. A vector $x$ is in $(\text{ran } T^*)^\perp$ if and only if $\langle Tx, y \rangle = \langle x, T^* y \rangle = 0$ for all $y$, i.e. if and only if $x$ is in the kernel of $T$. Use Lemma 1.6 for the second part. □

Implicitely, we used the following lemma in the proof above.

Lemma 1.28. Let $T \in B(H)$. If $\langle Tx, y \rangle = 0$ for all $y \in H$, then $Tx = 0$. In particular, $\langle Tx, y \rangle = \langle Sx, y \rangle$ for all $x, y \in H$ implies $S = T$.

Proof. Put $y = Tx$ for the first part and use the first part for the second. □
1.8. Algebraic structure of $B(H)$ and $C^*$-algebras. Let us now turn to the main structure of these lectures: to $C^*$-algebras. It turns out that it describes the algebraic structure of $B(H)$ pretty well.

**Definition 1.29.** We define the following algebraic notions.

(a) An algebra $A$ over $\mathbb{C}$ is a complex vector space equipped with a bilinear associative multiplication $\cdot : A \times A \to A$ satisfying $\lambda(xy) = (\lambda x)y = x(\lambda y)$ for $x, y \in A$ and $\lambda \in \mathbb{C}$. The algebra is **unital**, if it contains a unit 1 with respect to the multiplication, i.e. $1x = x1 = x$ for all $x \in A$.

(b) A normed algebra $A$ is an algebra which is also a normed vector space and whose norm is **submultiplicative**: It satisfies $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in A$.

(c) A Banach algebra is a normed algebra which is complete.

(d) An *involution* on an algebra $A$ is an antilinear map $^* : A \to A$ such that $(x^*)^* = x$ and $(xy)^* = y^*x^*$ for all $x, y \in A$.

(e) A $C^*$-algebra is a Banach algebra $A$ with an involution satisfying the $C^*$-identity $\|x^*x\| = \|x\|^2$ for all $x \in A$.

We conclude that a $C^*$-algebra combines algebraic structures (algebra with involution) with topological ones (norm and completion). The most important link between these two worlds is the $C^*$-identity, which turns $C^*$-algebras into a very special subclass of Banach algebras. We will see later how this identity comes into play. Also, we will discuss basic properties of the above definition in the next lecture. For now, let us be patient and let us only check that $B(H)$ is a $C^*$-algebra.

**Proposition 1.30.** Given a Hilbert space $H$, the map $T \mapsto T^*$ from Prop. 1.24 give rise to an involution and the composition of maps gives rise to a multiplication. Together with the operator norm, this turns $B(H)$ into a unital $C^*$-algebra.

**Proof.** Using Lemma 1.28, we may directly check that we have an involution on $B(H)$ given by the adjoints. For instance:

$$\langle (T^*)^* x, y \rangle = \langle x, T^* y \rangle = \langle T x, y \rangle, \quad \text{for all } x, y \in H$$

By Lemma 1.28 this yields $(T^*)^* = T$. Submultiplicativity of the norm follows from Lemma 1.23 when taking the supremum over $\|S(Tx)\| \leq \|S\|\|Tx\| \leq \|S\|\|T\|\|x\|$.

Let us now check that the involution is isometric (a fact that holds in general in $C^*$-algebras). Using Cauchy-Schwarz (Prop. 1.3), we have:

$$\|T^*x\|^2 = \langle T^*x, T^*x \rangle = \langle TT^*x, x \rangle \leq \|TT^*x\| \|x\| \leq \|T\|\|T^*x\| \|x\|$$

This implies $\|T^*x\| \leq \|T\|\|x\|$ and taking the supremum, we obtain $\|T^*\| \leq \|T\|$, by Lemma 1.23. On the other hand, $\|T\| = \|(T^*)^*\| \leq \|T^*\|$ which proves that the involution satisfies $\|T^*\| = \|T\|$.

We may now check the $C^*$-identity. Again, Cauchy-Schwarz yields

$$\|Tx\|^2 \leq \|T^*T\|\|x\|^2 = \|T^*T\|,$$
in case \( \|x\| = 1 \). Taking the supremum and using that the involution is isometric, we obtain:

\[
\|T\|^2 \leq \|T^*T\| \leq \|T^*\|\|T\| = \|T\|^2
\]

Hence, we have equality in the above computation.

As for the completeness of \( B(H) \) with respect to the operator norm, this is a general fact on Banach spaces, which we omit here.

The unit on \( B(H) \) is the identity map \( x \mapsto x \), denoted by 1.

We now have a good example of a \( C^* \)-algebra at hand: it is \( B(H) \), or \( M_N(\mathbb{C}) \), if you prefer the finite-dimensional setting. We may easily obtain further examples.

**Example 1.31.** Any closed \( * \)-subalgebra of \( B(H) \) is a \( C^* \)-algebra. More precisely, let \( A \subseteq B(H) \) be a linear subspace, which is closed under taking products and adjoints (i.e. it is a \( * \)-subalgebra), and which is also closed in the operator norm topology. Then, \( A \) is a \( C^* \)-algebra.

Finally, let us remark that \( B(H) \) is also closed under taking inverses with respect to the composition, i.e. the inverse as a map is also the inverse with respect to the multiplication.

**Proposition 1.32.** Let \( T \in B(H) \) be a bijective map. Then also \( T^{-1} \in B(H) \) and \( (T^{-1})^* = (T^*)^{-1} \).

**Proof (idea):** It is easy to see that \( T^{-1} \) is linear, but we need the Open Mapping Theorem for boundedness. The second assertion follows from Lemma 1.28.

### 1.9. Algebraic formulations of Hilbert space features

Being aware of the algebraic structure of \( B(H) \) has some advantages: We may express certain properties of operators by purely algebraic means.

**Definition 1.33.** Let \( A \) be a unital \( C^* \)-algebra. Let \( U, V, P \in A \).

(a) \( U \) is called unitary, if \( U^*U = UU^* = 1 \).
(b) \( V \) is called isometry, if \( V^*V = 1 \).
(c) \( P \) is called (orthogonal) projection, if \( P = P^* = P^2 \).

Let us take a look at the above definition in the special case \( A = B(H) \) and see how the naming is motivated. Recall that \( 1 \in B(H) \) denotes the identity map.

**Proposition 1.34.** Let \( U, V, P \in B(H) \).

(a) \( U \) is a unitary if and only if it is a Hilbert space isomorphism of \( H \).
(b) \( V \) is an isometry if and only if \( \langle Vx,Vy \rangle = \langle x,y \rangle \) for all \( x,y \in H \).
(c) \( P \) is a projection if and only if there is a closed subspace \( K \subseteq H \) such that \( P(x+y) = x \) for \( x+y \in K \oplus K^\perp = H \), i.e. \( \text{ran} \ P = K \).

**Proof.** Item (b) is an easy consequence of Lemma 1.28. As for (a), assume that \( U \) is a unitary. By (b), it is isometric, and from \( UU^* = 1 \) follows surjectivity. Hence, it is a Hilbert space isomorphism in the sense of Def. 1.17. Conversely, if \( U \) is a
Hilbert space isomorphism, we use (b) to deduce $U^*U = 1$. We prove $UU^* = 1$ as follows, making use of Lemma 1.28. Given $x, y \in H$ there is $x_0 \in H$ with $Ux_0 = x$ and hence:

$$
(UU^*x, y) = (UU^*Ux_0, y) = (Ux_0, y) = (x, y)
$$

Showing (c), let us first assume that $P$ is a projection. Put $K := \text{ran } P$, the range of $P$. Then $K$ is a linear subspace of $H$. Moreover, any $x \in \text{ran } P$ satisfies $Px = x$, since $P^2 = P$. Thus, for any sequence $x_n \to x$ with $x_n \in \text{ran } P$, we have $x_n = Px_n \to Px$ by continuity of $P$. As the limit is unique, we have $x = Px \in \text{ran } P$, which means that $K$ is closed. We may hence decompose $H = K \oplus K^\perp$ and we observe that $K^\perp = \ker P$ using Lemma 1.27 and $P = P^*$. Thus, $P(x + y) = x$ for $x \in K$ and $y \in K^\perp$.

Conversely, let $K \subseteq H$ be a closed subspace and $P(x + y) = x$ as in the assertion. Then $P^2 = P$. Moreover, $P^* = P$ holds, since for $x, x' \in K$ and $y, y' \in K^\perp$:

$$
(P^*(x + y), x' + y') = (x + y, P(x' + y')) = (x + y, x') = (x, x') = (x, x' + y')
$$

$$
= (P(x + y), x' + y')
$$

We then use Lemma 1.28 to finish the proof. \qed

We conclude, that even in an abstract $C^*$-algebra $A$ in the sense of Def. 1.29, we may define unitaries, isometries and projections as in Def. 1.33 - and this will allow us to deal abstractly with Hilbert space isomorphisms, the preservation of inner products and closed subspaces even if there is no underlying Hilbert space at hand!

**Example 1.35.** Let us briefly look at some examples of unitaries and isometries.

(a) In the finite dimensional setting, any isometry is automatically unitary. Indeed, by Prop. 1.34 we know that any isometry $V \in M_n(\mathbb{C})$ is injective: $Vx = 0$ implies $(V, Vx) = (x, Vx) = 0$. In finite dimensions, injectivity implies surjectivity, thus $V$ is a unitary.

(b) In the infinite dimensional setting, these two notions may differ. Consider the Hilbert space $\ell^2(\mathbb{N})$ with an orthonormal basis $\{e_n, n \in \mathbb{N}\}$, see Exm. 1.14 for instance. The *unilateral shift* $S \in B(\ell^2(\mathbb{N}))$ is defined by $Se_n := e_{n+1}$, for all $n \in \mathbb{N}$. It is easy to see that $S^*e_n = e_{n-1}$ for $n \geq 2$ and $S^*e_1 = 0$. So, $S^*S = 1$, but $SS^* \neq 1$. See also Exc. 1.17.

1.10. **Compact operators.** We have seen that $B(H)$ is a unital $C^*$-algebra. Let us come to another important example of a $C^*$-algebra, in fact a non-unital one.

**Definition 1.36.** An operator $T \in B(H)$ is *compact* if one of the following equivalent conditions is satisfied:

(a) For any bounded set $M \subseteq H$, the closed set $\overline{TM}$ is compact.

(b) The closed image $\overline{TB}(0, 1)$ of the unit ball $B(0, 1) := \{x \in H \mid \|x\| \leq 1\}$ is compact.

(c) For any bounded sequence $(x_n)_{n \in \mathbb{N}}$ in $H$, the sequence $(Tx_n)_{n \in \mathbb{N}}$ contains a convergent subsequence.
We denote by $K(H) \subseteq B(H)$ the set of all compact operators.

**Example 1.37.**
(a) In $M_N(\mathbb{C})$, any operator is compact (Heine-Borel).
(b) Let $H$ be infinite dimensional and assume that $T \in B(H)$ has finite rank, i.e. its image $\text{ran} T$ is finite dimensional. Then $T$ is compact. This follows again from some Heine-Borel argument, since $\overline{T B(0,1)}$ is contained in $\{ y \in \text{ran} T \mid \| y \| \leq C \}$ with $C = \| T \|$, by Lemma 1.23.
(c) Let $H$ be infinite dimensional. The operator $1 \in B(H)$ (i.e. the identity map) is not compact, since the closed unit ball is not compact. In fact, any normed vector space is finite-dimensional if and only if the closed unit ball is compact. We infer $K(H) \not\subseteq B(H)$ in infinite dimensions.

So, compact operators seem to be close to the finite dimensional setting – that is indeed the case: they may be approximated by finite rank operators as we will see in the next proposition. Thus, compact operators play the role of “small” operators.

**Proposition 1.38.** The compact operators have the following properties.
(a) $K(H)$ is a closed two-sided ideal of $B(H)$, i.e. it is a closed linear subspace satisfying $ST, TS \in K(H)$ for all $S \in K(H)$ and $T \in B(H)$.
(b) Given $T \in K(H)$, we may find a sequence $T_n \in B(H)$ of finite rank operators approximating $T$ in the operator norm.
(c) $K(H)$ is closed under taking adjoints.
(d) $K(H)$ is a $C^*$-algebra. It is non-unital, if and only if $H$ is infinite dimensional.

*Proof (idea):* The proof of (a) is no fun. That $K(H)$ is a linear subspace follows easily from the continuity of the addition. Also, the ideal property is doable. But showing that $K(H)$ is closed requires some tedious arguments with a diagonal sequence (but no magic).

In order to show (b), let us restrict to the case when $H$ is separable with orthonormal basis $e_n, n \in \mathbb{N}$. We denote by $E(H)$ the set of finite rank operators. By (a) and Exm. 1.37 we know $E(H) \subseteq K(H)$. For the converse inclusion, denote by $P_n$ the projection onto span$\{ e_1, \ldots, e_n \}$. Then $T_n := P_n T$ is of finite rank. One can then directly show that $T_n x \to T x$, using Lemma 1.12. But this is only pointwise convergence! In order to show convergence in the operator norm, we need to use that $T$ is compact.

Part (c) is known as Schauder’s Theorem (which holds for general Banach spaces $H$). In our case, it follows easily from (b) (but (b) is not true for general Banach spaces $H$): Let $T \in K(H)$ and pick a sequence $T_n$ of finite rank operators approximating $T$. Then $T_n^*$ is also of finite rank, since $P_m T_n = T_n$ for some $m \in \mathbb{N}$ and $T_n^* = T_m^* P_m$, i.e. $T_n^*$ acts only on a finite dimensional subspace. Now, the involution is isometric and hence continuous, i.e. $T_n^* \to T^*$ by (a) and $T^*$ is compact.

(d) We conclude that $K(H)$ is a closed *-subalgebra and hence it is a $C^*$-algebra by Exm. 1.31. Why isn’t it unital in the infinite dimensional case? Let $(e_i)_{i \in I}$ be an
orthonormal basis of $H$. Assume $P \in \mathcal{K}(H)$ was a unit for $\mathcal{K}(H)$, i.e. $PT = TP = T$ for all $T \in \mathcal{K}(H)$. Then also $PQ_i = Q_iP = Q_i$ where $Q_i$ is the projection onto $\mathbb{C}e_i$, the one-dimensional subspace spanned by the $i$-th basis vector. This implies $Pe_i = e_i$ for all $i \in I$, and hence $P = 1$. But $1 \notin \mathcal{K}(H)$ by Exm. 1.37. □

1.11. Exercises.

**Exercise 1.1.**

(a) Check that property (2) of Def. 1.1 may be derived from (1) and (3).

(b) Check that $\|x + y\|^2 = \|x\|^2 + 2\text{Re}\langle x, y \rangle + \|y\|^2$ holds, where Re is the real part of a complex number.

**Exercise 1.2.** Prove the Cauchy-Schwarz inequality (Prop. 1.3) and show that equality holds if and only if the vectors are linearly dependent.

**Exercise 1.3.** Show that the norm induced by an inner product is a norm indeed. Use Cauchy-Schwarz and Exc. 1.1(b). Show that the mapping $x \mapsto \|x\|$ is continuous. This turns a Hilbert space into a topological vector space.

**Exercise 1.4.** Show that $f_y(x) := \langle x, y \rangle$ is linear and bounded with norm $\|f_y\| = \|y\|$. Thus, $f_y$ is an element in the dual space of a Hilbert space $H$ and the inner product is continuous in the sense that $x \mapsto \langle x, y \rangle$ is continuous.

**Exercise 1.5.** In Def. 1.1 we defined Hilbert spaces only for complex vector spaces, but the definition of real Hilbert spaces is completely analogous. Let us consider $\mathbb{R}^2$ with the inner product $\langle x, y \rangle = \sum_{i=1}^{2} x_i y_i$.

(a) Describe all unit vectors (i.e. vectors with norm 1) with the help of sine and cosine.

(b) Describe all vectors that are orthogonal to a given vector $x = (x_1, x_2) \in \mathbb{R}^2$.

(c) Show that $\frac{\langle x, y \rangle}{\|x\|\|y\|} = \cos \varphi$, where $\varphi$ is the angle between $x$ and $y$.

(d) Convince yourself that Prop. 1.7 is really Pythagoras Theorem for $H = \mathbb{R}^2$.

**Exercise 1.6.** Prove Lemma 1.18.

**Exercise 1.7.** Consider the unilateral shift $S \in B(\ell^2(\mathbb{N}))$ from Exm. 1.35.

(a) Verify $S^*e_n = e_{n-1}$ for $n \geq 2$ and $S^*e_1 = 0$. Verify that $S$ is an isometry but no unitary.

(b) Now, consider the bilateral shift $\tilde{S} \in B(\ell^2(\mathbb{Z}))$ given by $\tilde{S}e_n = e_{n+1}$, where $e_n, n \in \mathbb{Z}$ is an orthonormal basis. How about this one, is it an isometry, is it a unitary?

(c) Which matrix is a reasonable analogue of $\tilde{S}$ in $M_N(\mathbb{C})$?

**Exercise 1.8.** An operator $V \in B(H)$ is called a partial isometry, if $VV^*V = V$.

(a) Show that $V$ is a partial isometry if and only if $V^*V$ is a projection (if and only if $VV^*$ is a projection) in the sense of Def. 1.33.
(b) Show that $V$ is a partial isometry if and only if there is a closed subspace $K \subseteq H$ such that $\langle Vx, Vy \rangle = \langle x, y \rangle$ for all $x, y \in K$ and $Vx = 0$ for $x \in K^\perp$. Compare with Prop. 1.34.
2. **C*-algebras, Banach algebras and their spectra**

**Abstract.** We consider Banach algebras and C*-algebras and study some of their basic properties. We then turn to the spectrum of an element and show that it is compact and non-empty. We define the spectral radius and we prove Beurling’s formula. We briefly introduce ideals and quotients for Banach algebras before we turn to unitizations of C*-algebras.

Throughout Lecture 2, we take a closer look at the special features of C*-algebras as a particular class of Banach algebras. The book [3] serves as a general reference for Lecture 2; further references are given at the end of the lecture.

**2.1. Banach algebras and C*-algebras.** Let us recall the definition of Banach and C*-algebras from Def. [1.29]

**Definition 2.1.** Building on Def. [1.29] we define the following notions for Banach and C*-algebras.

(a) An **involution** on a C-algebra A is an antilinear map *: A → A such that (x*)* = x and (xy)* = y*x. A *-algebra is an algebra A equipped with an involution; B ⊆ A is a *-subalgebra of A, if xy, λx + μy, x* ∈ B for all x, y ∈ B and λ, μ ∈ C.

(b) A **Banach algebra** is a normed C-algebra which is complete; its norm satisfies ∥xy∥ ≤ ∥x∥∥y∥. A Banach *-algebra is a Banach algebra with an involution.

(c) A C*-algebra is a Banach *-algebra A satisfying the C*-identity ∥x*x∥ = ∥x∥². A *-subalgebra B ⊆ A is a C*-subalgebra, if B is (topologically) closed.

(d) An algebra is **unital**, if it contains a unit with respect to the multiplication.

(e) An algebra A is **commutative**, if xy = yx for all x, y ∈ A.

(f) An element x ∈ A in a C*-algebra is **normal**, if x*x = xx*. It is selfadjoint, if x* = x.

We observe, that C*-algebras differ from Banach *-algebras only by the C*-identity. What is so special about it? Some of the immediate consequences are listed in the next remark; others will come up later, for instance when talking about positivity in C*-algebras. It is hard to believe at this stage, but it is exactly this C*-identity that turns the class of C*-algebras into a very well-behaved and very special subclass of Banach algebras.

**Remark 2.2.** Let A be a C*-algebra.

(a) Let x ∈ A. If x*x = 0, then x = 0. This follows from the C*-identity.

(b) The involution is bijective, since (x*)* = x. It is also isometric, since ∥x∥² = ∥x*x∥ ≤ ∥x*∥∥x∥ implies ∥x∥ ≤ ∥x*∥ ≤ ∥(x*)*∥ = ∥x∥.

(c) A Banach *-algebra satisfies the C*-identity if and only if it satisfies the following inequality:

∥x∥² ≤ ∥x*x∥
Indeed, assuming this inequality, we derive as in (b) that the involution is isometric. Then \( \|x\|^2 \geq \|x^*x\| \) follows from submultiplicativity.

(d) If \( A \) is unital (and \( A \neq 0 \)), then \( 1^* = 1 \) and \( \|1\| = 1 \). Indeed, \( 1^*x = (x^*1)^* = (x)^* = x \) and \( x1^* = x \) for all \( x \in A \), hence \( 1^* \) is also a unit for \( A \). Thus, \( 1^* = 1 \). Moreover, \( \|1\|^2 = \|1^*1\| = \|1\| \), by the \( C^* \)-identity. Thus, \( \|1\| \in \{0,1\} \) and we may exclude \( \|1\| = 0 \) if \( A \neq 0 \).

In fact, more generally, \( \|p\| = 1 \) for any non-trivial selfadjoint projection \( p \in A \), i.e. \( p = p^* = p^2 \) and \( p \neq 0 \), see Def. \ref{def:non_trivial_selfadjoint_projection}.

(e) If \( x \in A \) is invertible, then \( (x^{-1})^* = (x^*)^{-1} \), since \( (x^{-1})^*x^* = (xx^{-1})^* = 1 \).

(f) The algebraic operations, i.e. the addition, the multiplication and the involution are continuous, and also the norm is continuous.

(g) Clearly, any selfadjoint element is normal. Normal and selfadjoint elements will play an important role later.

Let us take a look at examples of \( C^* \)-algebras.

**Example 2.3.**

(a) From Prop. \ref{prop:bound_linear_operators_form_C*} we know that the algebra \( B(H) \) of bounded linear operators on a Hilbert space \( H \) is a unital \( C^* \)-algebra. This is \( M_N(\mathbb{C}) \) in the finite dimensional case.

(b) Any closed \( * \)-subalgebra of \( B(H) \) is a \( C^* \)-algebra (see Exm. \ref{exm:closed_subalgebra_of_B(H)}).

(c) If \( H \) is infinite dimensional, then also the compact operators \( \mathcal{K}(H) \) form a \( C^* \)-algebra, in fact a non-unital one (see Prop. \ref{prop:compact_operators_form_C*}); if \( H \) is finite dimensional, then \( \mathcal{K}(H) = B(H) \).

(d) Let \( X \) be a compact Hausdorff space. Then

\[
C(X) := \{ f : X \to \mathbb{C} \mid f \text{ is continuous} \}
\]

is a unital \( C^* \)-algebra with \( (f + g)(x) := f(x) + g(x) \), \( (\lambda f) := \lambda f(x) \), \( (fg)(x) := f(x)g(x) \), \( 1(x) := 1 \), \( f^*(x) := \overline{f(x)} \) for \( f, g \in C(X) \), \( x \in X \) and \( \lambda \in \mathbb{C} \) and the supremum norm

\[
\|f\|_\infty := \sup \{|f(x)| \mid x \in X \}.
\]

Compactness of \( X \) guarantees that \( \|f\|_\infty < \infty \) for all \( f \in C(X) \), i.e. the supremum norm is a norm indeed. Since \( |f(x)g(x)| = |f(x)||g(x)| \) and \( |f(x)\overline{f(x)}| = |f(x)|^2 \), it is easy to see that the norm is submultiplicative satisfying the \( C^* \)-identity. Completeness is a bit more elaborate, but you might recall a proof from your early analysis lectures regarding uniform convergence of sequences of functions.

(e) If \( X \) is not compact, then \( \|f\|_\infty = \infty \) may happen for some \( f \in C(X) \). However, if \( X \) is locally compact, we may restrict to a subalgebra of \( C(X) \) as follows. We say that a function \( f : X \to \mathbb{C} \) vanishes at infinity, if for all \( \varepsilon > 0 \) there exists a compact set \( K \subseteq X \) such that for all \( t \not\in K \) we have \( |f(t)| < \varepsilon \). Put

\[
C_0(X) := \{ f : X \to \mathbb{C} \mid f \text{ is continuous and vanishes at infinity} \}.
\]
We equip this set with the pointwise operations and the supremum norm as above. Then \( \|f\|_\infty < \infty \) for all \( f \in C_0(X) \) since \( f \) vanishes at infinity. One can check that \( C_0(X) \) is a \( C^* \)-algebra. If \( X \) is compact, then \( C_0(X) = C(X) \). If \( X \) is not compact, then \( C_0(X) \) is non-unital.

(f) Let us also give an example of a Banach \( * \)-algebra which is not a \( C^* \)-algebra. Let \( \mathbb{D} \) be the open unit disk in \( \mathbb{C} \). The disk algebra \( A(D) := \{ f : \mathbb{D} \to \mathbb{C} \mid f \text{ is continuous on } \mathbb{D} \text{ and holomorphic on } \mathbb{D} \} \) with the pointwise addition, multiplication and the involution \( f^*(z) := \overline{f(\bar{z})} \) is a Banach \( * \)-algebra with the supremum norm, but no \( C^* \)-algebra.

2.2. Spectrum of an element. We know from linear algebra that eigenvalues play an important role when studying matrices. In infinite dimensions, we need to consider spectral values instead of eigenvalues.

Definition 2.4. Let \( A \) be a unital Banach algebra and let \( x \in A \).

(a) The spectrum of \( x \) is defined as \( \text{sp}(x) := \{ \lambda \in \mathbb{C} \mid \lambda 1 - x \text{ is not invertible} \} \).

It is also denoted by \( \sigma(x) \) sometimes. We also write \( \lambda - x \) instead of \( \lambda 1 - x \).

(b) The resolvent set of \( x \) is the complement \( \rho(x) = \mathbb{C} \setminus \text{sp}(x) \).

Remark 2.5. (a) Recall the definition of an eigenvalue: Given an operator \( T \in B(H) \), a complex number \( \lambda \in \mathbb{C} \) is an eigenvalue, if \( \lambda 1 - T \) is not injective. Indeed, in that case, we find \( 0 \neq x \in H \) such that \( Tx = \lambda x \). Now, in finite dimensions, \( \lambda 1 - T \) is not injective if and only if \( \lambda 1 - T \) is not invertible. This is not true in the infinite dimensional setting (for instance, the unilateral shift from Exm. 1.35 is injective but not surjective and hence not invertible).

This means, that any eigenvalue is a spectral value, but the converse is not true. The set of eigenvalues \( \sigma_p(T) := \{ \lambda \in \mathbb{C} \mid \lambda 1 - T \text{ is not injective} \} \) is also called the point spectrum. The funny thing is, that there are even operators with no eigenvalues at all, i.e. \( \sigma_p(T) = \emptyset \) is possible for some \( T \in B(H) \) (not in finite dimensions, though).

(b) In linear algebra, there is the well-known spectral theorem. There are analogues in the infinite dimensional setting: a spectral theorem for compact operators as well as a spectral theorem for normal bounded linear operators making use of the above definition of the spectrum.

(c) If \( A \) is a unital \( C^* \)-algebra and \( x \in A \), then \( \text{sp}(x^*) = \{ \bar{\lambda} \mid \lambda \in \text{sp}(x) \} \). Indeed, from Rem. 2.2(e), we know that \( \lambda 1 - x \) is invertible if and only if \( (\lambda 1 - x)^* \) is invertible.

In \( M_N(\mathbb{C}) \), \( N \geq 1 \), the spectrum coincides with the set of eigenvalues – it is finite and non-empty in that case. What is the situation in \( B(H) \) for an infinite dimensional Hilbert space \( H \)? Let us prepare the investigation of this question.

Lemma 2.6. Let \( A \) be a unital Banach algebra.

(a) If \( x \in A \) with \( \|1 - x\| < 1 \), then \( x \) is invertible and \( x^{-1} = \sum_{n=0}^{\infty} (1 - x)^n \).
(b) If $x$ is invertible and $y \in A$ with $\|x - y\| < (\|x^{-1}\|)^{-1}$, then $y$ is invertible.

(c) $\text{GL}(A) := \{x \in A \mid x \text{ is invertible}\}$ is open, and $\text{GL}(A) \ni x \mapsto x^{-1}$ is continuous.

**Proof.** For the proof of (a), put $z := 1 - x$. Then, $\|z\| < 1$ and hence $\sum_{n=0}^{\infty} z^n$ is absolutely convergent (since $\|z^n\| \leq \|z\|^n$ by submultiplicativity). As a consequence, $\sum_{n=0}^{\infty} z^n$ is convergent. It yields the inverse of $x$:

$$x \sum_{n=0}^{\infty} z^n \leftarrow (1 - z) \sum_{n=0}^{N} z^n = \sum_{n=0}^{N} z^n - \sum_{n=1}^{N+1} z^n = 1 - z^{N+1} \to 1, \quad \text{as } N \to \infty.$$ 

As for (b), we have:

$$\|1 - yx^{-1}\| = \|(x - y)x^{-1}\| \leq \|x - y\||x^{-1}\| < 1$$

Thus, $yx^{-1}$ is invertible by (a) and hence so is $y$.

Finally, for showing (c), let $x \in \text{GL}(A)$ and $y \in A$ such that $\|x - y\| < \varepsilon < \|x^{-1}\|^{-1}$. By (b), this shows $y \in \text{GL}(A)$ and hence the $\varepsilon$-ball around $x$ is in $\text{GL}(A)$ proving that $\text{GL}(A)$ is open. Let us now show continuity of taking inverses. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence with $x_n \to x$ and $x_n, x \in \text{GL}(A)$, $n \in \mathbb{N}$. Then $\|x_n - x\| < \|x^{-1}\|^{-1} \frac{\varepsilon}{2}$ for $n \in \mathbb{N}$ large and $0 < \varepsilon < 1$. Hence:

$$\|1 - x_nx^{-1}\| = \|(x - x_n)x^{-1}\| \leq \frac{\varepsilon}{2} < 1$$

By (a), this shows that $x_nx^{-1}$ is invertible with inverse

$$xx_n^{-1} = (x_nx^{-1})^{-1} = \sum_{k=0}^{\infty} (1 - x_nx^{-1})^k = 1 + \sum_{k=1}^{\infty} (1 - x_nx^{-1})^k.$$ 

Using $\varepsilon^k \leq \varepsilon$, we conclude:

$$\|x_n^{-1} - x^{-1}\| = \|x^{-1}(xx_n^{-1} - 1)\| \leq \|x^{-1}\| \sum_{k=1}^{\infty} \|1 - x_nx^{-1}\|^k$$

$$\leq \|x^{-1}\| \sum_{k=1}^{\infty} \varepsilon \frac{1}{2^k} = \varepsilon \|x^{-1}\|$$

Thus, $x_n^{-1}$ converges to $x^{-1}$. \qed

As an immediate consequence, the spectrum is compact.

**Proposition 2.7.** Let $A$ be a unital Banach algebra and let $x \in A$. Then $\text{sp}(x)$ is compact and $\text{sp}(x) \subseteq \{\lambda \in \mathbb{C} \mid \|\lambda\| \leq \|x\|\}$.

**Proof.** Firstly, we notice that the resolvent set $\rho(x)$ is open since it can be written as $\rho(x) = f_x^{-1}(\text{GL}(A))$, where $f_x : \mathbb{C} \to A, \lambda \mapsto \lambda 1 - x$ is continuous and $\text{GL}(A)$ is open by Lemma 2.6. Thus $\text{sp}(x)$ is closed.
Secondly, if $|\lambda| > \|x\| \neq 0$, then $\|\frac{x}{\lambda}\| < 1$. Hence, $\lambda - x = \lambda(1 - \frac{x}{\lambda})$ is invertible by Lemma 2.6 which shows $\lambda \notin \text{sp}(x)$. Thus, $\text{sp}(x) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| \leq \|x\|\}$, which means that $\text{sp}(x)$ is bounded. In summary, $\text{sp}(x)$ is compact by Heine-Borel. □

Let us now prove that the spectrum is non-empty, a fact that some people call the Fundamental Theorem in Banach Algebras. It has been proven by Gelfand. Recall that we denote by $A'$ the dual space of $A$, see Sect. 1.3

**Theorem 2.8.** If $A$ is a unital Banach algebra, $A \neq 0$, then $\text{sp}(x) \neq \emptyset$ for all $x \in A$.

**Proof.** Let $x \in A$. For $\lambda \in \rho(x)$, put

$$R_\lambda(x) := (\lambda - x)^{-1}.$$ 

**Claim 1:** We have $R_\lambda(x) - R_\mu(x) = (\mu - \lambda)R_\lambda(x)R_\mu(x)$ for all $\lambda, \mu \in \mathbb{C}$.

For proving Claim 1, check that

$$R_\lambda(x) - R_\mu(x) = R_\lambda(x)R_\mu(x)(\mu - x) - (\lambda - x)R_\lambda(x)R_\mu(x) = (\mu - \lambda)R_\lambda(x)R_\mu(x),$$

where we used $R_\lambda(x)R_\mu(x)(\mu - x) = (\mu - x)R_\lambda(x)R_\mu(x)$. Note that in principle $ab \neq ba$ for $a, b \in A$, but here this does not cause issues because $(\lambda - x)(\mu - x) = (\mu - x)(\lambda - x)$ implies $(\mu - x)R_\lambda(x) = R_\lambda(x)(\mu - x)$. This proves Claim 1.

**Claim 2:** Assume that $x$ is invertible and let $f \in A'$ such that $f(x^{-1}) \neq 0$. Then $g: \rho(x) \rightarrow \mathbb{C}, g(\lambda) := f(R_\lambda(x))$ is holomorphic and $g(0) \neq 0$.

For proving Claim 2, note that $\lambda \mapsto R_\lambda(x)$ is continuous by Lemma 2.6. By Claim 1, we thus have for $\mu \rightarrow \lambda$:

$$g(\lambda) - g(\mu) \over \lambda - \mu = f \left( R_\lambda(x) - R_\mu(x) \over \lambda - \mu \right) = -f(\lambda R_\lambda(x)R_\mu(x)) \rightarrow -f(R_\lambda^2(x))$$

Thus $g$ is holomorphic with $g(0) = f(R_0(x)) = -f(x^{-1}) \neq 0$. This shows Claim 2.

Finally, assume that $\text{sp}(x) = \emptyset$. Then $\emptyset \notin \text{sp}(x)$, i.e., $0 - x$ is invertible and thus $x$ is invertible. By the Hahn-Banach Theorem, we find a functional $f \in A'$ with $f(x^{-1}) \neq 0$. Thus the function $g$ from Claim 2 is an entire function as $\rho(x) = \mathbb{C}$.

**Claim 3:** $g$ is bounded, because $g(\lambda) \rightarrow 0$ for $\lambda \rightarrow \infty$.

In order to show Claim 3, put $z := 1 - \lambda^{-1}x$. Then $\|1 - z\| = |\lambda|^{-1}\|x\| < 1$ for $|\lambda|$ large. Thus, $z$ is invertible by Lemma 2.6 with $z^{-1} = \sum_{n=0}^{\infty} (1 - z)^n$. Hence:

$$\|(1 - \lambda^{-1}x)^{-1}\| = \|z^{-1}\| \leq \sum_{n=0}^{\infty} \|1 - z\|^n = (1 - \|1 - z\|)^{-1} = \frac{1}{1 - \|x\| |\lambda|}$$

This implies:

$$\|R_\lambda(x)\| = \|(\lambda - x)^{-1}\| = |\lambda|^{-1}\|(1 - \lambda^{-1}x)^{-1}\| \leq \frac{1}{|\lambda|(1 - \|x\| |\lambda|)} = \frac{1}{|\lambda| - \|x\|} \rightarrow 0 \text{ as } |\lambda| \rightarrow \infty.$$

Claim 3 is proven.

As $g$ is a bounded, entire function, it is constant by Liouville’s Theorem. From $g(\lambda) \rightarrow 0$ as $|\lambda| \rightarrow \infty$, we infer $g = 0$, which contradicts $g(0) \neq 0$ from Claim 2. □
An easy consequence: The only Banach algebra which is also a skew field, is $\mathbb{C}$.

**Theorem 2.9** (Gelfand-Mazur). Let $A$ be a unital Banach algebra. If $A$ is also a skew field (i.e. every element $0 \neq a \in A$ is invertible), then $A = \mathbb{C}$.

**Proof.** Let $a \in A$. Then $\operatorname{sp}(a) \neq \emptyset$ by Thm. 2.8. Hence there is some $\lambda \in \mathbb{C}$ such that $\lambda a - a$ is not invertible. Since $A$ is a skew field, this implies $\lambda a - a = 0$. □

2.3. **Spectral radius.** An important information we can extract from the spectrum is the spectral radius. It is closely linked with the norm providing an attractive alternative way for computing the norm.

**Definition 2.10.** Let $A$ be a unital Banach algebra and let $x \in A$. The spectral radius of $x$ is defined as

$$r(x) := \sup\{||\lambda|| : \lambda \in \operatorname{sp}(x)\}.$$ 

**Remark 2.11.** From Prop. 2.7 we know $r(x) \leq ||x||$. Also, as the spectrum is compact (by the same proposition), the supremum is in fact a maximum.

**Example 2.12.** The spectral radius may or may not coincide with the norm:

(a) We have $r(f) = ||f||_{\infty}$ for any $f \in C(X)$ whenever $X$ is compact. Indeed, since $f$ is continuous and $X$ is compact, the image $f(X)$ is compact. Thus, there is some $x \in X$ with $|f(x)| = ||f||_{\infty}$, i.e. $f(x) = e^{i\alpha}||f||_{\infty}$ for some $\alpha \in [0, 2\pi)$. Then, $e^{i\alpha}||f||_{\infty} - f$ is not invertible and $e^{i\alpha}||f||_{\infty} \in \operatorname{sp}(f)$.

(b) Consider $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{C})$. Then, $\lambda - x = \begin{pmatrix} \lambda & -1 \\ 0 & \lambda \end{pmatrix}$ is invertible for all $\lambda \neq 0$. Thus $\operatorname{sp}(x) = \{0\}$ and $r(x) = 0$ while $||x|| \neq 0$.

So, when does the spectral radius coincide with the norm? Let us try to find out. The main ingredient is the following amazing formula by Beurling (also known as Gelfand-Beurling spectral radius formula). It relates an algebraic quantity (the spectral radius, speaking about invertibility in an algebraic sense) with a topological one (the norm).

**Theorem 2.13.** Let $A$ be a unital Banach algebra and $x \in A$. Then the spectral radius formula holds:

$$r(x) = \lim_{n \to \infty} \sqrt[n]{||x^n||}$$

**Proof.** Let $\lambda \in \operatorname{sp}(x)$. Then $\lambda^n \in \operatorname{sp}(x^n)$, since

$$\lambda^n - x^n = (\lambda - x)(\lambda^{n-1} + \lambda^{n-2}x + \cdots + \lambda x^{n-2} + x^{n-1})$$

cannot have an inverse. Hence, $|\lambda^n| \leq ||x^n||$ which means $|\lambda| \leq \sqrt[n]{||x^n||}$. We infer $r(x) \leq \liminf_{n \to \infty} \sqrt[n]{||x^n||}$. Thus, it remains to show $r(x) \geq \limsup_{n \to \infty} \sqrt[n]{||x^n||}$. As in the proof of Thm. 2.8 consider

$$R_z(x) = (z - x)^{-1} = \sum_{n=0}^{\infty} \frac{x^n}{z^{n+1}}$$
for $|x| < |z|$ (in particular $z \in \rho(x)$ by Prop. 2.7). If this was a power series in $\mathbb{C}$, its radius of convergence would be $\limsup_{n \to \infty} \sqrt[n]{|x^n|}$, which is a hint for us to be on the right track. However, it is a series in $A$ unfortunately, so we have to use the same trick as in the proof of Thm. 2.8 in order to “make it a series in $\mathbb{C}$”.

Let $C$ such that both $A$ and $\mathcal{C}$, $\mathcal{C} = \mathbb{C}$, are on the right track. However, it is a series in $A$, unfortunately, so we have to use the same trick as in the proof of Thm. 2.8 in order to “make it a series in $\mathbb{C}$”. Let $f \in A'$. Then, the function $g : \rho(x) \to \mathbb{C}, z \mapsto f(R_z(x))$ is holomorphic (see the proof of Thm. 2.8) and $g(z) = \sum_{n=0}^\infty \frac{f(x^n)}{z^n}$ for $|z| > \|x\|$; in fact even for $|z| > r(x)$ (using methods from complex analysis on the holomorphic domain of the Laurent series). Hence

$$\limsup_{n \to \infty} |f(x^n)|^{\frac{1}{n}} \leq r(x)$$

by the formula of Cauchy-Hadamard for convergence radii of power series. This is good, but now we have to get rid of $f$. For $r > r(x)$, we find some $N \in \mathbb{N}$ such that $|f(x^n)|^{\frac{1}{n}} < r$ for all $n \geq N$. Hence

$$\sup_{n \in \mathbb{N}} \left| \frac{f(x^n)}{r^n} \right| < \infty$$

for all $f \in A'$. By the Principle of Uniform Boundedness, we conclude that the set $\{ \frac{z^n}{n} \mid n \in \mathbb{N} \}$ is bounded. Hence there is some $C > 0$ such that $\|x^n\| \leq Cr^n$. Now, $\|x^n\|^{\frac{1}{n}} \leq C \frac{r}{n}$, which implies $\limsup_{n \to \infty} \|x^n\|^{\frac{1}{n}} \leq r$ for all $r > r(x)$ and thus also for $r = r(x)$.

This formula behaves particularly nice with respect to the $C^*$-identity as we will see in the next corollary. It also answers our question under which conditions the spectral radius and the norm coincide.

**Corollary 2.14.** Let $A$ be a unital $C^*$-algebra and let $x \in A$ be normal (i.e. $x^*x = xx^*$, see Def. 2.1). Then $r(x) = \|x\|$.

**Proof.** Using the $C^*$-identity and the fact that $x$ is normal, we have:

$$\|x^2\|^2 = \|(x^2)^*x^2\| = \|x^*x^*xx\| = \|x^*xx^*x\| = \|(x^*x)^*(x^*x)\| = \|x^*x\|^2 = \|x\|^4$$

Thus $\|x\|^2 = \|x^2\|$. Inductively, we see that $\|x^{2n}\| = \|x\|^{2n}$, hence

$$r(x) = \lim_{n \to \infty} \sqrt[2^n]{\|x^{2n}\|} = \|x\|.$$

Another surprising corollary is that a $C^*$-algebra cannot be equipped with another $C^*$-norm.

**Corollary 2.15.** Let $A$ be a unital $*$-algebra and let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on $A$ such that both $(A, \|\cdot\|_1)$ and $(A, \|\cdot\|_2)$ are $C^*$-algebras. Then $\|\cdot\|_1 = \|\cdot\|_2$.

**Proof.** Let $x \in A$. Then $\|x\|_i^2 = \|x^*x\|_i = r(x^*x)$ by Cor. 2.14 for $i = 1, 2$. 
2.4. Ideals and quotients. Let us now briefly discuss some further algebraic structure of Banach algebras and $C^*$-algebras: Ideals and quotients.

**Definition 2.16.** Let $A$ be an algebra.

(a) A (two-sided) ideal $I$ in $A$ (we write $I \triangleleft A$) is a linear subspace $I \subseteq A$ such that $xy, yx \in I$, for all $x \in I$ and $y \in A$.

(b) An ideal $I \neq A$ is called maximal, if for any other (two-sided) ideal $J \triangleleft A$ with $I \subseteq J \subseteq A$ either $J = I$ or $J = A$.

(c) If $A$ is a Banach algebra, we say that $I \triangleleft A$ is a closed ideal, if $I$ is a (two-sided) ideal and if it is closed with respect to the topology of $A$.

Throughout the lectures, all ideals will be two-sided unless specified otherwise. The main use of ideals is that we can take quotients, i.e. we may pass to some “rougher structures” in a way. Let us list some properties of ideals in Banach algebras.

**Proposition 2.17.** Let $A$ be a Banach algebra and let $I \triangleleft A$ be an ideal.

(a) If $I$ is a closed ideal, then the quotient $A/I$ is a Banach algebra.

(b) The closure $\bar{I}$ of $I$ is an ideal in $A$.

(c) For a unital Banach algebra $A$, the following are equivalent:

   (i) $I = A$
   
   (ii) $I \cap \text{GL}(A) \neq \emptyset$

   (iii) $1 \in \bar{I}$

(d) Let $A$ be unital. Then any maximal ideal is closed.

(e) Let $A$ be unital. Then any ideal $I \neq A$ is contained in a maximal ideal.

**Proof.** For (a), we first prove a more general (and probably well-known) statement: If $A$ is a Banach space and $I$ is a closed linear subspace, then $A/I$ is a Banach space.

For proving it, denote by $\hat{x} = x + I$ the elements in $A/I$. We define a vector space structure on $A/I$ by $\hat{x} + \hat{y} := (x + y)$ and $\lambda \hat{x} := (\lambda x)$. We define a norm on $A/I$ by $\|\hat{x}\| := \inf \{\|x + z\| \mid z \in I\}$. It is not too difficult to prove that this is a norm indeed; note that we need $I$ to be closed in order to deduce $\hat{x} = 0$ from $\|\hat{x}\| = 0$. It is a bit more technical to show that $A/I$ is again a Banach space, i.e. that it is complete with respect to this norm, but no magic is involved. Since $0 \in I$, we have $\|\hat{x}\| \leq \|x\|.$

We then turn back to the situation we are interested in and assume that $A$ is even a Banach algebra while $I \triangleleft A$ is a closed ideal. In addition to the above structure, we put $\hat{x}\hat{y} := (xy)$ turning $A/I$ into an algebra. This operation is well-defined since for all $a, b \in I$ and $x, y \in A$, the element $ay + xb + ab$ is in $I$ and hence:

$$((x + a)(y + b)) = (xy + ay + xb + ab) = (xy)$$

Moreover, the norm is submultiplicative: Given $\varepsilon > 0$, there are $a, b \in I$ with $\|x + a\| \leq \|\hat{x}\| + \varepsilon$ and $\|y + b\| \leq \|\hat{y}\| + \varepsilon$. Thus:

$$\|\hat{x}\hat{y}\| = \|(x + a)(y + b)\| \leq \|(x + a)(y + b)\| \leq \|x + a\|\|y + b\| \leq (\|\hat{x}\| + \varepsilon)(\|\hat{y}\| + \varepsilon)$$

As this holds true for all $\varepsilon > 0$, we just proved $\|\hat{x}\hat{y}\| \leq \|\hat{x}\|\|\hat{y}\|$, and $A/I$ is a Banach algebra.
As for (b), let \( x \in \bar{I} \) and \( y \in A \). We find a sequence \((x_n)_{n \in \mathbb{N}}\) with \( x_n \in I \) and \( x_n \to x \). Then \( x_n y \to xy \) with \( x_n y \in I \) for all \( n \in \mathbb{N} \). Hence \( xy \in \bar{I} \) and similarly \( yx \in \bar{I} \).

For item (c), it is easy to see that (iii) implies (i), while (i) implies (ii). As for (ii) to (iii), let \( x \in I \cap \text{GL}(A) \). Then \( 1 = xx^{-1} \in I \), since \( x \in I \).

For (d), let \( I \triangleleft A \) be a maximal ideal. Then \( I \subseteq \bar{I} \subseteq A \). We need to show \( \bar{I} \neq A \) in order to deduce \( I = \bar{I} \) from (b) and maximality of \( I \). Now, the complement \( \text{GL}(A)^c \) of \( \text{GL}(A) \) contains \( I \) by (c), since \( I \neq A \) by the definition of a maximal ideal. Moreover, \( \text{GL}(A)^c \) is closed by Lemma 2.6, so it also contains \( \bar{I} \). By (c), \( \bar{I} \neq A \).

Finally, (e) is a consequence of Zorn’s Lemma.

2.5. Unitization of \( C^*\)-algebras. In Exm. 2.3, we have seen some examples of non-unital \( C^*\)-algebras: \( \mathcal{K}(H) \), if \( H \) is infinite dimensional and \( C_0(X) \), if \( X \) is locally compact but not compact. In both cases, there are “would-be”-units: If the identity operator \( \text{id} : H \to H \) was compact, then it would be a unit for \( \mathcal{K}(H) \). Likewise, if the constant function \( 1 : X \to \mathbb{C} \) vanished at infinity, then it would be a unit for \( C_0(X) \). So, it seems that there is some unit in the background, if we enlarge our algebra! Let us do this systematically and study unitizations.

Our idea is to add a copy of a relatively small algebra – ideally: \( \mathbb{C} \) – to a non-unital \( C^*\)-algebra and to find a unit in this enlarged space. Let us take a look at the direct sum of \( C^*\)-algebras first.

**Lemma 2.18.** Let \( A, B \) be \( C^*\)-algebras. Put
\[
A \oplus B := \{(a, b) \mid a \in A, b \in B \}.
\]
This is a \( C^*\)-algebra with the entrywise operations and \( \|(x, y)\| := \max\{\|x\|, \|y\|\} \).

**Proof.** Straightforward. \( \square \)

When being equipped with the entrywise operations, the only chance for the direct sum of two \( C^*\)-algebras to be unital is when \( A \) and \( B \) are unital, too. In that case, \( (1, 1) \) is the unit. So we need to choose different kinds of operations, if we want to obtain some unitization given a non-unital \( C^*\)-algebra. As we are looking for a neutral element for the multiplication, we shall modify the multiplication accordingly.

**Definition 2.19.** Let \( A \) be a *-algebra. We define
\[
\tilde{A} := \{(a, \lambda) \mid a \in A, \lambda \in \mathbb{C} \}
\]
and we equip this set with the following operations, for \( a, b \in A, \lambda, \mu \in \mathbb{C} \).

(i) \( (a, \lambda) + (b, \mu) := (a + b, \lambda + \mu) \)
(ii) \( \mu(a, \lambda) := (\mu a, \mu \lambda) \)
(iii) \( (a, \lambda)(b, \mu) := (ab + \lambda b + \mu a, \lambda \mu) \)
(iv) \( (a, \lambda)^* := (a^*, \bar{\lambda}) \)
We also write \( \lambda + a \) for \((a, \lambda) \in \tilde{A}\).

It is immediately clear that \( \tilde{A} \) is a \(*\)-algebra again. The crucial point is: It is unital, the unit being \((0, 1)\). Observe that the notation \( \lambda + x \) really makes sense, in particular it helps to memorize the multiplication law. It is easy to equip \( \tilde{A} \) with a norm turning it into a Banach algebra (provided it helps to memorize the multiplication law. It is easy to equip \( \tilde{A} \) with a norm turning it into a Banach algebra (provided \( A \) is a Banach algebra): We may simply take \( \|(a, \lambda)\|_{\text{BA}} := \|a\| + |\lambda| \). Hence, the unitization of Banach algebras is not an issue. However, this norm does not satisfy the \( C^* \)-identity, so it does not turn \( \tilde{A} \) into a \( C^* \)-algebra (even if \( A \) is a \( C^* \)-algebra). We need to be more creative in finding a \( C^* \)-norm on \( \tilde{A} \).

**Proposition 2.20.** Let \( A \) be a \( C^* \)-algebra. There exists a unique norm on \( \tilde{A} \) turning \( \tilde{A} \) into a unital \( C^* \)-algebra. We have \( A \triangleleft \tilde{A} \) as an ideal, where \( A \) is identified with elements \((a, 0) \in \tilde{A} \) for \( a \in A \).

**Proof.** If a \( C^* \)-norm exists, it must be unique, by Cor. \([2.15]\). So, let us show existence.

In the first case, let us assume that \( A \) is already unital. Then \( A \oplus C \) is a unital \( C^* \)-algebra by Lemma \([2.18]\). The map \( \tilde{A} \to A \oplus C, (a, \lambda) \mapsto (\lambda 1 + a, \lambda) \) is a bijective map preserving the algebra operations such as multiplication, addition and involution and mapping \((0, 1)\) to \((1, 1)\). Thus, \( \tilde{A} \) and \( A \oplus C \) are isomorphic as unital \(*\)-algebras and we may define a \( C^* \)-norm on \( \tilde{A} \) simply by using the \( C^* \)-norm on \( A \oplus C \).

Now, let us assume that \( A \) is not unital. Consider

\[
L : \tilde{A} \to B(A) := \{ T : A \to A \mid T \text{ is linear and bounded} \}, \quad x \mapsto L_x,
\]

where \( L_x(b) := xb \) for \( b \in A \) and \( x \in \tilde{A} \). This is a left multiplication operator on \( A \): Given \( x = \lambda + a \), observe that \( xb = \lambda b + ab \in A \). Moreover, \( L_x \) is linear and bounded:

\[
\|L_x(b)\| = \|xb\| = \|\lambda b + ab\| \leq (|\lambda| + \|a\|) \|b\|
\]

This implies \( \|L_x\| \leq |\lambda| + \|a\| \). We conclude that \( L_x \in B(A) \) holds.

Now comes the crucial step: We define the norm on \( \tilde{A} \) via the operator norm of \( L_x \) by the following.

\[
\|x\|_{\tilde{A}} := \|L_x\| = \sup\|xb\|_A \mid b \in A, \|b\|_A \leq 1\}
\]

Let us now check a couple of properties of \( \|\cdot\|_{\tilde{A}} \).

Firstly, we have \( \|a\|_{\tilde{A}} = \|a\|_A \) for all \( a \in A \). Indeed, we have:

\[
\|a\|_{\tilde{A}} a^*_{\tilde{A}} = \|a^*\|_A^2 = \|aa^*\|_A = \|L_a(a^*)\| \leq \|L_a\| \|a^*\|_A
\]

This implies \( \|a\|_{\tilde{A}} \leq \|L_a\| \). On the other hand, we have \( \|a\|_A \geq \|L_a\| \), since \( \|L_a(b)\|_A = \|ab\|_A \leq \|a\|_A \|b\|_A \) for all \( b \in A \) with \( \|b\|_A \leq 1 \). Hence, \( \|a\|_A = \|a\|_{\tilde{A}} \).

Secondly, \( \|\cdot\|_{\tilde{A}} \) is a norm on \( \tilde{A} \). For proving it, the only non-trivial step is that \( \|x\|_{\tilde{A}} = 0 \) implies \( x = 0 \). We prove it by contraposition. Let \( x = \lambda + a \in \tilde{A} \) with \( x \neq 0 \). We may assume \( \lambda \neq 0 \) - otherwise \( x \in A \) and then \( \|x\|_{\tilde{A}} = \|x\|_A \neq 0 \), by the first step above. Let us assume \( \|L_x\| = 0 \). Then \( \lambda b + ab = xb = 0 \) for all \( b \in A \).

But this shows that \( e := -\frac{a}{\lambda} \) is a left unit, as \( eb = b \) for all \( b \in A \). Now, \( e^* \) is a right
unit, since \(b e^* = (eb)^* = b\) for all \(b \in A\). As, \(e = ee^* = e^*\), we infer that \(e\) is a unit. This contradicts the assumption that \(A\) is not unital and we conclude \(\|L_x\| \neq 0\).

Thirdly, \(\|\cdot\|_{\widetilde{A}}\) is submultiplicative. This follows immediately from \(L_{xy} = L_x L_y\) and the submultiplicativity of the operator norm:

\[
\|xy\|_{\widetilde{A}} = \|L_{xy}\| = \|L_x L_y\| \leq \|L_x\| \cdot \|L_y\|
\]

Fourthly, \(\|\cdot\|_{\widetilde{A}}\) satisfies the \(C^*\)-identity: Let \(x \in \widetilde{A}\) and \(\varepsilon > 0\). By the definition of the operator norm, there is some \(b \in A\) with \(\|b\| \leq 1\) and \(\|xb\|_A \geq \|L_x\| - \varepsilon\). Hence:

\[
(\|L_x\| - \varepsilon)^2 \leq \|xb\|_{\widetilde{A}}^2 = \|b^* x^* x b\|_A \leq \|b^*\|_A \cdot \|L_{x^* x}(b)\|_A \leq \|L_{x^* x}\|
\]

As this holds true for all \(\varepsilon > 0\), we deduce:

\[
\|x\|^2_{\widetilde{A}} = \|L_x\|^2 \leq \|L_{x^* x}\| = \|x^* x\|_{\widetilde{A}}
\]

From Remark 2.22c), we deduce that the \(C^*\)-identity holds.

Finally, \(\widetilde{A}\) is complete with respect to \(\|\cdot\|_{\widetilde{A}}\). For this, first note \(B(A)\) is complete by general Banach space arguments (note that \(A\) is in particular a Banach space). Furthermore, \(L(A) \subseteq B(A)\) is closed, since \(A\) is complete and \(\|L_a\| = \|a\|_A\) for all \(a \in A\). We write \(L(\widetilde{A}) = L(A) + C1 \subseteq B(A)\) and we observe that \(L(\widetilde{A})\) is the sum of a closed subspace and a finite-dimensional one. By general arguments from topology [28] Thm. 1.42, this shows that \(L(\widetilde{A})\) is closed; hence \(\widetilde{A}\) is complete.

**Remark 2.21.** The unitization \(\widetilde{A}\) of \(A\) is minimal in the following sense: Let \(B\) be a unital \(C^*\)-algebra and let \(A \triangleleft B\), then there is a unital *-homomorphism \(\varphi: \widetilde{A} \to B\) with \(\varphi(a) = a \in B\) for all \(a \in A\), i.e. \(\varphi\) respects \(A \triangleleft B\).

**Remark 2.22.** The main ingredient of the above unitization was the left multiplication operator \(L_b: A \to A, a \mapsto ba\). One might wonder whether the right multiplication operator is useful, too, and indeed the unitization may also be performed with the right multiplication operator instead. However, using both of them jointly (or rather an abstraction of them) yields yet another unitization, the so called “maximal” one. As the unitization from Prop. 2.20 will be more important for us, we only want to mention that one may define a multiplier algebra \(M(A)\) of \(A\) consisting of pairs \((L, R)\) called double centralizers. These double centralizers are an abstraction of the left and right multiplication. One may show that the multiplier algebra \(M(A)\) is a unital \(C^*\)-algebra and \(A \triangleleft M(A)\). The unitization \(M(A)\) of \(A\) is maximal in that sense: Let \(B\) be a unital \(C^*\)-algebra and let \(A \triangleleft B\), then there is a unital *-homomorphism from \(B\) to \(M(A)\) respecting \(A \triangleleft M(A)\).

Let us come back to the examples of Exm. 2.3.

**Example 2.23.** Here are the unitizations of \(\mathcal{K}(H)\) and \(C_0(X)\).

(a) Let \(H\) be an infinite dimensional Hilbert space. The minimal unitization of the algebra of compact operators is \(\widehat{\mathcal{K}(H)} = C^*(\mathcal{K}(H), 1) \triangleleft B(H)\), the smallest \(C^*\)-subalgebra of \(B(H)\) containing \(\mathcal{K}(H)\) and \(1 \in B(H)\). The maximal unitization (i.e. the multiplier algebra) is \(M(\mathcal{K}(H)) = B(H)\).
(b) Let $X$ be locally compact but not compact. The minimal unitization of the algebra of continuous functions on $X$ is $\widehat{C_0}(X) = C(\hat{X})$, where $\hat{X}$ is the one point compactification of $X$. The maximal unitization is $M(C_0(X)) = \tilde{C}(X) := \{ f : X \to \mathbb{C} \mid f \text{ is continuous and bounded} \}$ which is isomorphic to $C(\beta X)$, where $\beta X$ is the Stone-Čech compactification of $X$.

2.6. Exercises.

**Exercise 2.1.** Check that the algebraic structures of $C^*$-algebras are continuous, i.e. the addition, the multiplication, the involution and also the norm are continuous (see Rem. [2.2]).

**Exercise 2.2.** Convince yourself that $C(X)$ is a unital, commutative $C^*$-algebra (see Exm. [2.3]). Moreover, check that a continuous map $h : X \to Y$ between compact Hausdorff spaces induces a $^*$-homomorphism $\alpha_h : C(Y) \to C(X)$ by $f \mapsto f \circ h$. If $h$ is a homeomorphism, then $\alpha_h$ is even an isometric $^*$-isomorphism.

**Exercise 2.3.** Check that the function $f(z) = z$ in $\mathcal{A}(D)$ is selfadjoint and $\text{sp}(f) = \mathbb{D}$. Later, we will see that any selfadjoint element in a $C^*$-algebra has only real spectral values. Thus, $\mathcal{A}(D)$ is a Banach $^*$-algebra but no $C^*$-algebra (see Exm. [2.3]).

**Exercise 2.4.** Consider the unilateral shift $S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$, $S e_n = e_{n+1}$ from Exc. [1.7].

(a) Show that $\lambda - S$ is invertible for $|\lambda| > 1$.
(b) Show that $S$ has no eigenvalues, i.e. the point spectrum of $S$ is empty.
(c) Show that any $\lambda \in \mathbb{C}$ with $|\lambda| < 1$ is an eigenvalue for $S^*$.
(d) Deduce $\text{sp}(S) = \{ \lambda \mid |\lambda| \leq 1 \}$.

**Exercise 2.5.** A $C^*$-algebra is called **simple**, if it contains no proper closed ideals, i.e. for any closed ideal $I \triangleleft A$ we have $I = 0$ (shorthand for $I = \{0\}$) or $I = A$. Consider $A = M_N(\mathbb{C})$. By $E_{ij} \in M_N(\mathbb{C})$, $i, j = 1, \ldots, N$ we denote the matrix units, i.e. the $i$-$j$-th entry of $E_{ij}$ is one, and zero otherwise.

(a) Let $I \triangleleft M_N(\mathbb{C})$ be a (two-sided) ideal. Show that if $I$ contains a matrix $T = (t_{ij})$ with $t_{i_0j_0} \neq 0$ for some $i_0, j_0 \in \{1, \ldots, N\}$, then $I$ contains $E_{i_0j_0}$.
Multiply $T$ with appropriate matrix units in order to see this.
(b) Show that if $I$ contains some matrix unit $E_{i_0j_0}$, then it contains all matrix units $E_{ij} \in M_N(\mathbb{C})$, $i, j = 1, \ldots, N$. Deduce that $I$ contains $1 = \sum_i E_{ii}$.
(c) Deduce that $M_N(\mathbb{C})$ is simple.

**Exercise 2.6.** Recall the fact used in the proof of Prop. [2.17]: Given a Banach space $A$ and a closed linear subspace $I \subseteq A$, show that $A/I$ is a Banach space.

(a) Show that $\hat{x} + \hat{y}$ and $\lambda \hat{x}$ are well-defined.
(b) Show that $\|\hat{x}\|$ defines a norm and check $\|\hat{x}\| \leq \|x\|$.
(c) Show that $A/I$ is complete with respect to this norm. Hence $A/I$ is a Banach space.
2.7. Some references on $C^*$-algebras. A standard reference for the lectures on $C^*$-algebras could be the book by Blackadar [3]. Additionally, one could use the books by Murphy [20] or Pedersen [23]. The book by Davidson is a quite friendly approach, but slightly more based on examples [10]. Rather modern but more specialized on nuclearity or group actions is the book by Brown and Ozawa [6].

Historically, the books by Dixmier [11] and Sakai [29] are classic in the literature as well as the encyclopedic series of books by Kadison and Ringrose [18, 19, 16, 17] or by Takesaki [30, 31, 32].

The history of $C^*$-algebras goes back to the 1943 paper by Gelfand and Naimark [13], see also the 50 years celebration paper by Kadison [15].

As for the context of “quantum/noncommutative mathematics”, see also the books by Gracia-Bondía, Várilly and Figueroa [14], the one by Wegge-Olsen [36] or the epic book by Connes [7].
3. Gelfand-Naimark Theorem and functional calculus

Abstract. We begin this lecture with recalling the Stone-Weierstrass (Approximation) Theorem. We then turn to homomorphisms of Banach and $C^*$-algebras and the spectrum $\text{Spec}(A)$ of Banach algebras. We put a focus on the commutative case first: We investigate $\text{Spec}(A)$ assuming that $A$ is a commutative, unital Banach algebra. We learn that $\text{Spec}(A)$ provides the full description of the maximal ideal space in this case. We define the Gelfand transform and verify that it is a continuous algebra homomorphism which respects the spectrum $\text{sp}(x)$ of an element $x \in A$. If $A$ is even a commutative, unital $C^*$-algebra, we may prove our First Fundamental Theorem of $C^*$-Algebras (aka Gelfand-Naimark Theorem): The commutative, unital $C^*$-algebras are exactly the algebras of continuous functions on compact spaces. As a consequence, we obtain a very powerful tool for $C^*$-algebras: The (continuous) functional calculus. We investigate some properties of this functional calculus.

3.1. The Stone-Weierstrass Theorem. We begin this lecture with a little excursion to classical analysis. Initially, Weierstrass asked 1895: Given a continuous function $f \in C([0,1])$ – is there a way to approximate $f$ with respect to the supremum norm by “simpler” functions? The simplest functions we might have in mind are polynomials, and the answer is yes. This is the well-known Weierstrass Approximation Theorem. In 1948, Stone realized that very little of the particular structure of $C([0,1])$ was really needed in the proof and he extracted the main algebraic properties providing a way more general statement, which we now prepare.

Definition 3.1. Let $X$ be a compact Hausdorff space. A $^*$-subalgebra $A \subseteq C(X)$ separates the points, if for all $s,t \in X$, $s \neq t$ there is some $f \in A$ with $f(s) \neq f(t)$.

So point separation means that $A$ has sufficiently many functions to “see” that $s$ and $t$ are distinct.

Example 3.2. If $X \subseteq \mathbb{C}$ is compact, then the set of all polynomials in $x$ and its complex conjugate $\bar{x}$ is a unital $^*$-subalgebra of $C(X)$ separating the points.

Recall that $A$ is unital, if the constant function $1 \in C(X)$ is contained in $A$.

Theorem 3.3 (Stone-Weierstrass Theorem). Let $X$ be a compact Hausdorff space and $A \subseteq C(X)$ a closed, unital $^*$-subalgebra separating the points. Then $A = C(X)$.

Proof. We need to show that if $f \in C(X)$, then $f \in A$. In order to do so, let us first show that $A$ is closed under certain operations, the first one being the square root. So, assume that $f \in A$ with $0 \leq f(x) \leq 1$ for all $x \in X$. We put $g := 1 - f$ and we consider the Taylor series expansion of $\sqrt{1-z}$ around $z = 0$. Then

$$\sqrt{f(x)} = \sqrt{1 - g(x)} = 1 - \sum_{n=1}^{\infty} a_n g(x)^n$$

for all $x \in X$ with some coefficients $|a_n| < Cn^{-\frac{3}{2}}$ and some constant $C > 0$. The Taylor series of $\sqrt{1-z}$ converges uniformly on $[-1,1]$. Thus, $h_m := 1 - \sum_{n=1}^{m} a_n g^n$
Definition 3.5. Let \( A \) and \( B \) be Banach algebras and \( C \) valued functions \( C \). Weierstrass proved a real version of Thm. 3.3, so the result holds for \( C \), \( A \) valued functions \( A \). 

Proof. The set \( P \) of all polynomials is a unital \( A \)-subalgebra of \( C([0,1]) \) separating the points; its closure is all of \( C([0,1]) \) by Thm. 3.3. Note that we actually also proved a real version of Thm. 3.3 so the result holds for \( C_\mathbb{R}([0,1]) \), too. 

3.2. Homomorphisms for Banach algebras and \( C^* \)-algebras. We now want to specify the morphisms for our class of objects. Here they are in the case of Banach algebras and \( C^* \)-algebras.

Definition 3.5. Let \( A \) and \( B \) be Banach algebras.

(a) A map \( \varphi : A \to B \) is an (algebra) homomorphism, if \( \varphi \) is linear and multiplicative (i.e. \( \varphi(xy) = \varphi(x)\varphi(y) \) for all \( x, y \in A \)).
(b) Assume that $A$ and $B$ are Banach $*$-algebras. A $*$-homomorphism is a homomorphism $\varphi : A \to B$ which is also involutive, i.e. $\varphi(x^*) = \varphi(x)^*$ holds for all $x \in A$.

(c) A homomorphism is isometric, if $\|\varphi(x)\| = \|x\|$ for all $x \in A$.

(d) Assume that $A$ and $B$ are unital. A homomorphism is unital, if $\varphi(1) = 1$.

The correct notion of a morphism between Banach algebras is a homomorphism in the sense of Def. 3.5 which is also continuous; for $*$-Banach algebras, it is a continuous $*$-homomorphism. Interestingly, we do not have to require continuity for $C^*$-algebras – it is automatic, as we will show next. Let us first prepare a technical lemma on unitizations. Recall that $\tilde{A}$ denotes the (minimal) unitization of $A$, see Prop. 2.20.

**Lemma 3.6.** Let $A, B$ be $C^*$-algebras and $\varphi : A \to B$ a $*$-homomorphism. Then $\tilde{\varphi} : \tilde{A} \to \tilde{B}$ defined as $\lambda + a \mapsto \lambda + \varphi(a)$ is a unital $*$-homomorphism.

**Proof.** Straightforward. \(\square\)

**Definition 3.7** (Compare with Def. 2.4). Let $A$ be a $C^*$-algebra and let $x \in A$. We define the *spectrum* of $x$ as

$$\text{sp}(x) := \begin{cases} \text{sp}_A(x) & \text{if } A \text{ is unital}, \\ \text{sp}_{\tilde{A}}(x) & \text{if } A \text{ is non-unital.} \end{cases}$$

**Lemma 3.8.** Let $A$ and $B$ be $C^*$-algebras and let $\varphi : A \to B$ be a $*$-homomorphism. We have:

(a) If $A, B$ and $\varphi$ are unital, we have that $\text{sp}_B(\varphi(x))$ is contained in $\text{sp}_A(x)$.

(b) $\|\varphi(x)\| \leq \|x\|$ for all $x \in A$.

(c) $\varphi$ is continuous.

**Proof.** For (a), assume that $\lambda - \varphi(x) = \varphi(\lambda - x) \in B$ is not invertible. Hence, $\lambda - x \in A$ cannot be invertible as $\varphi$ maps invertible elements to invertible elements.

As for (b), assume first that $A, B$ and $\varphi$ are unital. We then have $r(\varphi(x)) \leq r(x)$ by (a). Thus, using also Cor. 2.14 we have:

$$\|\varphi(x)\|^2 = \|\varphi(x^*x)\| = r(\varphi(x^*x)) \leq r(x^*x) = \|x^*x\| = \|x\|^2$$

Hence, $\|\varphi(x)\| \leq \|x\|$.

Now, if $A, B$ and $\varphi$ are not necessarily unital, we know by (a) that $\|\tilde{\varphi}(x)\|_B \leq \|x\|_A$ holds for all $x \in \tilde{A}$. For $x \in A \subseteq \tilde{A}$, we infer $\|\varphi(x)\|_B = \|\tilde{\varphi}(x)\|_B \leq \|x\|_A = \|x\|_A$.

Item (c) is then a direct consequence of (b). \(\square\)

**Remark 3.9.** Note that the above statement holds true even if $A$ is just a Banach $*$-algebra whose involution is isometric. Indeed, in the proof of (b) we still have $r(\varphi(x^*x)) \leq r(x^*x)$ and we then use Rem. 2.11 and the submultiplicativity of the norm in order to deduce $r(x^*x) \leq \|x^*x\| \leq \|x\|^2$. With these modifications of the above proof, we infer the above statement (b) also in case $A$ is just a Banach $*$-algebra.
This has an interesting consequence highlighting again the special role of $C^*$-algebras amongst Banach algebras: Let us consider a unital $\ast$-algebra $A$ with two norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Assume that $(A, \|\cdot\|_1)$ is a Banach $\ast$-algebra with an isometric involution and assume that $(A, \|\cdot\|_2)$ is a $C^*$-algebra. Then, the identity map $\varphi : (A, \|\cdot\|_1) \to (A, \|\cdot\|_2)$ is norm decreasing, by the above discussion, i.e. $\|x\|_2 \leq \|x\|_1$. We may interpret this result in the sense that “the $C^*$-norm is the smallest Banach norm”.

It is good to know that $\ast$-homomorphisms between $C^*$-algebras are automatically continuous. They yield the correct notion of morphisms between $C^*$-algebras. In order to have a concept of isomorphism, we need isometric $\ast$-continuous. They yield the correct notion of morphisms between Banach $\ast$-algebras. To fully preserve the norm. Note that isometric $\ast$-homomorphisms are automatically injective – surprisingly, the converse is also true, as we will see later (Lecture 4): Any injective $\ast$-homomorphism between $C^*$-algebras is isometric! Thus, bijective $\ast$-homomorphisms are exactly isomorphisms of $C^*$-algebras.

3.3. Spectrum of a Banach algebra. Let us now turn to a special class of homomorphisms: those mapping to the complex numbers.

**Definition 3.10.** Let $A$ be a Banach algebra. A *character* is a homomorphism $\varphi : A \to \mathbb{C}$ with $\varphi \neq 0$. The set $\text{Spec}(A)$ of all characters is the *spectrum* of $A$.

We deduce some properties of a character directly from the definition.

**Lemma 3.11.** Let $A$ be a unital Banach algebra and let $\varphi \in \text{Spec}(A)$.

(a) $\varphi$ is unital ($\varphi(1) = 1$).
(b) If $x \in A$ is invertible, then $\varphi(x) \neq 0$.
(c) We have $\varphi(x) \in \text{sp}(x)$ for all $x \in A$.
(d) $\varphi$ is continuous and $\|\varphi\| \leq 1$.
(e) If $A$ is a $C^*$-algebra and $x \in A$ is selfadjoint, then $\varphi(x) \in \mathbb{R}$.
(f) If $A$ is a $C^*$-algebra, then $\varphi$ is a $\ast$-homomorphism with $\|\varphi\| = 1$.

**Proof.** For (a), let $x \in A$ with $\varphi(x) \neq 0$; it exists, since $\varphi \neq 0$. Then $\varphi(x) = \varphi(x1) = \varphi(x)\varphi(1)$. Hence $\varphi(1) = 1$.

Item (b) follows from the fact that homomorphisms map invertible elements to invertible elements.

Item (c) is a consequence of (a) and (b): We have $\varphi(\varphi(x)1 - x) = 0$ by (a) and hence $\varphi(x)1 - x$ cannot be invertible by (b).

Also, (d) is immediate: From (c) and Prop. [2.7], we infer $|\varphi(x)| \leq \|x\|$.

For (e), let $x = x^*$ and put $\varphi(x) = \alpha + i\beta \in \mathbb{C}$. Then $\varphi(x) + i\beta = \varphi(x + i\beta)$ by (a) and $|\varphi(x) + i\beta| \leq \|x + i\beta\|$ by (d), for all $\lambda \in \mathbb{R}$. Hence:

$$\alpha^2 + (\lambda + \beta)^2 = |\varphi(x) + i\beta|^2 \leq \|x + i\beta\|^2 = \|(x + i\lambda)^*(x + i\lambda)\| = \|x^2 + \lambda^2\| \leq \|x\|^2 + \lambda^2$$

Thus, $\alpha^2 + 2\lambda\beta + \beta^2 \leq \|x\|^2$, for all $\lambda \in \mathbb{R}$, which implies $\beta = 0$.

For (f), $\|\varphi\| = 1$ follows easily from $\|1\| = 1$ (see Rem. [2.2]) and (a) and (d). For proving that $\varphi$ is a $\ast$-homomorphism, let $x \in A$ and let $\varphi(x) = \alpha + i\beta$ and
\( \varphi(x^*) = \gamma + i\delta \). The elements \( x_1 := x + x^* \in A \) and \( x_2 := i(x - x^*) \in A \) are selfadjoint. We thus have by (e)

\[
(\alpha + \gamma) + i(\beta + \delta) = \varphi(x_1) \in \mathbb{R}, \quad i(\alpha - \gamma) + (\delta - \beta) = \varphi(x_2) \in \mathbb{R},
\]

which implies \( \beta = -\delta \) and \( \alpha = \gamma \) and hence \( \varphi(x)^* = \alpha - i\beta = \varphi(x^*) \). \( \square \)

We now show that \( \text{Spec}(A) \) is a nice topological space.

**Proposition 3.12.** Let \( A \) be a unital Banach algebra. Equipped with the topology of pointwise convergence, \( \text{Spec}(A) \) becomes a compact Hausdorff space.

*Proof (idea):* The proof is not difficult but rather technical. Let us sketch the main ingredients. Consider the pointwise convergence, i.e. a net \( (x_\lambda) \to x \) in \( A \). It is easy to check that if \( \varphi_\lambda \in \text{Spec}(A) \) for all \( \lambda \), then also \( \varphi \in \text{Spec}(A) \). Hence \( \text{Spec}(A) \) is a closed subset of the closed unit ball \( \{ \varphi \in A' \mid \|\varphi\| \leq 1 \} \) of the dual space \( A' \). The unit ball in turn is a closed subset of the product \( P := \prod_{x \in A_1} \{ \lambda \in \mathbb{C} \mid |\lambda| \leq 1 \} \), where \( A_1 \) denotes the closed unit ball in \( A \). Finally, Tychonoff’s Theorem asserts that the cartesian product of compact spaces is compact, hence \( P \) is compact – thus, \( \text{Spec}(A) \subseteq P \) is compact. \( \square \)

### 3.4. Spectrum of a commutative, unital Banach algebra.

Assuming that \( A \) is commutative, \( \text{Spec}(A) \) contains the information of the maximal ideal space of \( A \) as we will see next. Recall the definition of maximal ideals from Def. 2.16.

**Proposition 3.13.**[3] Let \( A \) be a commutative, unital Banach algebra. Then, the following assignment is bijective:

\[
\text{Spec}(A) \to \{ \text{maximal ideals in } A \}, \quad \varphi \mapsto \ker \varphi
\]

*Proof.* Before considering the assertion, let us prove that any maximal ideal in \( A \) can be written as the kernel of a character. In order to do so, we first show that \( A/I \) is a skew field, if \( I \triangleleft A \) is a maximal ideal. So, let \( \pi : A \to A/I \) be the quotient map, and \( a \in A \) with \( \pi(a) \neq 0 \). We need to show that \( \pi(a) \) is invertible. Put

\[
J_a := \{ ba + x \mid b \in A, x \in I \} \subseteq A.
\]

Then \( J_a \) is a two-sided ideal in \( A \), since for \( (ba + x), (b'a + x') \in J_a \) and \( c \in A \) we have \( (ba + x) + (b'a + x') = (b + b')a + (x + x') \in J_a \), \( c(ba + x) = (cb)a + cx \in J_a \) and \( (ba + x)c = (cb)a + xc \in J_a \) (recall that \( A \) is commutative). Furthermore \( J_a \) contains \( I \) (putting \( b = 0 \)) and \( I \neq J_a \) (putting \( b = 1 \) and \( x = 0 \); note that \( \pi(a) \neq 0 \), i.e. \( a \notin I \)). By the maximality of \( I \), we infer that \( J_a = A \), thus there are \( b \in A \) and \( x \in I \) such that \( 1 = ba + x \). This shows that \( \pi(a) \) is left invertible, since \( \pi(b)\pi(a) = \pi(ba + x) = \pi(1) = 1 \). By commutativity, it is also right invertible. Thus, \( \pi(a) \) is invertible for all \( a \in A \) which shows that \( A/I \) is a skew field. Moreover, it is a Banach algebra by Prop. 2.17 so by the Gelfand-Mazur Theorem, 2.9 \( A/I \) is isomorphic to \( \mathbb{C} \), i.e. the quotient map \( \pi \) is actually a character and \( I \) is its kernel.

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[3]This has been Prop. 3.16 in an earlier version.
We now turn to the assignment of the assertion. We first convince ourselves that 
\( \ker \varphi \triangleleft A \) is a maximal ideal, for all \( \varphi \in \text{Spec}(A) \). Given \( \varphi \in \text{Spec}(A) \), it is easy to see that \( \ker \varphi \triangleleft A \) is an ideal. Also, \( \varphi(1) = 1 \) by Lemma 3.11, so \( \ker \varphi \neq A \). By Prop. 2.17, we find a maximal ideal \( I \triangleleft A \) such that \( \ker \varphi \subseteq I \). By the above considerations, we know that \( I = \ker \psi \) for some character \( \psi \). We thus have \( \ker \varphi \subseteq \ker \psi \). But then \( \varphi(a)1 - a \in \ker \varphi \subseteq \ker \psi \) for all \( a \in A \), i.e. \( 0 = \psi(\varphi(a)1 - a) = \varphi(a) - \psi(a) \). Thus, \( \varphi = \psi \), which shows that \( \ker \varphi = I \) is a maximal ideal.

Finally, the assignment is bijective: given a maximal ideal \( I \triangleleft A \), we just showed that it can be written as the kernel of a character; this proves surjectivity. As, for injectivity, let \( \varphi, \psi \in \text{Spec}(A) \) with \( \ker(\varphi) = \ker(\psi) \). Then as above, \( \varphi(a)1 - a \in \ker(\varphi) = \ker(\psi) \) for all \( a \in A \), which implies \( \varphi = \psi \).

Let us harvest a little corollary: We relate the spectrum of a Banach algebra (Def. 3.10) with the spectrum of an element (Def. 2.4); this also justifies the usage of the same name for different objects.

**Corollary 3.14.** \(^2\) Let \( A \) be a commutative, unital Banach algebra and \( x \in A \). Then:

\[
\text{sp}(x) = \{ \varphi(x) \mid \varphi \in \text{Spec}(A) \}
\]

**Proof.** By Lemma 3.11, \( \varphi(x) \in \text{sp}(x) \) for any \( \varphi \in \text{Spec}(A) \). Conversely, let \( \lambda \in \text{sp}(x) \). Then \( I_\lambda := \{ b(\lambda - x) \mid b \in A \} \triangleleft A \) is an ideal in \( A \) (two-sided, by commutativity of \( A \)) and \( 1 \notin I_\lambda \) (since \( \lambda - x \) is not invertible). Hence, it is contained in a maximal ideal by Prop. 2.17, which means \( I_\lambda \subseteq \ker \varphi \) for some \( \varphi \in \text{Spec}(A) \), by Prop. 3.13. Hence, \( \varphi(\lambda - x) = 0 \), i.e. \( \lambda = \varphi(x) \).

### 3.5. Spectrum of a commutative \( C^* \)-algebra.

We investigate the case of commutative \( C^* \)-algebras in detail.

**Lemma 3.15.** Let \( A \) be a commutative \( C^* \)-algebra. Then \( \text{Spec}(A) \) and \( \text{Spec}(\tilde{A}) \setminus \{ \tilde{0} \} \) are homeomorphic, where \( \tilde{0} : \tilde{A} \to \mathbb{C} \) is given by \( \lambda + a \mapsto \lambda \).

**Proof.** We construct a map \( \Psi : \text{Spec}(\tilde{A}) \setminus \{ \tilde{0} \} \to \text{Spec}(A), \psi \mapsto \psi|_A \). Firstly, note that \( A \) is a maximal ideal in \( \tilde{A} \). This is by construction, but we can also argue that \( A = \ker \tilde{0} \) and use Prop. 3.13. Secondly, let \( \psi \in \text{Spec}(\tilde{A}) \), i.e. \( \psi : \tilde{A} \to \mathbb{C} \) is a homomorphism. Then, the restriction \( \psi|_A : A \to \mathbb{C} \) is a homomorphism, too. If \( \psi|_A = 0 \), then \( A \subseteq \ker \psi \). Moreover, \( \ker \psi \neq \tilde{A} \) by definition of \( \text{Spec}(\tilde{A}) \). As \( A \) is a maximal ideal, we infer \( A = \ker \psi \), and hence \( \psi = \tilde{0} \) as the assignment in Prop. 3.13 is bijective. Thus, \( \psi \in \text{Spec}(\tilde{A}) \setminus \{ \tilde{0} \} \) implies \( \psi|_A \neq 0 \) and hence \( \psi|_A \in \text{Spec}(A) \).

On the other hand, let \( \varphi \in \text{Spec}(A) \) and define \( \varphi'(a + \lambda) := \varphi(a) + \lambda \) for \( a \in A \) and \( \lambda \in \mathbb{C} \). This defines a map \( \Phi : \text{Spec}(A) \to \text{Spec}(\tilde{A}) \setminus \{ \tilde{0} \} \) which is inverse to \( \Psi \). Moreover, \( \Phi \) and \( \Psi \) are continuous, so they induce homeomorphisms between \( \text{Spec}(A) \) and \( \text{Spec}(\tilde{A}) \setminus \{ \tilde{0} \} \).

---

\(^2\)This has been Cor. 3.18 in an earlier version.
Proposition 3.16. Let $A$ be a commutative $C^*$-algebra. Then, its spectrum $\text{Spec}(A)$ is locally compact. If $A$ is unital, then $\text{Spec}(A)$ is compact.

Proof. By Prop. 3.12, $\text{Spec}(\tilde{A})$ is compact. Hence, $\text{Spec}(A)$ is locally compact by Lemma 3.15. The statement on the unital case is a special case of Prop. 3.12. □

Remark 3.17. One can define a unitization of a general Banach algebra $A$ and show that it contains $A$ as a maximal ideal. One can thus prove an analogue of Lemma 3.15 and Prop. 3.16 for commutative Banach algebras in general.

So, $\text{Spec}(A)$ seems to be all we need, given $A$ is a commutative $C^*$-algebra. Wait, let us do a quick check with a commutative $C^*$-algebra we know: Let $X$ be a compact Hausdorff space and consider $A = C(X)$, the algebra of continuous functions on $X$. By Exm. 2.3 and Exc. 2.2, we know that $C(X)$ is a commutative, unital $C^*$-algebra.

Proposition 3.18. Let $X$ be a compact Hausdorff space. Then $\text{Spec}(C(X))$ is homeomorphic to $X$.

Proof. The homeomorphism is given by $\Psi : X \to \text{Spec}(C(X))$, $t \mapsto \text{ev}_t$, with $\text{ev}_t : C(X) \to \mathbb{C}$ defined as $\text{ev}_t(f) := f(t)$ for $f \in C(X)$. The fact that it is a homeomorphism is left as Exc. 3.2 for the case of the metric space $X = ([0,1], d)$. For a general Hausdorff space $X$, the proof is analogous, but we need Urysohn’s Lemma for showing that $\Psi$ is injective. □

So, the spectrum contains all information about $C(X)$ and we may fully recover the $C^*$-algebra from its spectrum in that case, nice! As a caveat, note that the spectrum is less helpful, if $A$ is not commutative: For instance, $\text{Spec}(M_n(\mathbb{C})) = \emptyset$, as may be deduced easily from Exc. 2.5. So, the commutativity assumption is really crucial. The moral reason is: As $\varphi(xy) = \varphi(x)\varphi(y) = \varphi(y)\varphi(x) = \varphi(yx)$ for all $\varphi \in \text{Spec}(A)$ of an arbitrary $C^*$-algebra $A$, the spectrum does not "see" any noncommutativity in $A$. Thus, if $A$ is highly noncommutative, this particular information is not contained in $\text{Spec}(A)$.

3.6. Gelfand transform for commutative, unital Banach algebras. We now define one of the most important tools for commutative, unital Banach algebras. Note that the spectrum of such algebras is always non-empty, by Cor. 3.14.

Definition 3.19. Let $A$ be a commutative, unital Banach algebra. The Gelfand transform $\chi : A \to C(\text{Spec}(A))$ is defined by $\chi(x) := \hat{x}$ and $\hat{x}(\varphi) := \varphi(x)$ for $\varphi \in \text{Spec}(A)$.

Lemma 3.20. The Gelfand transform is a continuous, unital algebra homomorphism with $\|\hat{x}\|_{\infty} \leq \|x\|$.

Proof. First note that $\hat{x}$ is continuous with respect to the topology of pointwise convergence on $\text{Spec}(A)$: If $\varphi_\lambda \to \varphi$, then in particular $\varphi_\lambda(x) \to \varphi(x)$. So, $\hat{x} \in$ 

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3This has been Prop. 3.17 in an earlier version.
$C(\text{Spec}(A))$. Checking $\lambda \hat{x} + \mu \hat{y} = \lambda \hat{x} + \mu \hat{y}$ and $\hat{x} \hat{y} = \hat{\hat{xy}}$ is straightforward, since characters are additive and multiplicative; here $x, y \in A$ and $\lambda, \mu \in \mathbb{C}$. By Lemma [3.11] we have $|\hat{x}(\varphi)| = |\varphi(x)| \leq \|x\|$, which implies $\|\hat{x}\|_\infty \leq \|x\|$. The Gelfand transform is unital by Lemma 3.11. □

Lemma 3.21. Let $A$ be a commutative, unital Banach algebra. Then $\|\hat{x}\|_\infty = r(x)$ and $\hat{x}(\text{Spec}(A)) = \text{sp}(x)$.

Proof. By Cor. 3.14, $\hat{x}(\text{Spec}(A)) = \text{sp}(x)$. Thus, $r(x)$ is the supremum over all $|\hat{x}(\varphi)|$ for $\varphi \in \text{Spec}(A)$. This is exactly the definition of the supremum norm $\|\hat{x}\|_\infty$. □

3.7. Gelfand transform for commutative, unital $C^*$-algebras. If $A$ is in fact a commutative, unital $C^*$-algebra, the Gelfand transform is even nicer – it provides us a very powerful isomorphism. Let us quickly prove a lemma first.

Lemma 3.22. Let $A$ and $B$ be $C^*$-algebras and let $\varphi : A \to B$ be an isometric $^*$-homomorphism. Then $\varphi(A) \subseteq B$ is a closed $^*$-subalgebra; in particular $\varphi(A)$ is a $C^*$-algebra.

Proof. Since $\varphi$ is a $^*$-homomorphism, it is clear that $\varphi(A) \subseteq B$ is a $^*$-subalgebra. Moreover, $\varphi(A)$ is closed by a general fact for isometric maps: Let $(\varphi(x_n))_{n \in \mathbb{N}}$ be a Cauchy sequence in $\varphi(A)$. Then $\|x_n - x_m\| = \|\varphi(x_n) - \varphi(x_m)\|$, since $\varphi$ is isometric. Hence, $(x_n)$ is a Cauchy sequence in $A$ and we find a limit $x_n \to x$ since $A$ is complete. Then $\varphi(x_n) \to \varphi(x) \in \varphi(A)$, i.e. $\varphi(A)$ is complete and hence closed. □

We are now ready to prove the First Fundamental Theorem for $C^*$-Algebras. Note that this is not a very common name (we allow ourselves to use this in order to emphasize the role of this theorem) – it is more common to call it (commutative) Gelfand-Naimark Theorem. As both names of the mathematicians are Russian, you will also find Gel’fand or Neumark as alternative spellings.

Theorem 3.23 (1st Fundamental Theorem of $C^*$-Algebras, Gelfand-Naimark 1943). The Gelfand transform is an isometric $^*$-isomorphism for commutative, unital $C^*$-algebras. Hence, we have the following equivalence given a unital $C^*$-algebra $A$:

\[ A \text{ is commutative } \iff \exists X \text{ compact : } A \cong C(X) \]

The space $X$ is then given by $\text{Spec}(A)$. In the non-unital case, we have $A \cong C_0(X)$ for some locally compact space $X$.

Proof. Let $A$ be a commutative, unital $C^*$-algebra. By Lemma 3.20, the Gelfand transform $\chi : A \to C(\text{Spec}(A))$ is a unital algebra homomorphism. By Lemma 3.11 any character $\varphi \in \text{Spec}(A)$ is a $^*$-homomorphism. Hence,

\[ \hat{x}^* = \varphi(x^*) = \varphi(x)^* = (\hat{x}(\varphi))^* \]

Thus, the Gelfand transform is a unital $^*$-homomorphism. It is isometric (and hence also injective), since any element in $A$ is normal, thanks to commutativity. Thus, $\|\hat{x}\|_\infty = r(x)$ by Lemma 3.21 and $r(x) = \|x\|$ by Cor. 2.14.
As for surjectivity, we employ the Stone-Weierstrass (Approximation) Theorem (Thm. 3.3). Note that $X := \text{Spec}(A)$ is a compact Hausdorff space by Prop. 3.12. By Lemma 3.22, $B := \chi(A)$ is a closed *-subalgebra of $C(\text{Spec}(A))$. It remains to show that $B$ separates the points. Let $\varphi, \psi \in \text{Spec}(A)$ be two points in our compact space with $\varphi \neq \psi$. We then find some $\hat{x} \in B$ distinguishing these two points, i.e., $\hat{x}(\varphi) = \varphi(x) \neq \psi(x) = \hat{x}(\psi)$. Hence, we verified all conditions of the Stone-Weierstrass Theorem, and we conclude $\chi(A) = B = C(\text{Spec}(A))$, i.e., $\chi$ is surjective. This settles the unital case.

If $A$ is non-unital, we restrict the isomorphism $\chi : \hat{A} \to C(\text{Spec}(\hat{A}))$ to $A \subseteq \hat{A}$ and we observe that for $x \in A \subseteq \hat{A}$ we have $\hat{x}(0) = 0(x) = 0$. We then conclude that under the Gelfand isomorphism, $A$ is isomorphic to

$$\{ f : \text{Spec}(\hat{A}) \to \mathbb{C} \mid f \text{ is continuous and } f(\hat{0}) = 0 \} \subseteq C(\text{Spec}(\hat{A})).$$

This in turn is isomorphic to $C_0(\text{Spec}(A))$, see also Prop. 3.16. ∎

There is no way to overestimate the importance of the Gelfand-Naimark Theorem; we give a brief laudatio on this theorem at the end of this lecture, see Sect. 3.12.

**Remark 3.24.** The Gelfand-Naimark Theorem is not true for arbitrary Banach algebras. One can check that $\ell^1(\mathbb{Z})$ is a commutative, unital Banach algebra, the multiplication being the convolution and the unit being $(a_n)_{n \in \mathbb{N}}$ with $a_n := \delta_{n0}$. One can show that $\text{Spec}(\ell^1(\mathbb{Z})) = \mathbb{T}$ holds, where $\mathbb{T}$ is the closed unit circle in $\mathbb{C}$. Eventually, one can prove that the Gelfand transform maps $\ell^1(\mathbb{Z})$ to $C(\mathbb{T})$, and it is just the Fourier transform; and $\chi$ is no isomorphism here.

By the way, one can prove one of Wiener’s theorems using the Gelfand transform: If $f \in C(\mathbb{T})$ has an absolute convergent Fourier series and $f(z) \neq 0$ for all $z \in \mathbb{T}$, then also $\frac{1}{f}$ has an absolute convergent Fourier series. For the proof, one needs to investigate the image of $\ell^1(\mathbb{Z})$ under the Gelfand transform. See [28, Lemma 11.6].

### 3.8. $C^*$-subalgebras generated by subsets

Let us prepare some applications of the Gelfand-Naimark Theorem. For doing so, we need to study $C^*$-subalgebras generated by subsets.

**Definition 3.25.** Let $A$ be a $C^*$-algebra and let $M \subseteq A$ be a subset. We denote by $C^*(M)$ the smallest $C^*$-subalgebra of $A$ containing $M$. If $A$ is unital and $x \in A$, we denote by $C^*(x, 1)$ the smallest $C^*$-subalgebra of $A$ containing $x$ and $1$.

Note that $C^*(M)$ is the intersection of all $C^*$-subalgebras $B \subseteq A$ with $M \subseteq B$. If $M = \{x_1, \ldots, x_n\}$ is finite, then $C^*(M)$ is given by the closure of all a priori noncommutative polynomials in $x_i$ and $x_i^\ast$, $i = 1, \ldots, n$. More precisely, a \textit{noncommutative *-monomial} in $x_1, \ldots, x_n$ is an expression

$$x_{i_1}^{k_1}x_{i_2}^{k_2} \cdots x_{i_m}^{k_m},$$

where $i_1, \ldots, i_m \in \{1, \ldots, n\}$ and $k_1, \ldots, k_m \in \{1, *, \ldots, n\}$. In particular, note that $x_1x_2$ might differ from $x_2x_1$. A \textit{noncommutative *-polynomial} is a linear combination of noncommutative *-monomials.
Now, $C^*(x, 1)$ consists in limits of linear combinations of expressions of the form $x^{s_1}(x^*)^{s_2}x^{s_3}(x^*)^{s_4} \cdots x^{s_m}$ with $s_i \in \mathbb{N}_0$. So, we have:

$$C^*(x, 1) = \{ \text{noncommutative polynomials in } x \text{ and } x^* \}$$

We are going to prove that $C^*(x, 1)$ is particularly nice, if $x$ is normal. We first prove a preparatory lemma.

**Lemma 3.26.** Let $A$ be a $C^*$-algebra and let $M \subseteq A$ be a subset. Let $\varphi, \psi : A \to B$ be two $^*$-homomorphisms. If $\varphi(x) = \psi(x)$ for all $x \in M$, then $\varphi(x) = \psi(x)$ for all $x \in C^*(M)$.

**Proof.** Since $\varphi$ and $\psi$ are $^*$-homomorphisms, $B := \{ x \in A \mid \varphi(x) = \psi(x) \} \subseteq A$ is a $C^*$-subalgebra of $A$ containing $M$. Thus $C^*(M) \subseteq B$. \hfill $\square$

Thus, $^*$-homomorphisms are uniquely determined on the generators of $C^*$-algebras. Let us now study $C^*(x, 1)$ when $x$ is normal.

**Lemma 3.27.** Let $A$ be a unital $C^*$-algebra and let $x \in A$ be normal.

(a) $C^*(x, 1) \subseteq A$ is commutative.

(b) Let $y \in C^*(x, 1)$ and $\lambda \in \mathbb{C}$. If $\lambda - y$ is invertible (in $A$), then its inverse belongs to $C^*(x, 1)$. In particular, $\text{sp}_A(y) = \text{sp}_{C^*(x, 1)}(y)$.

(c) The map $\hat{x} : \text{Spec}(C^*(x, 1)) \to \text{sp}(x)$ mapping $\varphi \mapsto \varphi(x)$ is a homeomorphism, i.e. $\text{Spec}(C^*(x, 1)) \cong \text{sp}(x)$ as topological spaces. If $A$ is not necessarily unital, $\text{Spec}(C^*(x))$ is homeomorphic to $\text{sp}(x) \setminus \{0\}$.

**Proof.** For (a), if $x$ is normal, then the noncommutative monomials in $x$ and $x^*$ are of the form $x^k(x^*)^l$ for $k, l \in \mathbb{N}_0$ – and they actually commute. Since arbitrary elements in $C^*(x, 1)$ are limits of linear combinations of such monomials, all elements in $C^*(x, 1)$ commute.

As for (b), let $y \in C^*(x, 1)$ and $\lambda \in \mathbb{C}$ and assume that $\lambda - y$ is invertible, i.e. we have $(\lambda - y)^{-1} \in A$. We need to show $(\lambda - y)^{-1} \in C^*(x, 1)$. We do so by proving that $C^*(x, 1)$ coincides with $B := C^*(\{x, (\lambda - y)^{-1}, 1\}) \subseteq A$.

Observe that $x(\lambda - y) = (\lambda - y)x$, since $C^*(x, 1)$ is commutative by (a). We infer $x(\lambda - y)^{-1} = (\lambda - y)^{-1}x$. Thus, $B$ is commutative and unital. By Thm. 3.23 the Gelfand transform $\chi : B \to C(\text{Spec}(B))$ is an isomorphism, in particular, it is isometric. Thus, also the restriction to $C^*(x, 1) \subseteq B$ is isometric and we infer that $\chi(C^*(x, 1)) \subseteq C(\text{Spec}(B))$ is a closed $^*$-subalgebra by Lemma 3.22. Moreover, it separates the points: Let $\varphi$ and $\psi$ be in $\text{Spec}(B)$ with $\varphi \neq \psi$. Assuming that they coincide on $C^*(x, 1)$, we have $\varphi(\lambda - y) = \psi(\lambda - y)$ in particular. Hence, $\varphi((\lambda - y)^{-1}) = \psi((\lambda - y)^{-1})$, which implies that $\varphi$ and $\psi$ coincide on $B$, by Lemma 3.26 in contradiction to $\varphi \neq \psi$. We infer that $\varphi$ and $\psi$ differ on $C^*(x, 1)$, which means that $\chi(C^*(x, 1))$ separates the points.

By Stone-Weierstrass (Thm. 3.3), $\chi(C^*(x, 1)) = C(\text{Spec}(B))$. From surjectivity of $\chi$ we also have $\chi(B) = C(\text{Spec}(B))$, which yields $C^*(x, 1) = B$ by injectivity of $\chi$. Hence, $(\lambda - y)^{-1} \in B = C^*(x, 1)$. 

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In particular, if $\lambda \in \text{sp}_{C^*(x,1)}(y)$, then $\lambda - y$ has no inverse in $C^*(x,1)$ nor in $A$. Thus, $\text{sp}_{C^*(x,1)}(y) \subseteq \text{sp}_A(y)$, the other inclusion being trivial.

As for (c), note that $\hat{x}$ is continuous by definition. It is surjective by Lemma 3.21 and injective by Lemma 3.26. The inverse map $\hat{x}^{-1}$ is continuous by general facts from topology: Given a closed subset $M \subseteq \text{Spec}(C^*(x,1))$, the set $M$ is compact and so is $\hat{x}(M)$. Hence, $(\hat{x}^{-1})^{-1}(M) = \hat{x}(M) \subseteq \text{sp}(x)$ is closed and $\hat{x}^{-1}$ is continuous.

The non-unital case may be treated accordingly.

3.9. Functional calculus for continuous functions. The preceding lemma looks pretty technical and kind of boring. But it is the key to one of the most powerful applications of the Gelfand-Naimark Theorem: the functional calculus. How comes? Well, if $A$ is a noncommutative $C^*$-algebra, there is no chance to apply the Gelfand-Naimark Theorem. Really? From Lemma 3.27(a) we learn that $C^*(x,1) \subseteq A$ is commutative, if $x$ is normal – so, we may apply Gelfand-Naimark at least locally! And we even learned that $\text{Spec}C^*(x,1)$ is of a pretty nice form: It is $\text{sp}(x)$.

It seems to pay off to study such technicalities: we obtain a very useful tool.

**Theorem 3.28** (Continuous functional calculus). Let $A$ be a unital $C^*$-algebra and $x \in A$ be normal. There is an isometric $^*$-isomorphism $\Phi : C(\text{sp}(x)) \to C^*(x,1) \subseteq A$ mapping $\Phi(1) = 1$.

If $A$ is not unital, we have $\Phi : C_0(\text{sp}(x) \setminus \{0\}) \to C^*(x) \subseteq A$ mapping $\Phi(1) = x$.

**Proof.** By Lemma 3.27(c) and Exc. 2.2, $C(\text{sp}(x))$ is isomorphic to $C(\text{Spec}(C^*(x,1)))$ mapping $f \mapsto f \circ \hat{x}$. By the Gelfand-Naimark Theorem, Thm. 3.23 and Lemma 3.27, $C(\text{Spec}(C^*(x,1)))$ is isomorphic to $C^*(x,1)$ mapping $\hat{x}$ to $x$. The $^*$-isomorphism $\Phi$ (or equivalently $\Phi^{-1}$) is unique by Lemma 3.26. The non-unital case is similar.

Some quick comments: Firstly, note that this is the continuous functional calculus, since it allows us to apply continuous functions to normal elements in $C^*$-algebras.

There are many other functional calculi such as the measurable functional calculus (related to von Neumann algebras), the holomorphic functional calculus (for general Banach algebras) and many others – there has even been an Internetseminar on this subject, ISem21.

How about a polynomial functional calculus? Well, that is trivial. Of course, we are allowed to apply polynomials to elements in $A$: it is an algebra! This also explains the notation $f(x) = \Phi(f)$: For polynomials $f$, the element $\Phi(f) \in A$ is obtained exactly by plugging $x$ into the polynomial $f$. So, the continuous functional calculus is basically the evaluation homomorphism extended from polynomials to continuous functions. And if you trace back the proof of Thm. 3.28 and Thm. 3.23, you will see where this extension comes from: From the Stone-Weierstrass Theorem, Thm. 3.3, our core theorem of such extensions. It all adds up – nice, isn’t it?

Back to concrete math. Here are some properties of the functional calculus.

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4ISem21, Functional Calculus, virtual lectures by Markus Haase (Kiel), https://www.math.uni-kiel.de/isem21/en/course
Proposition 3.29. Let $A$ be a $C^*$-algebra, $x \in A$ be normal, $f,g \in C(\text{sp}(x))$. Then:

(a) $(f+g)(x) = f(x)+g(x)$, $(fg)(x) = f(x)g(x)$, $\bar{f}(x) = f(x)^*$. In particular, $f(x)$ is selfadjoint, if $f$ is real-valued.
(b) $\text{sp}(f(x)) = f(\text{sp}(x))$.
(c) For $h \in C(f(\text{sp}(x)))$, we have $(h \circ f)(x) = h(f(x))$.
(d) If $x \in A$ is selfadjoint, then $\text{sp}(x) \subseteq \mathbb{R}$ and we may decompose $x = x_+ - x_-$ with $x_+x_- = x_-x_+ = 0$ and $\text{sp}(x_+), \text{sp}(x_-) \subseteq [0, \infty)$ and $\|x_+\|, \|x_-\| \leq \|x\|$.
(e) Let $A,B$ be a unital and let $\varphi : A \to B$ be a unital $^*$-homomorphism. Then $\varphi(f(x)) = f(\varphi(x))$.

Proof. Item (a) follows directly from the fact that $\Phi$ from Thm. 3.28 is a $^*$-homomorphism. Items (b) and (c) are left as Exc. 3.3. As for (d), note that $\Phi(\text{id}) = \Phi(\text{id})^* = x^* = x = \Phi(\text{id})$. So $\text{id} = \Phi(\text{id})$, i.e. $\text{sp}(x) \subseteq \mathbb{R}$. Put:

$$h_+(t) := \begin{cases} t & t \geq 0 \\ 0 & \text{otherwise} \end{cases}, \quad h_-(t) := \begin{cases} -t & t \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

Then $h_+$ and $h_-$ are continuous functions on $\text{sp}(x)$ and we may put $x_+ := h_+(x)$, $x_- := h_-(x)$ by functional calculus. Then $\text{id} = h_+ - h_-$ and everything is transferred via $\Phi$ making use of (a) and (b).

Item (e) is clear for polynomials, as $\varphi$ is a $^*$-homomorphism. Let $p_n$ be polynomials approximating $f$ by Stone-Weierstrass. Then $p_n(x) = \Phi(p_n) \to \Phi(f) = f(x)$ and we apply $\varphi$. Note that $\text{sp}(\varphi(x)) \subseteq \text{sp}(x)$, thus $f(\varphi(x))$ exists. □

3.10. An application of the functional calculus. Let us end this lecture with an application, further ones will be treated for instance in the next lecture. Recall the definition of unitary elements from Def. 1.33

Proposition 3.30. Let $A$ be a unital $C^*$-algebra and let $u \in A$ be unitary.

(a) We have $\text{sp}(u) \subseteq S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$.
(b) If $\text{sp}(u) \neq S^1$, then there is a selfadjoint element $x \in A$ such that $u = e^{ix}$, i.e. $u$ can be written in polar coordinates.

Proof. Item (a) is part of Exc. 3.4. As for (b), let $\lambda_0 \in S^1$ with $\lambda_0 \notin \text{sp}(u)$. Let $f$ be a branch of the logarithm mapping $z \in S^1$ to $\vartheta \in \mathbb{R}$ with $e^{i\vartheta} = z$ such that $f$ is continuous on $\text{sp}(u)$. Recall that the logarithm on all of $S^1$ is not continuous, since it needs to jump at some point. But as $\text{sp}(u)$ is an honest subset of $S^1$, we may choose a branch of the logarithm avoiding this jump and we obtain a continuous function $f$ on $\text{sp}(u)$. This means, we are allowed to apply the functional calculus and we put $x := f(u) \in A$. The element $x$ is selfadjoint, by Prop. 3.29(a). Since $\text{id} = e^{if}$ as functions on $\text{sp}(u)$, we have $u = e^{ix}$. □
3.11. Exercises.

Exercise 3.1. Denote the circle by \( S^1 := \{ z \in \mathbb{C} \mid |z| = 1 \} \). Show that we may approximate functions in \( C(S^1) \) by polynomials \( p(z) = \sum_{n=-N}^{N} a_n z^n \), where \( a_n \in \mathbb{C} \) and \( z^{-n} = (\bar{z})^n \) for \( n > 0 \).

One can use this result to show that the functions \( e_n(t) := \frac{1}{\sqrt{2\pi}} e^{int} \), \( t \in [0, 2\pi] \), \( n \in \mathbb{Z} \) form an orthonormal basis for \( L^2([0, 2\pi]) \). This also shows that \( L^2([0, 2\pi]) \) is isomorphic to \( \ell^2(\mathbb{N}) \) as a Hilbert space.

Exercise 3.2. Consider the metric space \( ([0, 1], d) \) with the usual metric \( d(s, t) = |s - t| \). Then \( C([0, 1]) \) is a commutative, unital Banach algebra (in fact, even a \( C^* \)-algebra) by Exm. 2.3 and Exc. 2.2. We want to compute its spectrum. For \( t \in [0, 1] \), define \( ev_t : C([0, 1]) \to \mathbb{C} \) via \( ev_t(f) := f(t) \) for \( f \in C([0, 1]) \). Consider \( \Psi : [0, 1] \to \text{Spec}(C([0, 1])), t \mapsto ev_t \).

(a) Show that \( ev_t \) is in \( \text{Spec}(C([0, 1])) \) for all \( t \in [0, 1] \).
(b) Show that \( \Psi \) is injective. Use \( f_s(y) := d(s, y) = |s - y| \).
(c) Let \( I \triangleleft C([0, 1]) \) be an ideal such that for any \( t \in [0, 1] \) there is some \( f \in I \) with \( f(t) \neq 0 \). Show that \( I = C([0, 1]) \). In order to do so, use compactness of \([0, 1]\) to find an invertible element in \( I \); then use Prop. 2.17(c).
(d) Show that \( \Psi \) is surjective: Given \( \varphi \in \text{Spec}(C([0, 1])) \), use (c) and Prop. 3.13 in order to show that \( \ker \varphi = \ker ev_t \) for some \( t \in [0, 1] \). Observe that \( \ker ev_t = \{ f \in C([0, 1]) \mid f(t) = 0 \} \), for \( t \in [0, 1] \).
(e) Show that \( \Psi \) is continuous. And deduce that \( \Psi^{-1} \) is also continuous, by some general topological argument. Deduce that \([0, 1]\) and \( \text{Spec}(C([0, 1])) \) are homeomorphic (i.e. “the same” as topological spaces).

Exercise 3.3. Prove items (b) and (c) of Prop. 3.29 For (c), study \( A := \{ h \in C(\text{sp}(f(x))) \mid \Phi_1(h \circ f) = \Phi_2(h) \} \subseteq C(\text{sp}(f(x))) \), where \( \Phi_1 \) and \( \Phi_2 \) are suitable functional calculi.

Exercise 3.4. Let \( A \) be a unital \( C^* \)-algebra.

(a) Show that if \( x \in A \) is invertible, then \( \text{sp}(x^{-1}) = \{ \lambda^{-1} \mid \lambda \in \text{sp}(x) \} \).
(b) Show that if \( u \in A \) is unitary in the sense of Def. 1.33 then \( \|u\| = \|u^*\| = 1 \).

Use (a) to deduce that \( \text{sp}(u) \subseteq S^1 \).

Exercise 3.5. Let \( A \) be a unital \( C^* \)-algebra and let \( x \in A \) be selfadjoint. The following are easy statements needed in the next lecture.

(a) Recall that \( x \) is invertible if and only if \( 0 \not\in \text{sp}(x) \).
(b) Let \( x \) be invertible. Show that \( x^{-1} \) is selfadjoint.
(c) Let \( \text{sp}(x) \subseteq (0, \infty) \). Use the functional calculus to show \( \text{sp}(x^{-1}) \subseteq (0, \infty) \).
(d) Show that if \( f, g \in C(\text{sp}(x)) \), then \( f(x) \) and \( g(x) \) commute; in particular, \( f(x) \) and \( x \) commute.
(e) Show that \( \text{sp}(x - 1) \subseteq [0, \infty) \) if and only if \( \text{sp}(x) \subseteq [1, \infty) \).
(f) Show that if \( \text{sp}(x) \subseteq [1, \infty) \), then \( x \) is invertible and \( \text{sp}(1 - x^{-1}) \subseteq [0, \infty) \).
3.12. **Some comments on Gelfand duality.** Let us elaborate more on the philosophical impact of the Gelfand-Naimark Theorem, see also [34] for some easy account. What does Gelfand-Naimark say exactly? A unital $C^*$-algebra is commutative if and only if it is an algebra of continuous functions on a compact space. So, what does it say philosophically? It says that compact topological spaces are in “duality” with commutative, unital $C^*$-algebras. What is this famous “Gelfand duality” about?

Let us speak in the language of category theory first. Recall from Exc. 2.2 that given a continuous map $h : X \rightarrow Y$ between compact Hausdorff spaces, we obtain a $^*$-homomorphism $\alpha_h : C(Y) \rightarrow C(X)$ by composition with $h$: We map $f \mapsto f \circ h$. So, a morphism on the level of topological space induces a morphism on the level of $C^*$-algebras. One can push this further and show that there is an equivalence of categories between the category of commutative, unital $C^*$-algebras and the category of compact topological spaces, induced by the functors $A \mapsto \text{Spec}(A)$ on the one hand and $X \mapsto C(X)$ on the other. See [1] for more on this.

Now, in a more philosophical language, we may say that commutative $C^*$-algebras correspond to (classical) topology – while noncommutative $C^*$-algebras correspond to a kind of “noncommutative (or quantum) topology”. Indeed, Gelfand-Naimark tells us, that within the class of all (possibly noncommutative) $C^*$-algebras, the commutative ones are exactly those coming from classical topology. Hence, the others, noncommutative ones must correspond to some noncommutative topology. Therefore, we sometimes view noncommutative $C^*$-algebras as algebras of functions on some “noncommutative spaces”. And it makes sense to do so! As mathematicians, we need precise objects, precise definitions and precise theorems – but we also need some intuition! So, intuitively, we may want to think of noncommutative $C^*$-algebras as function algebras on some underlying noncommutative spaces – which do not “exist” in a precise way, but indirectly, via their function algebras.

Why is this way of thinking useful? Because it allows us to transfer concepts, ideas and possibly even techniques from the “classical” world to the “noncommutative” one. If you want to learn more about such a noncommutative topology, you may take a look at the nice noncommutative topology dictionaries in [14 Introduction to Ch. 1 + end of Sect. 1.3], [36 Sect. 1.11] revealing the dual concepts to connected components, closed subsets, compactifications etc. within noncommutative topology aka the theory of $C^*$-algebras.

Moreover, this Gelfand-Naimark philosophy (commutativity corresponds to the classical world, noncommutativity to the quantum world) is the basis also for other quantum theories: Murray-von Neumann’s von Neumann algebras, Voiculescu’s Free Probability Theory, Connes’s Noncommutative Geometry and Woronowicz’s Quantum Groups – they all share the same philosophy about the role of commutativity as a classical counterpart. Nowadays, the following areas may be counted to such a “non-commutative analysis” or “quantum mathematics”: 
Classical theory | Quantum/noncomm. version | Founders and pioneers
--- | --- | ---
Topology | $C^*$-Algebras | Gelfand-Naimark 1940s
Measure Theory | von Neumann Algebras | Murray-von Neumann 1930s
Probability Theory | Free Probability Theory, Quantum Probability Theory | Voiculescu 1980s

| Topology | $C^*$-Algebras | Gelfand-Naimark 1940s
--- | --- | ---
Measure Theory | von Neumann Algebras | Murray-von Neumann 1930s
Probability Theory | Free Probability Theory, Quantum Probability Theory | Voiculescu 1980s

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In many of the above theories, there is an analog of the Gelfand-Naimark Theorem. We conclude that quantum/noncommutative mathematics follows the philosophy:

\[
\text{commutative algebras} \leftrightarrow \text{classical situation} \\
\text{noncommutative algebras} \leftrightarrow \text{quantum/noncommutative situation}
\]

By the way, the name “$C^*$-algebra” has been coined by Segal in 1947, the letter “C” referring to “closed” *-subalgebras of $B(H)$ as a major example of $C^*$-algebras, see Exm. L3. Gelfand and Naimark themselves used the term “normed *-ring” in their 1943 article. For more on the history of $C^*$-algebras and the Gelfand-Naimark Theorem, see [15].
4. Positive elements, approximate units

Abstract. In this lecture, we turn to one of the key features of $C^*$-algebras: positivity. We define the notion of positive elements in a $C^*$-algebra and we prove the very important algebraic characterization that positive elements are exactly elements of the form $x^*x$. We show that the set of positive elements of a $C^*$-algebra forms a cone and we derive a number of useful observations on the induced order structure. From positivity, the amazing fact may be deduced that any injective $*$-homomorphism is already isometric. Besides, we observe that every positive element has a unique positive square root.

We then introduce the concept of approximate units and we show that any $C^*$-algebra and any ideal in a $C^*$-algebra possesses an approximate unit. This allows us to conclude that the quotient of a $C^*$-algebra by a closed two-sided ideal is a $C^*$-algebra again. Thus, the theory of $C^*$-algebras is admissible for homological tools such as short exact sequences.

From now on, we leave the general framework of Banach algebras and in the remaining lectures we will deal with the particular subclass of $C^*$-algebras only.

4.1. Definition of positive elements and sums of positive elements. In the first lecture, we briefly mentioned the importance of the $C^*$-identity $\|x\|^2 = \|x^*x\|$ for $C^*$-algebras. Let us now explore in detail some powerful consequences of this harmless identity. The main aspect is that it introduces positivity to $C^*$-algebras.

Definition 4.1. Let $A$ be a $C^*$-algebra. An element $x \in A$ is positive (we write $x \geq 0$), if $x = x^*$ and $\text{sp}(x) \subseteq [0, \infty)$. Recall from Prop. 3.29, that $\text{sp}(x) \subseteq \mathbb{R}$ whenever $x = x^*$. So, positive elements form a natural subclass of selfadjoint elements – their spectrum is positive! Note that we already encountered positivity in Prop. 3.29: We saw that any selfadjoint element may be decomposed as $x = x_+ - x_-$ with $x_+$ and $x_-$ being positive. So, if $x$ is positive itself, then $x = x_+$ in the sense of Prop. 3.29. Moreover, $*$-homomorphisms respect positivity, by Lemma 3.8.

Let us now characterize positivity via some norm estimate. This is basically the functional calculus point of view on positivity.

Lemma 4.2. Let $A$ be a unital $C^*$-algebra and let $x \in A$ be selfadjoint. Let $\lambda \geq \|x\|$. We have:

$$x \geq 0 \iff \|\lambda 1 - x\| \leq \lambda$$

In particular, if $x \geq 0$, then $\|x\| = \inf \{\lambda \geq 0 \mid \lambda 1 - x \geq 0\}$.

Proof. Let $x$ be selfadjoint and $\lambda \geq \|x\|$. Let us first work in $C(\text{sp}(x))$. Let $\text{id}$ be the identity function and $1$ the constant function both on $\text{sp}(x) \subseteq \mathbb{R}$. From Prop. 2.7, we infer that $\text{sp}(x) \subseteq [-\lambda, \lambda]$. We observe that $\text{sp}(x) \subseteq [0, \infty)$ holds if and only if $\|\lambda 1 - \text{id}\|_\infty = \sup \{|\lambda - \mu| \mid \mu \in \text{sp}(x)\} \leq \lambda$, as a statement on functions in $C(\text{sp}(x))$. 

Coming back to $x \in A$, we employ the functional calculus. As it is isometric, we conclude that $x$ is positive if and only if $\| \lambda 1 - x \| = \| \lambda 1 - \text{id} \|_\infty \leq \lambda$.

If now $x \geq 0$ – i.e. $\text{sp}(x) \subseteq [0, \infty)$ – then $\lambda 1 - \text{id} \geq 0$ as a function on $\text{sp}(x)$ if and only if $\lambda \geq \| \text{id} \|_\infty$. Thus, $\| \text{id} \|_\infty = \inf \{ \lambda \geq 0 \mid \lambda 1 - \text{id} \geq 0 \}$. Now, by Prop. 3.29 $\lambda 1 - \text{id} \geq 0$ if and only if $\lambda 1 - x \geq 0$ and we infer $\| x \| = \inf \{ \lambda \geq 0 \mid \lambda 1 - x \geq 0 \}$. □

As an easy consequence, we see that sums of positive elements are positive.

**Lemma 4.3.** Let $A$ be a $C^*$-algebra and $x, y \in A$ be positive. Then $x + y \geq 0$.

**Proof.** Assume first that $A$ is unital. Put $\lambda := \| x \| + \| y \|$. By the triangle inequality, $\lambda \geq \| x + y \|$. By Lemma 4.2, $\| \| x \| - x \| \leq \| x \|$ and $\| \| y \| - y \| \leq \| y \|$. Then
\[
\| \lambda - (x + y) \| \leq \| \| x \| - x \| + \| \| y \| - y \| \leq \| x \| + \| y \| = \lambda
\]
and we apply Lemma 4.2 again.

If $A$ is not unital, we view $x$ and $y$ as elements in $\tilde{A}$. Then, they are also positive in $\tilde{A}$, see also Def. 3.7, and we conclude that $x + y \in A \subseteq \tilde{A}$ is positive. □

**4.2. Positive square root.** While the functional calculus was the key to the above characterization of positivity, it also allows us to write expressions such as
\[
\sqrt{T} = \sqrt{\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}} \quad \text{for} \quad T = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} \in M_2(\mathbb{C}).
\]

What exactly do we mean by it? Let us clarify it in the next proposition.

**Proposition 4.4.** Let $A$ be a $C^*$-algebra and $x \in A$ be positive. Then there is a unique positive element $y \in A$ such that $y^2 = x$, i.e. any positive element $x$ possesses a unique positive square root $\sqrt{x}$.

**Proof.** The existence is by functional calculus: Since $\sqrt{\cdot}$ is continuous on $[0, \infty)$, we are allowed to put $y := \sqrt{x}$ by functional calculus and we check $y = y^*$ and $\text{sp}(y) = \sqrt{\text{sp}(x)} \subseteq [0, \infty)$ and $y^2 = x$ using Prop. 3.29.

As for uniqueness, assume first that $A$ is unital. Let $\tilde{y} \in A$ be positive with $\tilde{y}^2 = x$. If $\tilde{y} \in C^*(x, 1)$, then we may find a positive function $\tilde{f} \in C(\text{sp}(x))$ with $\tilde{f}(x) = \tilde{y}$ by the Gelfand-Naimark Thm. But then $\tilde{f}^2 = \text{id}$, i.e. $\sqrt{\cdot}$ and $\tilde{y} = y$. We are left with proving that for general $\tilde{y} \in A$, we always have $\tilde{y} \in C^*(x, 1)$.

It is clear that $C^*(x, 1) \subseteq C^*(\tilde{y}, 1)$ as $x = \tilde{y}^2$. Now, $\tilde{y}$ is normal and we have a functional calculus $\tilde{\Phi} : C(\text{sp}(\tilde{y})) \to C^*(\tilde{y}, 1)$. Then, $\tilde{\Phi}^{-1}(C^*(x, 1)) \subseteq C(\text{sp}(\tilde{y}))$ is a unital closed *-subalgebra by Lemma 3.22. Let us show that $\tilde{\Phi}^{-1}(C^*(x, 1))$ separates the points. Recall $\text{sp}(\tilde{y}) \cong \text{Spec}(C^*(\tilde{y}, 1))$ from Lemma 3.27. So, let $\varphi_1, \varphi_2 \in \text{Spec}(C^*(\tilde{y}, 1))$ with $\varphi_1 \neq \varphi_2$. Then $\varphi_1(\tilde{y}) \neq \varphi_2(\tilde{y})$ by Lemma 3.26. This implies $\varphi_1(x) \neq \varphi_2(x)$ since $\tilde{y}^2 = x$. We conclude that $\tilde{\Phi}^{-1}(C^*(x, 1))$ separates the points. Thus, $\tilde{\Phi}^{-1}(C^*(x, 1)) = C(\text{sp}(\tilde{y})) = \tilde{\Phi}^{-1}(C^*(\tilde{y}, 1))$ by Stone-Weierstrass (Thm. 3.3) and hence $C^*(x, 1) = C^*(\tilde{y}, 1)$ which shows $\tilde{y} \in C^*(x, 1)$. 

Now, assume that $A$ is not unital. We then consider $x \in A \subseteq \tilde{A}$ and we just showed, that $x$ has as a unique positive square root in $\tilde{A}$ – so, there cannot be another positive square root in $A \subseteq \tilde{A}$. □

Coming back to our above example, our knowledge from linear algebra allows us to compute the eigenvalues (i.e. the spectrum) of the given $2 \times 2$ matrix $T$ and we deduce that it is positive. We then have by Prop. 4.4

$$\sqrt{T} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \in M_2(\mathbb{C}).$$

4.3. **Algebraic characterization of positivity.** We now turn to the most important subsection of Lecture 4: The algebraic characterization of positivity. We want to show that positive elements are exactly those of the form $x^*x$.

As a motivation, we interpret Exc. 4.1 as a hint. There, we show that an operator $T \in B(H)$ is positive if and only if $\langle Tx, x \rangle \geq 0$ for all $x \in H$. As a consequence, any operator $T = S^*S \in B(H)$ is positive. The same holds true in arbitrary $C^*$-algebras, as we are going to show soon. Let us prepare some technical tools.

**Lemma 4.5.** Let $A$ be a $C^*$-algebra and let $x, y \in A$. Then

$$\text{sp}(xy) \cup \{0\} = \text{sp}(yx) \cup \{0\}.$$  

**Proof.** The proof is left as Exc. 4.3 □

**Definition 4.6.** Let $A$ be a $C^*$-algebra and let $x \in A$. The real part $\text{Re}(x)$ and the imaginary part $\text{Im}(x)$ of $x$ are defined as follows:

$$\text{Re}(x) := \frac{x + x^*}{2}, \quad \text{Im}(x) := \frac{x - x^*}{2i}.$$  

We may then decompose $x = \text{Re}(x) + i\text{Im}(x)$.

Note that the real and imaginary parts are defined in clear analogy to the situation in $\mathbb{C}$. The whole point of the above decomposition of $x$ is that $\text{Re}(x)$ and $\text{Im}(x)$ are selfadjoint. We learned already that selfadjoint (or weaker: normal) elements behave particularly nice – for instance, they are admissible for functional calculus!

**Lemma 4.7.** Let $A$ be a $C^*$-algebra and $x \in A$. If $-x^*x \geq 0$, then $x = 0$.

**Proof.** We may assume that $A$ is unital; otherwise, we consider $x \in \tilde{A}$. We decompose $x = x_1 + ix_2$ in its real and imaginary part as in Def. 4.6. We then have $x^* = x_1 - ix_2$ and thus:

$$x^*x + xx^* = (x_1^2 + ix_1x_2 - ix_2x_1 + x_2^2) + (x_1^2 + ix_2x_1 - ix_1x_2 + x_2^2) = 2x_1^2 + 2x_2^2$$

By functional calculus and Prop. 3.29(b), we know that $\text{sp}(2x_j^2) \subseteq [0, \infty)$ for $j = 1, 2$. Using Lemma 4.3, we then infer that $xx^* = 2x_1^2 + 2x_2^2 + (-x^*x)$ is positive as a sum of three positive elements. By Lemma 4.5, we thus infer

$$\text{sp}(x^*x) \subseteq \text{sp}(x^*x) \cup \{0\} = \text{sp}(xx^*) \cup \{0\} \subseteq [0, \infty).$$
On the other hand, $-x^*x \geq 0$, i.e. $\text{sp}(x^*x) \subseteq (-\infty, 0]$ using Prop. 3.29(b) again. Thus, $\text{sp}(x^*x) = \{0\}$ which implies

$$\|x\|^2 = \|x^*x\| = r(x^*x) = 0$$

by Cor. 2.14. This shows $x = 0$. □

We are now ready for the main result of this lecture: The algebraic characterization of positivity.

**Theorem 4.8.** Let $A$ be a C*-algebra and $x \in A$. The following are equivalent.

(a) $x \geq 0$.

(b) There is a selfadjoint element $y \in A$ with $y^2 = x$.

(c) There is an element $z \in A$ with $x = z^*z$.

**Proof.** The implication from (a) to (b) is by Prop. 4.4, putting $y := \sqrt{x}$. The step from (b) to (c) is trivial, putting $z := y$.

Now assume that $x$ is of the form $x = z^*z$. We want to show that $x$ is positive.

We decompose $x$ as $x = x_+ - x_-$ with $x_+, x_- \geq 0$ and $x_+ x_- = x_- x_+ = 0$, see Prop. 3.29. By functional calculus, we infer $x_+^3 \geq 0$. Put $y := zx_-$. Then:

$$-y^*y = -x_- z^* zx_- = -x_-(x_+ - x_-)x_- = x_-^3 \geq 0$$

By Lemma 4.7, $y = 0$, which implies $x_-^3 = 0$. Thus also $x_- = 0$ by functional calculus (or as $\|x_-\|^4 = \|x_-^4\|$, see for instance the proof of Cor. 2.14). We conclude $x = x_+ \geq 0$. □

4.4. **Induced partial order structure.** As a corollary of the above theorem, we obtain a partial order structure on C*-algebras.

**Corollary 4.9.** Let $A$ be a C*-algebra. We put

$$A_{sa} := \{x \in A \mid x = x^*\} \subseteq A, \quad A_+ := \{x \in A \mid x \geq 0\} \subseteq A_{sa} \subseteq A.$$  

Then $A_+$ is a convex cone, i.e. we have:

(i) If $x \in A_+$ and $\lambda \geq 0$, then $\lambda x \in A_+$.

(ii) If $x, y \in A_+$, then $x + y \in A_+$.

Moreover, $A_+ \cap (-A_+) = \{0\}$, $A_{sa} = A_+ - A_+$ and $A_+$ is (topologically) closed.

**Proof.** Item (i) is by functional calculus. Item (ii) is by Lemma 4.3. The assertion $A_+ \cap (-A_+) = \{0\}$ is by Lemma 4.7 and Thm. 4.8. The decomposition $A_{sa} = A_+ - A_-$ is by Prop. 3.29. The cone $A_+$ is closed as an easy consequence of Lemma 4.2. □

**Definition 4.10.** For elements $x, y \in A$ in a C*-algebra, we write $x \leq y$, if $y - x \geq 0$.

The order structure defined above is a partial order (reflexive, antisymmetric, transitive), by Cor. 4.9.

Note that *-homomorphisms respect the order structure. Indeed, let us recap that *-homomorphisms preserve positivity, as we mentioned before. We can employ
Lemma [3.8] for a proof, but with the characterization in Thm. 4.8 it is even easier: Since any positive element is of the form $x^*x$, it is clear that also $\varphi(x^*x) = \varphi(x)^*\varphi(x)$ is positive, provided that $\varphi : A \to B$ is a *-homomorphism and $x \in A$. Then, of course, preservation of positivity implies preservation of the order structure, so $x \leq y$ implies $\varphi(x) \leq \varphi(y)$.

Remark 4.11. For some instances, the preservation of the positivity structure (aka the order structure) on $C^*$-algebras is so important that also weakenings of *-homomorphisms are considered: A linear map $\varphi : A \to B$ between $C^*$-algebras is called positive, if it maps positive elements to positive elements, so $x \geq 0$ implies $\varphi(x) \geq 0$. If such a positive map respects some matrix structure over $A$ and $B$, it is called completely positive. Completely positive maps are generalizations of *-homomorphisms and a key ingredient in the theory of nuclear $C^*$-algebras [6].

Let us now prove some properties of the order structure.

Proposition 4.12. Let $A$ be a $C^*$-algebra and let $x, y \in A$.

(a) If $x \leq y$, then $z^*xz \leq z^*yz$ for all $z \in A$.
(b) If $x \geq 0$, then $\|x\| = \inf\{\lambda \geq 0 \mid \lambda x \geq x\}$. In particular $x \leq \|x\|1$.
(c) If $0 \leq x \leq y$, then $\|x\| \leq \|y\|$.
(d) If $A$ is unital, $0 \leq x \leq y$ and $x, y$ are invertible, then $0 \leq y^{-1} \leq x^{-1}$.
(e) If $0 \leq x \leq y$ and $\beta \in [0, 1]$, then $0 \leq x^\beta \leq y^\beta$, in particular $0 \leq \sqrt{x} \leq \sqrt{y}$.

Proof. Item (a) is easy: We use Thm. 4.8 in order to write $y - x = w^*w$ and we infer $z^*yz - z^*xz = z^*(y - x)z = (wz)^*(wz) \geq 0$, again by Thm. 4.8.

Item (b) follows directly from Lemma 4.2. Note that we view $A \subseteq \hat{A}$ here, in case $A$ is not unital.

Item (c) follows from (b).

As for (d), we use Exc. 3.5(d) in order to see $x^{-1} \geq 0$. Thus, the expression $\sqrt{x^{-1}}$ makes sense and this element commutes with $x$, by functional calculus (see Exc. 3.5(d)). Thus,

$$1 = \sqrt{x^{-1}}x\sqrt{x^{-1}} \leq \sqrt{x^{-1}y\sqrt{x^{-1}}}$$

by (a). Put $z := \sqrt{xy^{-1}}\sqrt{x}$ and observe that

$$z = \sqrt{xy^{-1}}\sqrt{x} = (\sqrt{x^{-1}y\sqrt{x^{-1}}})^{-1} \leq 1.$$

For the latter inequality, we used that $w \geq 1$ implies $w^{-1} < 1$ for any self-adjoint element $w$, again by functional calculus, see Exc. 3.5(f). We conclude $y^{-1} = \sqrt{x^{-1}}z\sqrt{x^{-1}} \leq \sqrt{x^{-1}}\sqrt{x^{-1}} = x^{-1}$, where we used (a).

Finally, (e) is more complicated and we omit a complete proof; see [23, Prop. 1.3.8], [3, Prop. II.3.1.10]. The idea is to write the real-valued function $[0, \infty) \ni t \mapsto g(t) := t^\beta$ as an integral $g(t) = \frac{1}{\gamma} \int_0^\infty f_\alpha(t) \alpha^{-\beta} \, \text{d} \alpha$, where $f_\alpha(t) = \frac{t}{1+\alpha t}$, $\alpha > 0$ and $\gamma > 0$. One may then check $f_\alpha(y) - f_\alpha(x) \geq 0$ and derive $y^\beta - x^\beta \geq 0$ from the integral presentation. \[\Box\]
Remark 4.13. Item (e) of Prop. 4.12 is not true in general for $\beta > 1$. In particular, an implication from $0 \leq x < y$ to $0 \leq x^2 \leq y^2$ is wrong in general. We even have: If there is some $\beta > 1$ such that for all $x, y \in A$ with $0 \leq x \leq y$ the inequality $x^\beta \leq y^\beta$ holds, then $A$ must be commutative. So, we must be careful in our intuition for order relations as in item (e) and we shall not be driven by our understanding from the commutative case.

4.5. Injective $^*$-homomorphisms are isometric. Let us now come to an amazing fact: For $^*$-homomorphisms between $C^*$-algebras, injectivity implies the preservation of the norm! Before we prove this nice result, let us mention a little lemma.

Lemma 4.14. Let $A$ be a unital $C^*$-algebra and let $x \in A$ with $x \geq 0$. Then $\|x\| \in \text{sp}(x)$.

Proof. Since $x$ is selfadjoint, we have $r(x) = \|x\|$ by Cor. 2.14. Hence, $\|x\| \in \text{sp}(x)$ or $-\|x\| \in \text{sp}(x)$ by the definition of the spectral radius. Now, $\text{sp}(x) \subseteq [0, \infty)$ since $x$ is positive, so $\|x\| \in \text{sp}(x)$ must hold. \qed

Proposition 4.15. Let $A, B$ be unital $C^*$-algebras and $\varphi : A \to B$ a $^*$-homomorphism. Then, $\varphi$ is injective if and only if $\varphi$ is isometric (i.e. $\|\varphi(x)\| = \|x\|$ for all $x \in A$).

Proof. If $\varphi$ is isometric, then it is clearly also injective. So, let us prove the converse, by contraposition. Assume that $\varphi$ is not isometric. Hence, there is some $x \in A$ such that $\|\varphi(x^*)\| = \|\varphi(x)\|^2 < \|x\|^2 = \|x^*x\|$. Here, we used that $\|\varphi(x)\| \leq \|x\|$ holds by Lemma 3.8.

Let $f : \text{sp}(x^*) \to \mathbb{R}$ be a continuous function with $0 \leq f \leq 1$ which is zero on $[0, \|\varphi(x^*)\|]$ and $f(\|x^*x\|) = 1$. Then $\|f\|_\infty = 1$, since the supremum $\|f\|_\infty$ is taken over the set $\text{sp}(x^*) \subseteq [0, \|x^*x\|]$ and we have $\|x^*x\| \in \text{sp}(x^*)$ by Lemma 4.14. Then $\|f(x^*)\| = \|f\|_\infty = 1$ by the functional calculus for $x^*x$. In particular $f(x^*) \neq 0$ holds.

It remains to show that $\varphi(f(x^*)) = 0$, which then implies that $\varphi$ is not injective. From, $\text{sp}(\varphi(x^*)) \subseteq [0, \|\varphi(x^*)\|]$ we infer that $\text{sp}(f(\varphi(x^*))) = f(\text{sp}(\varphi(x^*))) = \{0\}$ holds by Prop. 3.29(b). Then $\|f(\varphi(x^*))\| = r(f(\varphi(x^*))) = 0$ using Cor. 2.14. Hence, $\varphi(f(x^*)) = f(\varphi(x^*)) = 0$ by Prop. 3.29(e). \qed

4.6. Definition of approximate units. We now come to a different topic: to approximate units. We learned already that some $C^*$-algebras do not possess a unit, see Exm. 2.3. In the sequel, we will show that they always possess at least an approximate unit. We first recall the notion of a net.

Definition 4.16. Let $X$ be a topological space. A family $(x_\lambda)_{\lambda \in \Lambda} \subseteq X$ is called a net, if $\Lambda$ is a partially ordered, directed set, i.e. there is a relation $\leq$ on $\Lambda$ such that for all $\lambda, \mu, \nu \in \Lambda$:

1. $\lambda \leq \lambda$
2. If $\lambda \leq \mu$ and $\mu \leq \lambda$, then $\lambda = \mu$. 
(iii) If \( \lambda \leq \mu \) and \( \mu \leq \nu \), then \( \lambda \leq \nu \).
(iv) For all \( \lambda, \mu \in \Lambda \) there is some \( \nu \in \Lambda \) with \( \lambda \leq \nu \) and \( \mu \leq \nu \).

A net \((x_\lambda)\) converges to \( x \in X \) (we write \( x_\lambda \to x \)), if for any neighborhood \( U \) of \( x \) there is some \( \lambda_0 \in \Lambda \) with \( x_\lambda \in U \) for all \( \lambda \geq \lambda_0 \).

Our favourite choice is \( \Lambda = \mathbb{N} \) with its natural order. Then nets are simply sequences.

**Definition 4.17.** Let \( A \) be a \( C^* \)-algebra and \( I \subseteq A \) be a subset. An approximate unit for \( I \) is a net \((u_\lambda)_{\lambda \in \Lambda} \subseteq I \) such that:

(i) \( 0 \leq u_\lambda \) and \( \|u_\lambda\| \leq 1 \) for all \( \lambda \in \Lambda \).
(ii) If \( \lambda \leq \mu \), then \( u_\lambda \leq u_\mu \).
(iii) We have \( u_\lambda x \to x \) and \( xu_\lambda \to x \) for all \( x \in I \).

It is clear, that the unit 1 in a unital \( C^* \)-algebra is an approximate unit.

**Example 4.18.** Again, we take a look at our favourite non-unital \( C^* \)-algebras.

(a) For \( C_0(\mathbb{R}) \), we choose functions \( 0 \leq u_n \leq 1 \) being 1 on \([-n, n]\) and zero outside of \([-n-1, n+1]\). They form an approximate unit with \( \Lambda = \mathbb{N} \) and its natural order.

(b) For the compact operators \( \mathcal{K}(H) \) on an infinite dimensional separable Hilbert space with orthonormal basis \((e_n)_{n \in \mathbb{N}}\), we let \( u_n \) be the projection onto the finite dimensional subspace of \( H \) spanned by \( e_1, \ldots, e_n \). By Prop. 1.38, \((u_n)\) is an approximate unit.

### 4.7. Existence of approximate units

Let us now show that approximate units always exist. We prepare some technical lemma first. Recall that \( I \triangleleft A \) denotes a two-sided ideal in a \( C^* \)-algebra \( A \). If in addition \( I \) is closed (in topology) and closed under taking adjoints, it is a \( C^* \)-algebra itself. Later in this lecture, we will see that a two-sided closed ideal is automatically closed under taking adjoints.

**Lemma 4.19.** Let \( A \) be a \( C^* \)-algebra and let \( I \triangleleft A \) be a closed ideal. We put

\[ \Lambda := \{ h \in I \mid h \geq 0, \|h\| < 1 \}. \]

(a) The set \( \Lambda \) is a partially ordered, directed set.
(b) Let \( h \in I \) with \( h \geq 0 \) and \( n \in \mathbb{N} \). Then \( h(1 - h(1/n + h)^{-1})h \leq 1/n \|h\| \).
(c) Let \( h \in I \) with \( h \geq 0 \) and \( n \in \mathbb{N} \). Let \( g \in \Lambda \) with \( h(1/n + h)^{-1} \leq g \), then \( \|h - gh\|^2 \leq 1/n \|h\| \) and \( \|h - hg\|^2 \leq 1/n \|h\| \).

**Proof.** We begin with (a). Items (i), (ii) and (iii) of Def. 4.16 follow from the fact that we have a partial order on \( C^* \)-algebras, see Sect. 4.4. We are left with proving directedness, i.e. item (iv) of Def. 4.16. Let \( a, b \in \Lambda \). We need to find some \( c \in \Lambda \) with \( a \leq c \) and \( b \leq c \).
Note that $\|a\| < 1$ implies $\text{sp}(a) \subseteq [0, 1)$. The function $z \mapsto z(1 - z)^{-1}$ is continuous and positive on $[0, 1)$. We may therefore define
\[
a' := a(1 - a)^{-1}, \quad b' := b(1 - b)^{-1}
\]
and derive $a' \geq 0$ and $b' \geq 0$ by functional calculus, see also Ex. 4.5. Hence also $c' := a' + b'$ is positive, by Lemma 4.3. We may thus define
\[
c := c'(1 + c')^{-1}
\]
again by functional calculus. We need to show that $c \in \Lambda$, $a \leq c$ and $b \leq c$ hold.

Firstly, the function $z \mapsto z(1 + z)^{-1}$ is positive on $[0, \infty)$ and strictly smaller than 1. Thus, by functional calculus, $c \in \Lambda$. Secondly, we have $0 \leq a' \leq c'$ and hence also $0 \leq 1 + a' \leq 1 + c'$. By Prop. 4.12, $(1 + c')^{-1} \leq (1 + a')^{-1}$. Thus:
\[
a = a'(1 + a')^{-1} = 1 - (1 + a')^{-1} \leq 1 - (1 + c')^{-1} = c'(1 + c')^{-1} = c
\]
Similarly, $b \leq c$ and we conclude that $\Lambda$ is directed.

Let us comment on some subtlety of the proof. We did not assume that $A$ is unital, nor that $I$ is. Nevertheless, we were using the symbol 1 all the time – weren’t we mistaken to do so? No, we were not. We may view $I$ as a two-sided closed ideal in $\tilde{A}$ and let $\Lambda$ be involved, that the resulting element was in $A$ or in $I$ resp., simply, because $A$ and $I$ are ideals in $\tilde{A}$. We address these issues in Ex. 4.5.

For (c), note that $(1 - g) - (1 - g)^2 = g(1 - g) \geq 0$ by functional calculus, since $\text{sp}(g) \subseteq [0, 1)$. Thus $0 \leq (1 - g)^2 \leq 1 - g$, as a comparison of elements in $\tilde{A}$. By Prop. 4.12, we then have $0 \leq h(1 - g)^2 h \leq h(1 - g)h$. Moreover, $0 \leq 1 - g \leq 1 - h(\frac{1}{n} + h)^{-1}$ by assumption, which yields $0 \leq h(1 - g)h \leq h(1 - h(\frac{1}{n} + h)^{-1})h$. Applying Prop. 4.12(c) twice and using (b), this implies:
\[
\|h - gh\|^2 = \|h(1 - g)^2 h\| \leq \|h(1 - g)h\| \leq \|h(1 - h(\frac{1}{n} + h)^{-1})h\| \leq \frac{1}{n}\|h\|
\]
Similarly, $\|h - hg\|^2 \leq \frac{1}{n}\|h\|$.

Theorem 4.20. Let $A$ be a $C^*$-algebra and let $I \lhd A$ be a two-sided closed ideal (possibly $I = A$). Then, $I$ possesses an approximate unit.

Proof. The idea is to take all positive elements in $I$ (which have a small norm) as an approximate unit and to index this set by itself. The technical part of the proof has been shifted to the previous lemma, so we may now simply put everything together. We define $\Lambda$ as in Lemma 4.19 and we put $u_\lambda := \lambda$ for $\lambda \in \Lambda$. Then, (i) and (ii) of Def. 4.17 are satisfied.

As for Def. 4.17(iii), let $x \in I$ and let $\varepsilon > 0$. Put $h := xx^* \geq 0$. Let $n \in \mathbb{N}$ such that $\frac{1}{n}\|h\| < \varepsilon^2$. Put $\lambda_0 := h(\frac{1}{n} + h)^{-1}$. Then $\lambda_0 \in \Lambda$, by Lemma 4.19. Let $g \in \Lambda$ with $g \geq \lambda_0$. Then $\|h - gh\| \leq \left(\frac{1}{n}\|h\|\right)^{-\frac{1}{2}} < \varepsilon$ by Lemma 4.19. Using the
\[ \|x - gx\|^2 = \|(1-g)h(1-g)\| \leq \|h - gh\|(1 + \|g\|) < 2\|h - gh\| < 2\varepsilon \]

This shows that \( u_g = g \in U \) for an \( \varepsilon \)-neighborhood \( U \) of \( x \) and hence \( u_g x - x \to 0 \). Similarly \( xu_g - x \to 0 \).

Just to make sure, note that an approximate unit of \( I \triangleleft A \) approximates only elements \( x \) from \( I \) in the sense of \( u_\lambda x \to x \) and \( xu_\lambda \to x \) – we cannot say anything about the approximation of elements \( x \) from \( A \).

**Remark 4.21.** If \( A \) is a separable C*-algebra (i.e. there is a countable, dense subset in \( A \)), then \( A \) admits a countable approximate unit (i.e. an approximate unit \( (u_n)_{n \in \mathbb{N}} \) with \( u_1 \leq u_2 \leq u_3 \leq \ldots \)). Indeed, by Thm. 4.20 there is an approximate unit \( (u_\lambda)_{\lambda \in \Lambda} \) in \( A \). Given a dense subset \( \{x_n \mid n \in \mathbb{N}\} \subseteq A \), we choose a sequence \( (\lambda_n)_{n \in \mathbb{N}} \subseteq \Lambda \) with \( \lambda_{n+1} \geq \lambda_n \) and \( \| u_\lambda x_i - x_i \|, \| x_i u_\lambda - x_i \| < \frac{1}{n} \) for all \( \lambda \geq \lambda_n \) and all \( i = 1, \ldots, n \). Put \( u_n := u_{\lambda_n} \).

### 4.8. Consequence for ideals of C*-algebras

The existence of approximate units has some consequences for the ideal structure of C*-algebras.

**Lemma 4.22.** Let \( A \) be a C*-algebra.

(a) Any closed ideal \( I \triangleleft A \) is closed under taking adjoints. Hence, \( I \) is a C*-algebra.

(b) If \( I \triangleleft J \triangleleft A \) are closed ideals, then \( I \triangleleft A \).

**Proof.** For (a), let \( (u_\lambda) \) be an approximate unit for \( I \) (which exists by Thm. 4.20) and let \( x \in I \). We use the convergence \( u_\lambda x \to x \) and the fact that the involution is continuous in order to derive \( x^* u_\lambda \to x^* \). As all elements \( x^* u_\lambda \) are in \( I \) and since \( I \) is closed, we infer \( x^* \in I \). Thus, \( I \subseteq A \) is a closed *-subalgebra and hence it is a C*-algebra.

For (b), let again \( (u_\lambda) \) be an approximate unit for \( I \), \( x \in I \) and \( a \in A \). Now, \( a \in A \) and \( u_\lambda \in I \subseteq J \) implies \( u_\lambda a \in J \), since \( J \) is an ideal in \( A \). But as \( x \in I \) and \( I \) is an ideal in \( J \), we infer \( x(u_\lambda a) \in I \). Then, \( xu_\lambda \to x \) implies \( I \ni xu_\lambda a \to xa \) and we infer \( xa \in I \), since \( I \) is closed. Likewise \( ax \in I \).

### 4.9. Quotients of C*-algebras

Did you ever wonder whether a C*-algebraic version of Prop. 2.17(a) holds? Why shall we investigate ideals in C*-algebras if we are not allowed to take quotients? Well, we are! Let’s prove it. Note that approximate units are a crucial ingredient in the proof.

**Theorem 4.23.** Let \( A \) be a C*-algebra and let \( I \triangleleft A \) be a closed ideal. Then \( A/I \) is a C*-algebra.

**Proof.** By Prop. 2.17 \( A/I \) is a Banach algebra via \( \dot{x} + \dot{y} := (x + y) \), \( \lambda \dot{x} := (\lambda x) \), \( \dot{x}y := (xy) \) and \( \|\dot{x}\| := \inf\{\|x + z\| \mid z \in I\} \).
We equip it with an involution $\dot{x}^* := (x^*)$. By Lemma 4.22, this is well-defined: Assume $\dot{x} = \dot{y}$. Then $x - y \in I$, but also $x^* - y^* \in I$, since $I$ is closed under taking adjoints. Hence $(x^*) = (y^*)$.

It remains to show that the norm satisfies the $C^*$-identity. Let $(u_\lambda)$ be an approximate unit for $I$ and let $x \in A$. We then have

$$||\dot{x}|| = \lim_{\lambda \in \Lambda} ||x - u_\lambda x||.$$ 

Let’s prove this description of the norm. Let $\varepsilon > 0$. By the definition of the norm, we find some $z \in I$ with $||x + z|| \leq ||\dot{x}|| + \varepsilon$. On the other hand, there is some $\lambda_0 \in \Lambda$ with $||z - u_\lambda z|| < \varepsilon$ for all $\lambda \geq \lambda_0$, due to the convergence $u_\lambda z \to z$. Note that $u_\lambda x \in I$, so $||\dot{x}|| \leq ||x - u_\lambda x||$. Moreover, $||1 - u_\lambda|| \leq 1$ since $\text{sp}(u_\lambda) \subseteq [0,1]$ implies $\text{sp}(1 - u_\lambda) \subseteq [0,1]$. Thus, for all $\lambda \geq \lambda_0$:

$$||\dot{x}|| \leq ||x - u_\lambda x|| \leq ||1 - u_\lambda||x + z|| + ||(1 - u_\lambda)z|| \leq ||1 - u_\lambda|| ||x + z|| + \varepsilon \leq ||\dot{x}|| + 2\varepsilon$$

We infer $||\dot{x}|| = \lim_{\lambda \in \Lambda} ||x - u_\lambda x||$. This implies, for any $z \in I$:

$$||\dot{x}||^2 = \lim_{\lambda \in \Lambda} ||x - u_\lambda x||^2$$

$$= \lim_{\lambda \in \Lambda} ||(1 - u_\lambda)x^*x(1 - u_\lambda)||$$

$$= \lim_{\lambda \in \Lambda} ||(1 - u_\lambda)(x^*x + z)(1 - u_\lambda)||$$

$$\leq ||x^*x + z||$$

Taking the infimum over all $z \in I$, we infer $||\dot{x}||^2 \leq ||\dot{x}^*||$. By Remark 2.2(c), we thus obtain the $C^*$-identity. $\square$

Theorem 4.23 showed that we found the right concept of ideals, when investigating closed two-sided ideals. Together with the fact that injective $^*$-homomorphisms are already isometric, we may prove that images of $C^*$-algebras under $^*$-homomorphisms are $C^*$-algebras again.

**Proposition 4.24.** Let $A, B$ be $C^*$-algebras and let $\varphi : A \to B$ be a $^*$-homomorphism. Then, $\varphi(A)$ is a $C^*$-algebra which is isomorphic to $A/\ker \varphi$.

**Proof.** It is easy to check that $\ker \varphi \triangleleft A$ is an ideal. By Thm. 4.23, $A/\ker \varphi$ is a $C^*$-algebra. Let us denote the quotient map by $\pi : A \to A/\ker \varphi$, $x \mapsto \dot{x}$. We define a map $\dot{\varphi} : A/\ker \varphi \to B$ by $\dot{\varphi}(\dot{x}) := \varphi(x)$. This is an injective $^*$-homomorphism with range $\varphi(A)$. By Prop. 4.13, $\dot{\varphi}$ is even isometric. Hence, any Cauchy sequence in $\varphi(A)$ is then also a Cauchy sequence in $A/\ker \varphi$. This implies that $\varphi(A)$ is complete. Hence $\varphi(A) \subseteq B$ is a closed $^*$-subalgebra and hence it is a $C^*$-algebra. We conclude that $\dot{\varphi} : A/\ker \varphi \to \varphi(A)$ is a $^*$-isomorphism. $\square$

**Remark 4.25.** The above proposition has some nice homological consequences for $C^*$-algebras: We may work with short exact sequences. Recall the concept of exact sequences from homological algebra. In our context, it means the following. Let $(A_n)_{n \in J}$ be $C^*$-algebras for $J = \{1, \ldots, N\}$, and let $\varphi_n : A_n \to A_{n+1}$
be \(\ast\)-homomorphisms for \(n = 1, \ldots, N - 1\). This chain of \(C^*\)-algebras and \(\ast\)-homomorphisms is called exact, if \(\operatorname{ker} \varphi_{n+1} = \operatorname{ran} \varphi_n\) for all \(n = 1, \ldots, N - 1\). A short exact sequence is a sequence
\[
0 \to I \to A \to B \to 0
\]
of \(C^*\)-algebras \(I\), \(A\) and \(B\). This encodes exactly the situation of ideals and quotients: If \(I \triangleleft A\), then
\[
0 \to I \to A \to A/I \to 0
\]
is a short exact sequence. See also Exc. 4.6.

4.10. Exercises.

Exercise 4.1. Let \(H\) be a Hilbert space and let \(T \in \mathcal{B}(H)\). Show that \(T\) is positive if and only if \(\langle Tx, x \rangle \geq 0\) for all \(x \in H\). In order to do so, show that \(T - \lambda I\) is bounded from below and surjective for \(\lambda \notin [0, \infty)\).

Exercise 4.2. Let \(H\) be a Hilbert space and let \(T \in \mathcal{B}(H)\). We want to show that \(T\) admits a polar decomposition \(T = V|T|\).

(a) Convince yourself that \(|T| := \sqrt{T^*T}\) exists by functional calculus.
(b) Show that \(\operatorname{ker}|T| = \operatorname{ker} T\) and that the map \(V_0 : \operatorname{ran}|T| \to \operatorname{ran} T\) given by \(V_0 : \operatorname{ran}|T| \to \operatorname{ran} T\) is well-defined and isometric. It thus has an isometric extension \(\overline{V_0} : \overline{\operatorname{ran}|T|} \to \overline{\operatorname{ran} T}\) and we may define \(V(x_1 + x_2) := \overline{V_0 x_1}\) for \(x_1 + x_2 \in \overline{\operatorname{ran}|T|} \oplus \overline{\operatorname{ran}|T|}^\perp\).
(c) Show that \(V\) is a partial isometry in the sense of Exc. 1.8. Show that \(V^* V\) is the projection onto \((\operatorname{ker} T)^\perp\) while \(V V^*\) is the projection onto \(\overline{\operatorname{ran} T}\).
(d) Show that \(T = V|T|\). Show that \(V\) is the unique partial isometry with \(T = V|T|\) and \(\operatorname{ker} V = \operatorname{ker} T\).
(e) Show that \(V\) is unitary in the sense of Def. 1.33 if \(T\) is invertible.
(f) What is the polar decomposition of \(T\) in the one-dimensional case \(H = \mathbb{C}\)?

Exercise 4.3. Let \(A\) be a \(C^*\)-algebra and let \(x, y \in A\). Show that
\[
\operatorname{sp}(xy) \cup \{0\} = \operatorname{sp}(yx) \cup \{0\}.
\]
Find an example for \(\operatorname{sp}(xy) \neq \operatorname{sp}(yx)\).

Exercise 4.4. Let \(A\) be a \(C^*\)-algebra and \(x, h \in A\) be selfadjoint with \(h \geq 0\) and \(h \geq x\). The positive part \(x_+ \geq 0\) of \(x\) is defined as in Prop. 3.29.

(a) Show that \(h \geq x_+\), if \(A\) is commutative.
(b) Show that \(h \geq x_+\) in general by giving a counterexample in \(A = M_2(\mathbb{C})\).

Exercise 4.5. Let \(A\) be a \(C^*\)-algebra and let \(I \triangleleft A\) be a closed ideal in \(A\). Let \(a, b \in I\) with \(a, b \geq 0\) and \(\|a\|, \|b\| < 1\).

(a) Show that we may define \(a' := a(1 - a)^{-1}\). Be careful: We do not assume that \(A\) is unital! Why can we still write down the expression \(a'\) and why do we even have \(a' \in I\)?
(b) Let $b' := b(1 - b)^{-1}$, $c' := a' + b'$ and $c := c'(1 + c')^{-1}$. Show that we may define $b', c'$ and $c$ and that they all lie in $I$ (again, taking care of the issue with the unit).

(c) Show that $a', b', c'$ and $c$ are positive. Show $\|c\| < 1$.

(d) Show $1 - (1 + a')^{-1} \leq 1 - (1 + c')^{-1}$.

(e) Show $a = a'(1 + a')^{-1}$.

(f) Show that $0 \leq b(1 - a)^2 b \leq b(1 - a) b$ holds. Again, why are all these elements in $I$?

**Exercise 4.6.** Let $I$, $A$ and $B$ be $C^*$-algebras and let $\iota : I \to A$ and $\pi : A \to B$ be $^*$-homomorphisms. Show that the sequence

$$0 \to I \to A \to B \to 0$$

is exact if and only if $\iota$ is injective, $\pi$ is surjective, $\iota(I) \lhd A$ is an ideal in $A$ and $B \cong A/\iota(I)$. 

5. States, representations and the GNS construction

Abstract. We introduce and study positive linear functionals and states. We briefly recall the Hahn-Banach Theorem. We then turn to representations of $C^*$-algebras and we discuss the famous GNS construction. This yields our Second Fundamental Theorem of $C^*$-Algebras: Any abstractly defined $C^*$-algebra may be represented concretely on a Hilbert space – as a subalgebra of all bounded linear operators.

5.1. Positive linear functionals. In the last lecture, we discussed positivity for elements in a $C^*$-algebra and the induced order structure. We now turn to a class of linear functionals preserving these structures.

Definition 5.1. Let $A$ be a $C^*$-algebra. A linear functional $\varphi : A \to \mathbb{C}$ is positive (we write $\varphi \geq 0$), if $\varphi(x) \geq 0$ for all $x \in A$ with $x \geq 0$.

We infer that positive linear functionals preserve the order structure: Given $x, y \in A$, the relation $x \leq y$ implies $\varphi(x) \leq \varphi(y)$. Indeed, recall that $x \leq y$ holds if and only if $z := y - x \geq 0$. Thus, $z \geq 0$ implies $\varphi(z) \geq 0$ from which we deduce $\varphi(x) \leq \varphi(y)$, see also the discussion around Rem. 4.11.

Example 5.2. Let us take a look at some examples.

(a) Let $A = C([0,1])$ as in Exm. 2.3 and let $t \in [0,1]$. Then $ev_t : C([0,1]) \to \mathbb{C}$ given by $ev_t(f) := f(t)$ is a positive linear functional. In fact, it is even an algebra homomorphism, see Prop. 3.18 and Exc. 3.2, but forgetting the multiplicative structure, we obtain a positive linear functional. More generally, let $\mu$ be a Radon measure on $[0,1]$ (i.e. for all $x \in [0,1]$ there is an open neighborhood $U_x$ such that $\mu(U_x) < \infty$; and for any Borel set $B$, the value $\mu(B)$ is the supremum of all $\mu(K)$, where $K \subseteq B$ is compact). Then

$$\varphi(f) := \int_0^1 f(t)d\mu(t)$$

is a positive linear functional, $f \in C([0,1])$. In fact, any positive linear functional is of exactly this form! You might have met the Representation Theorem by Riesz-Markov in some of your analysis lectures. It states that the positive linear functionals on $[0,1]$ are in bijection with Radon measures on $[0,1]$ via the above correspondence. If $\mu$ is the Dirac measure on $t \in [0,1]$, we obtain $ev_t$ under this correspondence.

(b) In Exc. 5.3 we will see the following. The trace $Tr : M_N(\mathbb{C}) \to \mathbb{C}$ with

$$Tr(T) = Tr((t_{ij})) := \sum_{i=1}^N t_{ii}, \quad T = (t_{ij}) = (t_{ij})_{i,j=1,...,N} \in M_N(\mathbb{C})$$
is a positive linear functional on $M_N(\mathbb{C})$. Likewise, the normalized trace
\[
\text{tr}(T) = \text{tr}((t_{ij})) := \frac{1}{N} \sum_{i=1}^{N} t_{ii}
\]
is a positive linear functional. More generally, if $B \in M_N(\mathbb{C})$ is a positive matrix, then
\[
\tau_B(T) := \text{tr}(BT)
\]
is a positive linear functional. In fact, any positive linear functional on $M_N(\mathbb{C})$ of exactly this form, see Exc. 5.3.

(c) Let $H$ be a Hilbert space and let $x \in H$. Consider $A = B(H)$. Then
\[
\varphi_x : B(H) \to \mathbb{C}, \quad \varphi_x(T) := \langle Tx, x \rangle, \quad T \in B(H)
\]
is a positive linear functional. Indeed, recall from Thm. 4.8 that positive elements $T$ are of the form $T = S^*S$. Thus, $\varphi_x(S^*S) = \langle Sx, Sx \rangle \geq 0$.

If $H = \mathbb{C}^N$ is finite dimensional with the standard basis $e_1, \ldots, e_N$, and $x = \sum_j x_j e_j \in H$, then the above positive linear functional $\varphi_x$ is of the form $\varphi_x(T_t) = \tau_B(T)$ with $B = X^*X$. Here, $X = (x_{ij}) \in M_N(\mathbb{C})$ is the matrix given by $x_{ij} = \sqrt{N}\delta_{i1}x_j$. The matrix $T_t$ is the transpose of $T \in M_N(\mathbb{C})$, so $T_t = (t_{ji})$, if $T = (t_{ij})$.

We learn from Exm. 5.2(a), that positive linear functionals may be seen as generalized evaluation maps.

5.2. Induced positive sesquilinear form. Before we proceed investigating some properties of positive linear functionals, let us make an important observation: Positive linear functionals induce positive sesquilinear forms.

Lemma 5.3. Let $A$ be a $C^*$-algebra, $\varphi : A \to \mathbb{C}$ a positive linear functional. Then
\[
\langle x, y \rangle := \varphi(y^*x)
\]
is a positive sesquilinear form on $A$, i.e. we have:

(i) $\langle \lambda x_1 + \mu x_2, y \rangle = \lambda \langle x_1, y \rangle + \mu \langle x_2, y \rangle$ for all $x_1, x_2, y \in A, \lambda, \mu \in \mathbb{C}$.

(ii) $\langle x, \lambda y_1 + \mu y_2 \rangle = \lambda \langle x, y_1 \rangle + \mu \langle x, y_2 \rangle$ for all $x, y_1, y_2 \in A, \lambda, \mu \in \mathbb{C}$.

(iii) $\langle x, x \rangle \geq 0$ for all $x \in A$.

As a consequence, we have $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for all $x, y \in A$.

Proof. It is immediately clear that $\langle \cdot, \cdot \rangle$ is a positive sesquilinear form. A direct computation shows that any sesquilinear form satisfies the polarisation identity, see also Prop. 1.4
\[
\langle x, y \rangle = \frac{1}{4} \sum_{k=0}^{3} i^k \langle x + i^k y, x + i^k y \rangle
\]
Here, $i \in \mathbb{C}$ denotes the imaginary unit and $x, y \in A$. We infer $\langle x, y \rangle = \overline{\langle y, x \rangle}$.

These sesquilinear forms will be important later in the present lecture.
5.3. **Properties of positive linear functionals.** Positive linear functionals are automatically continuous and involutive (i.e. they preserve the involution).

**Lemma 5.4.** Let $A$ be a $C^*$-algebra and let $\varphi : A \to \mathbb{C}$ be a positive linear functional.

(a) $\varphi$ is bounded (and hence continuous), i.e. $|\varphi(x)| \leq \|\varphi\|\|x\|$ for all $x \in A$.

(b) $\varphi$ is involutive, i.e. $\varphi(x^*) = \overline{\varphi(x)}$ for all $x \in A$.

(c) We have $|\varphi(x)|^2 \leq \|\varphi\|\varphi(x^*x)$ for all $x \in A$.

**Proof.** For (a), let $S := \{x \in A \mid x \geq 0, \|x\| \leq 1\}$. We first show, that $\varphi$ is bounded on $S$. Assume the converse. Hence, we find a sequence $(x_n)_{n \in \mathbb{N}}$ with $x_n \in S$ and $\varphi(x_n) \geq 2^n$, for all $n \in \mathbb{N}$. For $N \in \mathbb{N}$ put $s_N := \sum_{n=1}^{N} \frac{1}{2^n} x_n \in A$ and $x := \lim_{N \to \infty} s_N$. All these elements are positive by Cor. 4.9 and we have $x \geq s_N$ for all $N \in \mathbb{N}$. Thus, $\varphi(x) \geq \varphi(s_N) \geq N$ for all $N \in \mathbb{N}$ which is a contradiction.

Now, as $\varphi$ is bounded on $S$, we find some constant $K \geq 0$ such that $\varphi(x) \leq K$ for all $x \in S$. This implies for $x \in A$ with $x \geq 0$ and $x \neq 0$ that $x^0 := \|x\|^{-1} x \in S$ and thus $\varphi(x) = \|x\| \varphi(x^0) \leq \|x\|K$. Finally, let us consider an arbitrary element $x \in A$. By Prop. 3.29 and Def. 4.6 we may decompose it into a linear combination of four positive elements each with norm less or equal to $\|x\|$

$$x = (\text{Re}(x))_+ - (\text{Re}(x))_- + i(\text{Im}(x))_+ - i(\text{Im}(x))_-$$

Then $|\varphi(x)| \leq 4K\|x\|$.

Let us prove (b). Let $(u_\lambda)$ be an approximate unit for $A$, which exists by Thm. 4.20. Using (a) and the induced sesquilinear form from Lemma 5.3 we deduce:

$$\varphi(x^*) \leftarrow \varphi(x^*u_\lambda) = \langle u_\lambda, x \rangle = \overline{\langle x, u_\lambda \rangle} = \overline{\varphi(u_\lambda^*x)} = \overline{\varphi(u_\lambda x)} \to \varphi(x)$$

As for (c), this is basically the Cauchy-Schwarz inequality (Prop. 1.3) for positive sesquilinear forms. Let again $(u_\lambda)$ be an approximate unit for $A$. We deduce $\varphi(u_\lambda^2) \leq \|\varphi\|$ from (a), since $\|u_\lambda^2\| = \|u_\lambda\|^2 \leq 1$ by the $C^*$-identity and Def. 4.17. Then:

$$|\varphi(x)|^2 \leftarrow |\varphi(u_\lambda x)|^2 = \langle x, u_\lambda \rangle^2 \leq \langle x, x \rangle \langle u_\lambda, u_\lambda \rangle = \varphi(x^*x)\varphi(u_\lambda^2) \leq \|\varphi\|\varphi(x^*x)$$

$\Box$

Approximate units were used in the previous lemma and they also help us to give a characterization of positive linear functionals in the next proposition. Let us insert two preparatory lemmas first.

**Lemma 5.5.** Let $A$ be a $C^*$-algebra and let $x, y \in A$ be positive with $\|x\|, \|y\| \leq 1$. Then $\|x - y\| \leq 1$.

**Proof.** We have $\text{sp}(x) \subseteq [0, 1]$. Thus, $1 - x \geq 0$ by functional calculus, when viewing $1 - x$ as an element in the unitization $\hat{A}$, in case $A$ does not have a unit. Hence, $(1 - x) + y \geq 0$, if $y \geq 0$, by Lemma 4.3. This shows $x - y \leq 1$. The argument is symmetric and we have $-1 \leq x - y \leq 1$. Thus $\|x - y\| \leq 1$ by functional calculus.

$\Box$

The next lemma is a modification of Lemma 3.11(e).
Lemma 5.6. Let $A$ be a $C^*$-algebra and let $\varphi : A \to \mathbb{C}$ be linear and continuous with $\| \varphi \| = 1$. Assume there is some approximate unit $(u_\lambda)_{\lambda \in \Lambda}$ of $A$ with $\| \varphi \| = \lim_{\lambda \to \infty} \varphi(u_\lambda)$. Then, $\varphi(x) \in \mathbb{R}$, for all selfadjoint elements $x \in A$.

Proof. We mimic the proof of Lemma 3.11, but we may not assume that $\varphi$ is unital. So, let $\alpha, \beta \in \mathbb{R}$ with $\varphi(x) = \alpha + i\beta$, for a given selfadjoint element $x \in A$. Let $\mu \in \mathbb{R}$. Then, for all $\lambda \in \Lambda$:

$$\| \varphi(x + i\mu u_\lambda) \| ^2 \leq \| x + i\mu u_\lambda \| ^2$$

$$= \| (x + i\mu u_\lambda)^*(x + i\mu u_\lambda) \|$$

$$\leq \| x \| ^2 + |\mu| \| x u_\lambda - u_\lambda x \| + \mu^2 \| u_\lambda \| ^2$$

As $\| x u_\lambda - u_\lambda x \| \to 0$ and $\| u_\lambda \| \leq 1$, we infer:

$$\alpha^2 + \mu^2 + 2\mu\beta + \beta^2$$

$$= |\varphi(x) + i\mu|^2 \leq \| x \| ^2 + |\mu| \| x u_\lambda - u_\lambda x \| + \mu^2 \to \| x \| ^2 + \mu^2$$

Since this holds true for all $\mu \in \mathbb{R}$, we conclude $\beta = 0$, i.e. $\varphi(x) \in \mathbb{R}$. \qed

Proposition 5.7. Let $A$ be a $C^*$-algebra and let $\varphi : A \to \mathbb{C}$ be linear and continuous. The following are equivalent:

(i) $\varphi$ is positive.

(ii) For all approximate units $(u_\lambda)$ of $A$, we have $\| \varphi \| = \lim_{\lambda \to \infty} \varphi(u_\lambda)$.

(iii) For some approximate unit $(u_\lambda)$ of $A$, we have $\| \varphi \| = \lim_{\lambda \to \infty} \varphi(u_\lambda)$.

Proof. The case $\varphi = 0$ is trivial, so let us assume $\| \varphi \| = 1$; otherwise we work with the positive linear functional $\varphi' := \| \varphi \| ^{-1} \varphi$.

We begin with proving the implication from (i) to (ii). Assume that $\varphi$ is positive. Let $(u_\lambda)$ be an approximate unit for $A$. Then, $\lambda \leq \mu$ implies $u_\lambda \leq u_\mu$ by Def. 4.17 and hence $\varphi(u_\mu) \leq \varphi(u_\lambda)$ as positive maps preserve the order structure. Moreover, $\varphi(u_\lambda) \leq 1$ for all $\lambda \in \Lambda$. Hence $(\varphi(u_\lambda))$ is an increasing, bounded net in $\mathbb{C}$. We may thus find some $\alpha \leq 1$ such that $\varphi(u_\lambda) \to \alpha$ converges from below. Note that we have $u_\lambda^2 \leq u_\lambda$ by functional calculus and Def. 4.17, hence, $\varphi(u_\lambda^2) \leq \varphi(u_\lambda)$. Employing the Cauchy-Schwarz inequality as in the proof of Lemma 5.4, we deduce for all $x \in A$ with $\| x \| \leq 1$:

$$|\varphi(x)|^2 \leq |\varphi(u_\lambda x)|^2 = |\langle x, u_\lambda \rangle|^2 \leq \langle x, x \rangle < u_\lambda, u_\lambda > = \varphi(x^{*}x)\varphi(u_\lambda^2) \leq \varphi(x^{*}x)\varphi(u_\lambda) \leq \alpha$$

Hence, $1 = \| \varphi \| \leq \sqrt{\alpha} \leq 1$, which shows $\| \varphi \| = 1 = \alpha = \lim_{\lambda \to \infty} \varphi(u_\lambda)$.

The implication from (ii) to (iii) is trivial (provided the existence of approximate units is ensured, which it is, by Thm. 4.20).

As for (iii) to (i), assume that there is an approximate unit $(u_\lambda)$ with $\varphi(u_\lambda) \to 1$. Let $x \in A$ with $x \geq 0$ and $x \neq 0$. Assume $\| x \| \leq 1$; otherwise we work with $x' := \| x \| ^{-1} x$. By Lemma 5.5, $\| u_\lambda - x \| \leq 1$ for all $\lambda \in \Lambda$; hence $|\varphi(u_\lambda - x)| \leq 1$. Moreover, $\varphi(u_\lambda - x) \to 1 - \varphi(x)$, as $\lambda \to \infty$, by assumption. This shows $|1 - \varphi(x)| \leq 1$. On the other hand, $1 - \varphi(x) \in \mathbb{R}$ by Lemma 5.6. Thus, $1 - \varphi(x) \leq |1 - \varphi(x)| \leq 1$. This shows $\varphi(x) \geq 0$. \qed
In the unital case, this gives a very easy – and amazing! – characterization of positive linear functionals: Positivity is encoded in $\varphi(1)$.

**Corollary 5.8.** Let $A$ be a unital $C^*$-algebra and let $\varphi : A \to \mathbb{C}$ be linear and continuous. Then $\varphi$ is positive if and only if $\varphi(1) = \|\varphi\|$.

Surprisingly, the norm behaves additive on positive linear functionals turning the space of positive linear functionals on $C^*$-algebras into an abstract $L$-space (or $AL$-space).

**Corollary 5.9.** Let $A$ be a $C^*$-algebra and let $\varphi, \psi : A \to \mathbb{C}$ be two positive linear functionals. Then
\[
\|\varphi + \psi\| = \|\varphi\| + \|\psi\|.
\]

**Proof.** Let $(u_\lambda)$ be an approximate unit of $A$. By Prop. 5.7, we have:
\[
\|\varphi + \psi\| \leftarrow (\varphi + \psi)(u_\lambda) = \varphi(u_\lambda) + \psi(u_\lambda) \to \|\varphi\| + \|\psi\| \quad \square
\]

5.4. **Hahn-Banach Theorem.** Let us quickly recall a classic theorem from functional analysis. In fact, it exists in hundreds of variations, so let us state a suitable one for our purposes.

**Theorem 5.10** (Hahn-Banach). Let $E$ be a normed complex vector space and let $F \subseteq E$ be a linear subspace. Let $f : F \to \mathbb{C}$ be linear and continuous. Then, there is a linear and continuous map $\tilde{f} : E \to \mathbb{C}$ extending $f$ (i.e. $\tilde{f}(x) = f(x)$ for all $x \in F$) such that $\|\tilde{f}\| = \|f\|$.

The crucial point is, that we may extend linear, continuous maps from subspaces to the whole space – in a norm preserving way!

5.5. **States.** We now turn to a special subclass of positive linear functionals: states.

**Definition 5.11.** Let $A$ be a $C^*$-algebra. A state on $A$ is a positive linear functional $\varphi : A \to \mathbb{C}$ with $\|\varphi\| = 1$.

So, states are simply positive linear functionals with some normalization. In view of Cor. 5.8, this normalization is reasonable.

**Proposition 5.12.** Let $A$ be a unital $C^*$-algebra and let $\varphi : A \to \mathbb{C}$ be a linear functional. Then, $\varphi$ is a state if and only if $\varphi$ is positive and unital (i.e. $\varphi(1) = 1$).

**Proof.** Follows immediately from Cor. 5.8 and Def. 5.11 $\square$

Next, we formulate some Hahn-Banach Theorem for $C^*$-algebras.

**Theorem 5.13.** Let $A$ be a $C^*$-algebra and let $x \in A$ be normal. There is a state $\varphi : A \to \mathbb{C}$ with $|\varphi(x)| = \|x\|$.
Proof. The proof is beautiful. We consider \( C^*(x,1) \subseteq \tilde{A} \). This is a unital Banach algebra and we consider the Gelfand transform \( \chi : C^*(x,1) \to C(\text{Spec}(C^*(x,1))) \). Now, for \( \hat{x} = \chi(x) \in C(\text{Spec}(C^*(x,1))) \) we find some character \( \varphi_0 \in \text{Spec}(C^*(x,1)) \) such that \( |\hat{x}(\varphi_0)| = \|\hat{x}\|_{\infty} \). As \( \varphi_0 \) is a character, we also have \( \varphi_0(1) = 1 \). Moreover, \( \varphi_0 \) is in particular a positive linear functional, so \( \|\varphi_0\| = \varphi_0(1) = 1 \), by Cor. 5.8.

Since \( x \) is normal, \( C^*(x,1) \subseteq \tilde{A} \) is in fact a commutative, unital \( C^* \)-algebra. Hence, \( \chi \) is an isometric \(^*\)-isomorphism by the Gelfand-Naimark Theorem (Thm. 3.23), which means that \( \|\hat{x}\|_{\infty} = \|x\| \). Now comes the first funny aspect: Forgetting some of the structure of \( \varphi_0 \), we conclude that we found a linear and continuous map \( \varphi_0 : C^*(x,1) \to \mathbb{C} \) with \( \varphi_0(1) = 1 \) and

\[ |\varphi_0(x)| = |\hat{x}(\varphi_0)| = \|\hat{x}\|_{\infty} = \|x\|. \]

We may hence apply the Hahn-Banach Theorem (Thm. 5.10), and find a linear and continuous extension \( \tilde{\varphi} : \tilde{A} \to \mathbb{C} \) with \( \|\tilde{\varphi}\| = \|\varphi_0\| \). Is this extension still positive? It is, surprisingly: since \( \tilde{\varphi} \) coincides with \( \varphi_0 \) on \( C^*(x,1) \), we have

\[ \tilde{\varphi}(1) = \varphi_0(1) = \|\varphi_0\| = \|\tilde{\varphi}\|. \]

By Cor. 5.8 we obtain that \( \tilde{\varphi} : \tilde{A} \to \mathbb{C} \) is a positive linear functional. Thus, the restriction of \( \tilde{\varphi} \) to \( A \subseteq \tilde{A} \) yields a positive linear functional \( \varphi : A \to \mathbb{C} \) with

\[ |\varphi(x)| = |\tilde{\varphi}(x)| = |\varphi_0(x)| = \|x\|. \]

Moreover, \( \|\varphi\| \leq \|\tilde{\varphi}\| = 1 \) and \( |\varphi(x)| = \|x\| \) imply \( \|\varphi\| = 1 \), i.e. \( \varphi \) is a state. \( \square \)

5.6. Representations. Having discussed positive linear functionals and states in detail, we now come to a different subject: Representations. In the next definition, we define representations and some related notions.

Recall the definition of a unitary \( U : H \to H \) on a Hilbert space from Def. 1.33. From Prop. 1.34 we know that unitaries are exactly isomorphisms of the Hilbert space \( H \). In accordance with Def. 1.17 we adapt the notion and call a map \( U : H_1 \to H_2 \) between two Hilbert spaces \( H_1 \) and \( H_2 \) unitary, if \( U \) is surjective and \( \langle Ux, Uy \rangle_{H_2} = \langle x, y \rangle_{H_1} \) for all \( x, y \in H_1 \).

Recall the definition of the direct sum of Hilbert spaces: Given Hilbert spaces \( H_i \), \( i \in I \), we define \( \oplus_{i \in I} H_i \) as the set of all families \( (x_i)_{i \in I} \) with \( x_i \in H_i \) for \( i \in I \), and \( \sum_{i \in I} \|x_i\|_{H_i}^2 < \infty \). The inner product is given by \( \langle (x_i), (y_i) \rangle := \sum_{i \in I} \langle x_i, y_i \rangle_{H_i} \).

**Definition 5.14.** Let \( A \) be a \( C^* \)-algebra.

(a) Let \( H \) be a Hilbert space. A representation of \( A \) on \( H \) is a \(^*\)-homomorphism \( \pi : A \to B(H) \).

(b) Two representations \( \pi_i : A \to B(H_i) \), \( i = 1, 2 \) are equivalent, if there is a unitary \( U : H_1 \to H_2 \) such that \( \pi_2(x) = U \pi_1(x) U^* \) for all \( x \in A \).

(c) Given representations \( \pi_i : A \to B(H_i) \), \( i \in I \), let \( \oplus_{i \in I} \pi_i : A \to B(\oplus_{i \in I} H_i) \) be the representation given by \( (\oplus_{i \in I} \pi_i)(x_i) := \pi_j(x_j) \) for \( x_j \in H_j \), \( j \in I \).

(d) A representation \( \pi : A \to B(H) \) is non-degenerate, if we have \( \overline{\pi(A)H} = H \) for the closure of the range of \( \pi \).
(e) A representation \( \pi : A \to B(H) \) is cyclic if there is some \( x \in H \) (a cyclic vector) such that \( \pi(A)x = H \).

(f) Given a closed linear subspace \( K \subseteq H \), we say that \( K \) is invariant under a representation \( \pi : A \to B(H) \), if \( \pi(A)K \subseteq K \), i.e. \( \pi(a)x \in K \) for all \( a \in A \) and all \( x \in K \).

(g) A representation is faithful, if it is injective.

Before we comment a bit on the above definition, let us prove a little lemma.

**Lemma 5.15.** Let \( A \) be a \( C^* \)-algebra, \( \pi : A \to B(H) \) a representation and \( K \subseteq H \) a closed linear subspace. If \( K \) is invariant under \( \pi \), then also its orthogonal complement \( K^\perp \) is invariant under \( \pi \). We may then write \( \pi = \pi_1 \oplus \pi_2 \), where \( \pi_1 : A \to B(K) \) and \( \pi_2 : A \to B(K^\perp) \) are defined as restrictions of \( \pi \).

**Proof.** Let \( x \in K \) and \( a \in A \). Then \( \pi(a^*)x \in K \) since \( K \) is invariant under \( \pi \). Hence, \( \langle x, \pi(a)y \rangle = \langle \pi(a^*)x, y \rangle = 0 \) for \( y \in K^\perp \). As this is true for all \( x \in K \), this shows \( \pi(a)y \in K^\perp \) for all \( y \in K^\perp \). Thus, we have \( \pi(a) : K \to K \) and \( \pi(a) : K^\perp \to K^\perp \) for the restrictions and thus \( \pi = \pi_1 \oplus \pi_2 \) on the Hilbert space \( H = K \oplus K^\perp \). \( \square \)

**Remark 5.16.**

(a) The notion in Def. 5.14(b) is an equivalence relation indeed, as can be checked easily.

(b) Non-degeneracy means that \( \pi \) transports vectors to “all of \( H \)”. Put a bit differently, any representation \( \pi : A \to B(H) \) may be written as a direct sum of a non-degenerate representation and a zero representation.

Indeed, \( K := \overline{\pi(A)H} \subseteq H \) is a closed linear subspace which is invariant under \( \pi \), since \( \pi(A)K \subseteq \pi(A)H \subseteq K \). This also shows that the restriction of \( \pi \) to \( K \) is non-degenerate. By Lemma 5.15, also \( K^\perp \) is invariant and we have \( \pi = \pi_{|K} \oplus \pi_{|K^\perp} \). Now, \( \pi_{|K^\perp} = 0 \), since for \( a \in A \) and \( x \in K^\perp \), the element \( \pi(a)x \) is in \( K^\perp \) (by invariance), but also in \( K \) (by definition of \( K \)). Hence, \( \pi(a)x = 0 \).

(c) A cyclic representation transports the cyclic vector to any place in \( H \). In particular, cyclic representations are non-degenerate. One may show that any non-degenerate representation \( \pi : A \to B(H) \) is the direct sum of cyclic representations. This follows from Zorn’s Lemma (note that any vector \( x \in H \) is cyclic for the restriction of \( \pi \) to \( K := \overline{\pi(A)x} \subseteq H \)).

(d) Let \( \pi : A \to B(H) \) be a representation. Then \( \pi(A) \subseteq B(H) \) is a \( C^* \)-subalgebra by Prop. 4.24. If \( \pi \) is faithful, then \( A \) is isomorphic to \( \pi(A) \).

(e) Let \( (u_\lambda) \) be an approximate unit for \( A \) and let \( \pi : A \to B(H) \) be a non-degenerate representation. Then \( \pi(u_\lambda)x \to x \) for all \( x \in H \). Indeed, let \( a \in A \) and \( y \in H \). Let \( x := \pi(a)y \). Then \( \pi(u_\lambda)x = \pi(u_\lambda)a \to \pi(a)y = x \). Since such vectors \( x \) are dense in \( H \), thanks to the non-degeneracy, this shows \( \pi(u_\lambda)x \to x \) for all \( x \in H \). We conclude that \( (\pi(u_\lambda)) \) approximates the unit on \( B(H) \).
5.7. **GNS construction.** Given a representation \( \pi : A \to B(H) \) of a \( C^* \)-algebra, and a cyclic vector \( x \in H \), it is easy to see that
\[
\varphi(a) := \langle \pi(a)x, x \rangle, \quad a \in A,
\]
defines a positive linear functional \( \varphi : A \to \mathbb{C} \). Moreover, \( \|\varphi\| = \|x\|^2 \), as a combination of Prop. 5.7 and Rem. 5.16(e). Interestingly, given a positive linear functional, we can also go the way back: We will find a representation such that \( \varphi \) can be written as above. This is the famous GNS construction, which we now prepare. Let us prove a lemma on uniqueness first.

**Lemma 5.17.** Let \( A \) be a \( C^* \)-algebra and let \( \pi_i : A \to B(H_i) \), \( i = 1, 2 \), be two cyclic representations with cyclic vectors \( x_i \in H_i \), \( i = 1, 2 \). Let \( \varphi_i : A \to \mathbb{C} \) be the positive linear functionals given by \( \varphi_i(a) = \langle \pi_i(a)x_i, x_i \rangle \), for \( a \in A \). If \( \varphi_1 = \varphi_2 \), then there exists a unitary \( U : H_1 \to H_2 \) such that \( \pi_2(a) = U\pi_1(a)U^* \) for all \( a \in A \), and \( Ux_1 = x_2 \).

**Proof.** Put \( H_i^0 := \pi_i(A)x_i \subseteq H_i \), \( i = 1, 2 \), and let \( \lambda \in A \). Then:
\[
\|\pi_2(a)x_2\|^2 = \langle \pi_2(a)x_2, \pi_2(a)x_2 \rangle = \langle \pi_2(a^*a)x_2, x_2 \rangle = \varphi_2(a^*a) = \varphi_1(a^*a) = \|\pi_1(a)x_1\|^2
\]
We define \( U_0 : H_1^0 \to H_2^0 \) by \( U_0\pi_1(a)x_1 := \pi_2(a)x_2 \), for \( a \in A \). This is well-defined, since given some \( a, b \in A \) with \( \pi_1(a - b)x_1 = 0 \), we have \( \pi_2(a - b)x_2 = 0 \), by the above equation. Moreover, \( U_0 : H_1^0 \to H_2^0 \) is surjective. Furthermore, \( U_0 \) preserves the inner product, as can be seen directly from the polarization identity (Prop. 1.4).

Finally, \( U_0\pi_1(a)U_0^* = \pi_2(a) \) holds for all \( a \in A \), since
\[
(U_0\pi_1(a)U_0^*)(\pi_2(b)x_2) = U_0\pi_1(a)\pi_1(b)x_1 = U_0\pi_1(ab)x_1 = \pi_2(ab)x_2 = (\pi_2(a)(\pi_2(b)x_2)
\]
for any \( b \in A \). Now, \( H_i^0 \) is dense in \( H_i \), as \( x_i \) is cyclic, for \( i = 1, 2 \). As \( U_0 \) preserves the norm, it can be extended to \( U : H_1 \to H_2 \). This operator is a unitary with \( \pi_2(a) = U\pi_1(a)U^* \) for all \( a \in A \), as all these properties hold on a dense subset.

Finally, let \( (u_\lambda) \) be an approximate unit for \( A \). By Rem. 5.16
\[
Ux_1 \leftarrow U\pi_1(u_\lambda)x_1 = U\pi_1(u_\lambda)x_1 = \pi_2(u_\lambda)x_2 \to x_2 \tag*{\Box}
\]

We are ready for the GNS construction. It can be found in the 1943 Gelfand-Naimark article [13], but it has been refined by Segal a few years later; the letters GNS stand for Gelfand-Naimark-Segal.

**Theorem 5.18.** Let \( A \) be a \( C^* \)-algebra and let \( \varphi : A \to \mathbb{C} \) be a state. There are a Hilbert space \( H_\varphi \), a representation \( \pi_\varphi : A \to B(H_\varphi) \), and a cyclic vector \( x_\varphi \in H_\varphi \), such that \( \varphi(a) = \langle \pi_\varphi(a)x_\varphi, x_\varphi \rangle \) for all \( a \in A \). With these properties, the triple \((H_\varphi, \pi_\varphi, x_\varphi)\) is unique up to equivalence (by Lemma 5.17).

**Proof.** We construct the triple \((H_\varphi, \pi_\varphi, x_\varphi)\) step by step.

1. **Construction of the Hilbert space \( H_\varphi \).** We consider the \( C^* \)-algebra \( A \) as a vector space. The state \( \varphi \) induces a positive sesquilinear form \( \langle x, y \rangle := \varphi(y^*x) \) on \( A \), by Lemma 5.3. It might fail to satisfy the implication from \( \langle x, x \rangle = 0 \) to \( x = 0 \), so let us mod out the bad elements: Consider the closed linear subspace
$N_\varphi := \{ x \in A \mid \langle x, x \rangle = 0 \} \subseteq A$ and let $K_\varphi := A/N_\varphi$ be the quotient space, $\gamma : A \to K_\varphi$ be the quotient map.

Then $K_\varphi$ is a pre Hilbert space with $\langle \gamma(x), \gamma(y) \rangle := \langle x, y \rangle$. Using Cauchy-Schwarz, we infer that this is a well-defined inner product, see Exc. \[.\] Besides, note that the quotient map is continuous, since $\|\gamma(x)\|^2 = \varphi(x^*x) \leq \|x\|^2$, by Lemma 5.4.

Finally, the completion of $K_\varphi$ is our Hilbert space $H_\varphi$. Summarizing:

$$H_\varphi := \overline{K_\varphi} = A/\{ x \in A \mid \varphi(x^*x) = 0 \}$$

(2) Construction of the representation $\pi_\varphi : A \to B(H_\varphi)$. The most natural representation of $A$ on itself is left multiplication. As $H_\varphi$ is basically $A$ itself (up to the defect $N_\varphi$), the idea is to define $\pi_\varphi$ as a left multiplication operator, which goes as follows. For $a \in A$, we first consider

$$\pi_\varphi^0(a) : K_\varphi \to K_\varphi, \quad \pi_\varphi^0(a)(\gamma(y)) := \gamma(ay), y \in A.$$ 

Note that $a^*a \leq \|a^*a\|1$ (in $\overline{A}$) by Prop. 4.12(b), and hence $y^*a^*ay \leq \|a^*a\|y^*y$ (in $A$) by Prop. 4.12(a). Thus, the map $\pi_\varphi^0(a)$ is continuous, as

$$\|\pi_\varphi^0(a)(\gamma(y))\|^2 = \|\gamma(ay)\|^2 = \varphi(y^*a^*ay) \leq \|a^*a\|\varphi(y^*y) = \|a\|^2\|\gamma(y)\|^2,$$

for all $y \in A$; this shows $\|\pi_\varphi^0(a)\| \leq \|a\|$ for all $a \in A$. In particular, $\pi_\varphi^0(a)$ is well-defined: If $\gamma(y) = \gamma(y')$, then

$$\|\gamma(ay) - \gamma(ay')\|^2 = \|\gamma(a(y - y'))\|^2 \leq \|a\|^2\|\gamma(y - y')\|^2 = 0.$$ 

Moreover, $a \mapsto \pi_\varphi^0(a)$ is obviously linear and multiplicative. It is involutive, as

$$\langle \pi_\varphi^0(a)\gamma(y), \gamma(z) \rangle = \langle \gamma(ay), \gamma(z) \rangle = \varphi(z^*ay) = \varphi((a^*z)^*y) = \langle \gamma(y), \pi_\varphi^0(a^*)(\gamma(z)) \rangle,$$

for all $y, z \in A$; this shows $\pi_\varphi^0(a)^* = \pi_\varphi^0(a^*)$ for all $a \in A$. Finally, for $a \in A$, we extend $\pi_\varphi^0(a) : K_\varphi \to K_\varphi$ to $\pi_\varphi(a) : H_\varphi \to H_\varphi$. Then $\|\pi_\varphi(a)\| \leq \|a\|$. Summarizing,

$$\pi_\varphi : A \to B(H_\varphi), \quad a \mapsto \pi_\varphi(a), \quad \text{where } \pi_\varphi(a)(\gamma(y)) = \gamma(ay), \quad y \in A,$$

is a *-homomorphism, i.e. it is a representation of $A$ on $H_\varphi$.

(3) Construction of the cyclic vector $x_\varphi \in H_\varphi$. Let $(u_\lambda)_{\lambda \in \Lambda}$ be an approximate unit for $A$ (which exists by Thm. \[.\]). Then $(\gamma(u_\lambda))_{\lambda \in \Lambda}$ is a Cauchy net in $K_\varphi$. Indeed, let $\lambda \geq \mu$. Then $u_\lambda \geq u_\mu$ by Def. \[.\] and we have $1 \geq u_\lambda \geq u_\lambda - u_\mu \geq 0$.

We infer $(u_\lambda - u_\mu)^2 \leq u_\lambda - u_\mu$ by functional calculus. As $\varphi$ is order preserving, this implies

$$\|\gamma(u_\lambda) - \gamma(u_\mu)\|^2 = \|\gamma(u_\lambda - u_\mu)\|^2 = \varphi((u_\lambda - u_\mu)^2) \leq \varphi(u_\lambda - u_\mu).$$

Since $(\varphi(u_\lambda))_{\lambda \in \Lambda}$ is an increasing bounded net in $\mathbb{C}$, see the proof of Prop. \[.\] we conclude that $(\gamma(u_\lambda))_{\lambda \in \Lambda}$ is a Cauchy net in $K_\varphi$. Hence, we may define:

$$x_\varphi := \lim_{\lambda} \gamma(u_\lambda) \in H_\varphi$$
Now, for any \( a \in A \), the net \( \pi_\varphi(a)\gamma(u_\lambda) = \gamma(au_\lambda) \) converges to \( \gamma(a) \), from which we deduce \( \pi_\varphi(a)x_\varphi = \gamma(a) \) for all \( a \in A \). Thus, \( K_\varphi \subseteq \pi_\varphi(A)x_\varphi \), which shows that \( x_\varphi \) is a cyclic vector.

(4) We have \( \varphi(a) = \langle \pi_\varphi(a)x_\varphi, x_\varphi \rangle \) for all \( a \in A \). Indeed, we check
\[
\langle \pi_\varphi(a)x_\varphi, x_\varphi \rangle = \lim_{\lambda} \langle \pi_\varphi(a)\gamma(u_\lambda), \gamma(u_\lambda) \rangle = \lim_{\lambda} \varphi(u_\lambda au_\lambda) = \varphi(a)
\]
for all \( a \in A \).

We thus have a one-to-one correspondence between states and cyclic representations (up to equivalence classes of representations).

5.8. Existence of faithful representations. We derive the Second Fundamental Theorem of \( C^* \)-algebras as a consequence of the GNS construction. As in Lecture 3, note that the naming as “Fundamental Theorem” is not common – we only use it here in these lectures. The theorem below is sometimes called (noncommutative) Gelfand-Naimark Theorem or 2nd Gelfand-Naimark Theorem.

**Theorem 5.19.** Any \( C^* \)-algebra \( A \) possesses a faithful representation \( \pi : A \to B(H) \) on some Hilbert space \( H \). Thus, \( A \) is isomorphic to a \( C^* \)-subalgebra of \( B(H) \).

**Proof.** Let \( a \in A \) with \( a \neq 0 \). Then \( a^*a \in A \) is normal and by Thm. [5.13](#) we find a state \( \varphi : A \to \mathbb{C} \) with \( \varphi(a^*a) = \|a^*a\| = \|a\|^2 \neq 0 \). By the GNS construction (Thm. [5.18](#)), we obtain a Hilbert space \( H_\varphi \) and a representation \( \pi_\varphi : A \to B(H_\varphi) \) with a cyclic vector \( x_\varphi \in H_\varphi \) such that
\[
\|\pi_\varphi(a)x_\varphi\|^2 = \langle \pi_\varphi(a^*a)x_\varphi, x_\varphi \rangle = \varphi(a^*a) \neq 0.
\]
Hence, \( \pi_\varphi(a) \neq 0 \).

We then put \( H := \bigoplus_{\varphi \in S(A)} H_\varphi \) and \( \pi := \bigoplus_{\varphi \in S(A)} \pi_\varphi \), where \( S(A) \) is the set of all states on \( A \). By the previous consideration, we have \( \pi(a) \neq 0 \) for all \( a \in A \) with \( a \neq 0 \), i.e. \( \pi \) is faithful. By Prop. [4.24](#), \( A \) is then isomorphic to \( \pi(A) \subseteq B(H) \).

We conclude that any \( C^* \)-algebra may be represented concretely on a Hilbert space. This is a quite deep insight, for various reasons. On a practical level, it allows us to add a Hilbert space structure to our given \( C^* \)-algebra, if we need it. Thus, we may also use Hilbert space techniques when working with \( C^* \)-algebras. On a more philosophical level, it means that we could also define \( C^* \)-algebras as norm closed \( * \)-subalgebras of \( B(H) \) – and this definition would be equivalent to the more abstract, axiomatic one given in Def. [2.1](#). Some remarks on these possible definitions and the history behind the GNS construction may be found in Sect. [5.10](#).

**Remark 5.20.** If \( A \) is a separable \( C^* \)-algebra, then there is a faithful representation \( \pi : A \to B(H) \) on a separable Hilbert space \( H \), see Exc. [5.5](#).

**Remark 5.21.** In Def. [5.14](#), we learned that a closed subspace \( K \subseteq H \) of a Hilbert space is invariant under a representation \( \pi : A \to B(H) \), if \( \pi(A)K \subseteq K \). In that case, we say that \( K \) reduces \( \pi \). If \( \pi \) has no reducing subspaces (i.e. the only invariant subspaces are 0 and \( H \)), we say that \( \pi \) is irreducible.
The theory of irreducible representations is a subject of its own and we could easily devote a whole lecture to it. However, we decided to skip this part of the theory as it won’t be really necessary for the remainder of the lectures. Just to mention a few facts on irreducible representations: We can write representations as direct sums of irreducible representations; there is a definition of a pure state and the corresponding GNS construction yields an irreducible representation; on $C(X)$, the pure states correspond exactly to Dirac measures, see also Exm. 5.2(a); the pure states (together with 0) form the extremal points of all positive linear functionals with norm less or equal to one, on a given $C^*$-algebra, by the Krein-Milman Theorem, we may show that for any non-zero element $x \in A$ in a $C^*$-algebra $A$, there is an irreducible representation $\pi$ such that $\|\pi(x)\| = \|x\|$, in analogy to Thm. 5.13 and Thm. 5.19.

5.9. Exercises.

**Exercise 5.1.** Verify the details of the proof of Lemma 5.3. Let $A$ be a $C^*$-algebra and let $\varphi : A \to \mathbb{C}$ be a positive linear functional. Put $\langle x, y \rangle := \varphi(y^*x)$, for $x, y \in A$.

(a) Check that $\langle \cdot, \cdot \rangle$ is a positive sesquilinear form.
(b) Check that $\langle \cdot, \cdot \rangle$ satisfies the polarisation identity and that we have $\overline{\langle x, y \rangle} = \langle y, x \rangle$ for all $x, y \in A$. (This holds true for any positive sesquilinear form.)
(c) Convince yourself of a Cauchy-Schwarz inequality in the following form: $|\langle x, y \rangle|^2 \leq \varphi(x^*x)\varphi(y^*y)$, for all $x, y \in A$.
(d) Show that $N_\varphi := \{ x \in A \mid \langle x, x \rangle = 0 \} \subseteq A$ is a closed linear subspace of $A$.
(e) Let $\gamma : A \to A/N_\varphi$ be the quotient map. Let $x, x' \in A$ with $x-x' \in N_\varphi$. Show that $\langle x, y \rangle = \langle x', y \rangle$ for all $y \in A$ using (c). Deduce that $\langle \gamma(x), \gamma(y) \rangle := \langle x, y \rangle$ is a well-defined inner product on $A/N_\varphi$.

**Exercise 5.2.** Let $A$ be a $C^*$-algebra and let $\varphi : A \to \mathbb{C}$ be a state. Let $\pi_\varphi : A \to B(H_\varphi)$ be the associated GNS representation. A state $\varphi$ is called faithful, if $\varphi(x^*x) = 0$ implies $x = 0$, for $x \in A$.

(a) Let $I \lhd A$ be a closed ideal. Show that $I \subseteq \ker \pi_\varphi$ holds if and only if $I \subseteq \ker \varphi$.
(b) Show that $\pi_\varphi$ is faithful, if $\varphi$ is faithful.

**Exercise 5.3.** Let $\text{tr} : M_N(\mathbb{C}) \to \mathbb{C}$, $(t_{ij}) \mapsto \frac{1}{N} \sum_i t_{ii}$ be the normalized trace as in Exm. 5.2. Let $B \in M_N(\mathbb{C})$. We define $\tau_B(T) := \text{tr}(BT)$ for $T \in M_N(\mathbb{C})$.

(a) Let $B$ be positive. Show that $\tau_B$ is a positive linear functional with $\|\tau_B\| = \text{tr}(B)$.
(b) Compute the values $\tau_B(E_{ij})$, $i, j = 1, \ldots, N$, where $E_{ij} \in M_N(\mathbb{C})$ are the matrix units, i.e. the $i$-$j$-th entry of $E_{ij}$ is one, and zero otherwise.
(c) Let $\tau$ be a positive linear functional on $M_N(\mathbb{C})$. Show that $\tau = \tau_B$ for some matrix $B \in M_N(\mathbb{C})$ (in fact, it can be shown that $B$ must be positive). Use (b) in order to find $B$.
(d) Characterize all states on $M_N(\mathbb{C})$. 
(e) Show that the trace $\text{tr}$ on $M_N(\mathbb{C})$ is a faithful state.

**Exercise 5.4.** Consider $A = M_N(\mathbb{C})$ and its trace $\text{tr} : M_N(\mathbb{C}) \to \mathbb{C}$.

(a) What does the GNS construction yield for $\text{tr}$? Determine all components: $(H_{\text{tr}}, \pi_{\text{tr}}, x_{\text{tr}})$. Is this GNS representation faithful?

(b) Let $B = E_{11} \in M_N(\mathbb{C})$, see Exc. 5.3 for a definition. Consider $\tau_B$ as in Exc. 5.3. Determine all components of its GNS construction. Is this GNS representation faithful?

5.10. **Comments on the Second Fundamental Theorem of $C^*$-algebras.** Let us comment a bit on the history of the Second Fundamental Theorem of $C^*$-algebras. Representing certain algebras on Hilbert spaces (i.e. as subalgebras of some $B(H)$) is an old business. For instance, let $G$ be a locally compact group. If $G$ is abelian, its dual, consisting in all continuous group homomorphisms $\varphi : G \to S^1 \subseteq \mathbb{C}$ (characters) forms a locally compact group again, the so called dual group $\hat{G}$. Here, $S^1$ is seen as a multiplicative group. Now, we can take the dual group of the dual group – and we obtain $\hat{\hat{G}} \cong G$, i.e. we may reconstruct $G$ from its dual group. This is the famous Pontryagin Duality.

Now, how about a non-abelian locally compact group $G$? Unfortunately, the duality principle breaks down. So, we need to come up with some more sophisticated notion of a dual, if we want to have some object from which we may reconstruct $G$. One idea is to replace group homomorphisms $\varphi : G \to S^1 \subseteq \mathbb{C}$ (characters) forms a locally compact group again, the so called dual group $\hat{G}$. Here, $S^1$ is seen as a multiplicative group. Now, we can take the dual group of the dual group – and we obtain $\hat{\hat{G}} \cong G$, i.e. we may reconstruct $G$ from its dual group. This is the famous Pontryagin Duality.

Next, how about a non-abelian locally compact group $G$? Unfortunately, the duality principle breaks down. So, we need to come up with some more sophisticated notion of a dual, if we want to have some object from which we may reconstruct $G$. One idea is to replace group homomorphisms $\varphi : G \to S^1 \subseteq \mathbb{C}$ by group homomorphisms $\varphi : G \to M_N(\mathbb{C})$. Or $\varphi : G \to B(H)$, if you want. The philosophy is then to “understand” $G$ by “understanding” all of its representations.

One may associate a group algebra $\mathbb{C}G$ to $G$. Interestingly, representations of $G$ on $B(H)$ correspond to representations of $\mathbb{C}G$ on $B(H)$. In other words: The representation theory of groups (on Hilbert spaces) boils down to the representation theory of group algebras on Hilbert spaces.

This was the starting point for Gelfand’s work on $C^*$-algebras: Being interested in the representation theory of groups on Hilbert spaces, he wanted to understand the theory of subalgebras of $B(H)$. As we are dealing with unitary representations, these subalgebras should be closed under taking adjoints (i.e. they are $^*$-subalgebras) and since we are interested in topological groups, we also want to require some topological closure. Choosing the operator norm topology, we end up with: $C^*$-algebras! These
are the right subalgebras of $B(H)$ to consider. By the way, choosing the closure in the weak or the strong operator topology, we obtain von Neumann algebras.

Summarizing: From the perspective of representation theory of groups, we might be interested in norm closed *-subalgebras of $B(H)$. Shouldn’t this be our definition of a $C^*$-algebra then? Why do we define $C^*$-algebras in an abstract way instead, as certain *-Banach algebras obeying some strange $C^*$-identity? Because it is more conceptual in its axiomatic nature – and it is equivalent!

The latter is the content of Thm. 5.19: Given any abstractly defined $C^*$-algebra (in the sense of Def. 2.1), it is isomorphic to some concrete $C^*$-algebra, i.e. to some $C^*$-subalgebra of some $B(H)$. Conversely, any such concrete $C^*$-algebra is also an abstract $C^*$-algebra in the sense of Def. 2.1. We conclude: The axiomatic definition of $C^*$-algebras (Def. 2.1) and the concrete one (as norm closed *-subalgebra of some $B(H)$) are equivalent.

In that respect, one could say in retrospective: the merit of the seminal Gelfand-Naimark article [13] is to provide an alternative, axiomatic definition of $C^*$-algebras – which is equivalent to the concrete definition (2nd Fundamental Theorem). And it includes classical topology in form of commutative algebras (1st Fundamental Theorem).
6. Universal $C^*$-algebras

Abstract. We introduce the concept of universal $C^*$-algebras. We show that the following $C^*$-algebras may be viewed as universal $C^*$-algebras: The algebra of functions on the circle $S^1$, the matrix algebras $M_N(\mathbb{C})$, the algebra of compact operators $\mathcal{K}(H)$ on a separable Hilbert space, as well as the so-called Toeplitz algebra. We explain how the latter one may be seen as an extension of the function algebra $C(S^1)$ by the compact operators $\mathcal{K}(H)$.

6.1. Definition of universal $C^*$-algebras. In this lecture, we turn to a quite modern way of dealing with $C^*$-algebras: universal $C^*$-algebras. First, recall the definition of a $C^*$-algebra (Def. 2.1): a $C^*$-algebra is a complex algebra equipped with an involution; moreover, it possesses a norm which is submultiplicative and which satisfies the $C^*$-identity; finally, the algebra is complete with respect to this norm. This provides us with a recipe how to cook up $C^*$-algebras abstractly: we begin with a $\ast$-algebra; we find a good norm on it; and we complete. Let’s do this systematically in terms of generators and relations.

Definition 6.1. Let elements $E = \{x_i \mid i \in I\}$ be given, where $I$ is some index set.

(a) A noncommutative monomial in $E$ is a word $x_{i_1} \cdots x_{i_m}$ with $i_1, \ldots, i_m \in I$ and $m \in \mathbb{N}$.

(b) A noncommutative polynomial in $E$ is a complex linear combination of noncommutative monomials: $\sum_{k=1}^{N} \alpha_k y_k$ with $N \in \mathbb{N}$, $\alpha_k \in \mathbb{C}$ and $y_1, \ldots, y_N$ being noncommutative monomials in $E$.

(c) On noncommutative monomials, we consider the concatenation of words, i.e.

$$(x_{i_1} \cdots x_{i_m}) \cdot (x_{j_1} \cdots x_{j_n}) := x_{i_1} \cdots x_{i_m} x_{j_1} \cdots x_{j_n},$$

where $x_{i_1} \cdots x_{i_m}$ and $x_{j_1} \cdots x_{j_n}$ are two monomials.

(d) The free (complex) algebra on the generator set $E$ is given as the set of noncommutative polynomials in $E$ together with the canonical addition and scalar multiplication, and the multiplication of elements given by the concatenation.

Note that the order of the elements plays a role for such noncommutative monomials, i.e. $x_1 x_2 \neq x_2 x_1$ in the free algebra. Moreover, the algebra is “free” in the sense that the elements $x_i$ satisfy no relations, i.e. the only polynomial in the generators which is zero, is the zero polynomial itself. Hence, the free algebra has the following universal property: Whenever $B$ is some algebra containing elements $\{y_i \mid i \in I\}$ (where we even allow $y_i = y_j$ for some $i, j \in I$), there is a replacement homomorphism from the free algebra to $B$ sending $x_i$ to $y_i$, for all $i \in I$.

Given $E = \{x_i \mid i \in I\}$, we may add another set of generators $E^* := \{x_i^* \mid i \in I\}$ and we may define an involution on the free algebra on $E \cup E^*$ by extending

$$(\alpha x_{i_1}^{\epsilon_1} \cdots x_{i_m}^{\epsilon_m})^* := \bar{\alpha} x_{i_m}^{\epsilon_m} \cdots x_{i_1}^{\epsilon_1}$$
to linear combinations; here \( \alpha \in \mathbb{C}, \varepsilon_k \in \{1, *\} \) and \( \tilde{\varepsilon}_k := \begin{cases} 1 & \text{if } \varepsilon_k = * \\ * & \text{if } \varepsilon_k = 1 \end{cases} \). In this way, we obtain the free \(*\)-algebra \( P(E) \) on the generator set \( E \). Note that any polynomial \( p \in P(E) \) can be viewed as an algebraic relation when considering the equation \( p = 0 \), see also Exm. 6.3.

**Definition 6.2.** We consider the following data:

(i) Let \( E = \{x_i \mid i \in I\} \) be a set elements, \( I \) some index set.

(ii) Let \( R \subseteq P(E) \) be a set of polynomials.

Let \( J(R) \subseteq P(E) \) be the two-sided ideal generated by \( R \). The universal \(*\)-algebra with generators \( E \) and relations \( R \) is defined as the quotient \( A(E \mid R) := P(E)/J(R) \).

The image of an element \( x_i \in E \) in \( A(E \mid R) \) is denoted by \( x_i \) again, by some slight (but very common) abuse of notation.

**Example 6.3.** Let \( E = \{x\} \) and \( R = \{x^2, xx^*x - x\} \subseteq P(E) \). Then \( A(E \mid R) \) is the universal \(*\)-algebra with generator \( x \) such that the relations \( x^2 = 0 \) and \( xx^*x = x \) hold. Using these relations, we see that the only monomials in \( A(E \mid R) \) are \( x, x^*, xx^* \) and \( xx^*x \), hence \( A(E \mid R) \) is four-dimensional.

We now want to find a \( C^*\)-norm on \( A(E \mid R) \). Let's consider \( C^*\)-seminorms first.

**Definition 6.4.** Let \( A \) be a \(*\)-algebra (i.e. a complex algebra with an involution).

A \( C^*\)-seminorm on \( A \) is a map \( p : A \to [0, \infty) \) such that

(i) \( p(\lambda x) = |\lambda|p(x) \) and \( p(x + y) \leq p(x) + p(y) \) for all \( x, y \in A, \lambda \in \mathbb{C} \),

(ii) \( p(xy) \leq p(x)p(y) \) for all \( x, y \in A \),

(iii) and \( p(x^*x) = p(x)^2 \) for all \( x \in A \).

We are now ready for the main definition of today’s lecture.

**Definition 6.5.** Let \( E \) be a set of generators and \( R \subseteq P(E) \) be relations. Put

\[
\|x\| := \sup\{p(x) \mid p \text{ is a } C^*\text{-seminorm on } A(E \mid R)\}.
\]

If \( \|x\| < \infty \) for all \( x \in A(E \mid R) \), it is not difficult to show that \( \|\cdot\| \) is a \( C^*\)-seminorm (see Lemma 6.6) and we define the universal \( C^*\)-algebra \( C^*(E \mid R) \) as the completion with respect to \( \|\cdot\| \):

\[
C^*(E \mid R) := \overline{A(E \mid R)/\{x \in A(E \mid R) \mid \|x\| = 0\}}^{\|\cdot\|}
\]

In the same way, we may define enveloping \( C^*\)-algebras for any given \(*\)-algebra \( A \). Let us quickly check that our definition makes sense.

**Lemma 6.6.** Let \( E = \{x_i \mid i \in I\} \) be a set of generators and \( R \subseteq P(E) \) be relations.

(a) If \( \|x\| < \infty \) for all \( x \in A(E \mid R) \), then \( C^*(E \mid R) \) is a \( C^*\)-algebra and we say that the universal \( C^*\)-algebra of \( E \) and \( R \) exists.

(b) If there is a constant \( C > 0 \) such that \( p(x_i) < C \) for all \( i \in I \) and all \( C^*\)-seminorms \( p \) on \( A(E \mid R) \), then \( \|x\| < \infty \) for all \( x \in A(E \mid R) \).
Proof. By definition, it is clear that \( \| \cdot \| \) is a \( C^* \)-seminorm; hence we obtain a norm on the quotient \( A(E | R)/\{ x \in A(E | R) \mid \| x \| = 0 \} \) and the completion yields a \( C^* \)-algebra. This shows (a). As for (b), the norm of any monomial in \( E \) of length \( N \) is bounded by \( C^N \) and hence any polynomial in \( A(E | R) \) has bounded norm. \( \square \)

6.2. The universal property. So, we have some criterion for the existence of \( C^*(E | R) \), see Lemma 6.6. However, it could still be the case, that the construction yields the trivial \( C^* \)-algebra: we could have \( C^*(E | R) = 0 \). In order to exclude triviality, we need to find a non-trivial \( * \)-homomorphism from our universal \( C^* \)-algebra to another (non-trivial) \( C^* \)-algebra. For this the following property is very useful, ensuring the existence of many \( * \)-homomorphisms.

Let \( E = \{ x_i \mid i \in I \} \) be a set of generators and \( R \subseteq P(E) \) be relations. We say that elements \( \{ y_i \mid i \in I \} \) in some \( * \)-algebra \( B \) satisfy the relations \( R \), if all polynomials \( p \in R \) are zero, when we replace each \( x_i \) by \( y_i \), for all \( i \in I \).

**Proposition 6.7.** Let \( E = \{ x_i \mid i \in I \} \) be a set of generators and \( R \subseteq P(E) \) be relations. Let \( B \) be a \( C^* \)-algebra containing a subset \( E' = \{ y_i \mid i \in I \} \). If the elements \( E' \) satisfy the relations \( R \), then there is a unique \( * \)-homomorphism \( \varphi : C^*(E | R) \to B \) sending \( x_i \) to \( y_i \), for all \( i \in I \).

Proof. The ideal generated by \( R \) vanishes in \( B \) by assumption. Hence, the replacement homomorphism from the free \( * \)-algebra \( P(E) \) to \( B \), sending \( x_i \in P(E) \) to \( y_i \in B \), for all \( i \in I \), induces a homomorphism \( \varphi_0 : A(E | R) \to B \) sending \( x_i \in A(E | R) \) to \( y_i \in B \), for all \( i \in I \). For \( x \in A(E | R) \), put \( p(x) := \| \varphi_0(x) \|_B \). This is a \( C^* \)-seminorm and we conclude \( \| \varphi_0(x) \|_B \leq \| x \| \), by Def. 6.5. Hence, \( \varphi_0 \) is continuous and we may extend it to a \( * \)-homomorphism \( \varphi : C^*(E | R) \to B \). Uniqueness is by Lemma 3.26. \( \square \)

From the proof above, we understand in retrospective, why we defined the norm in Def. 6.5 exactly this way: any \( * \)-homomorphism \( \varphi : A \to B \) between \( C^* \)-algebras yields a \( C^* \)-seminorm \( p(x) := \| \varphi(x) \| \) on \( A \), where \( x \in A \). So, since \( * \)-homomorphisms on \( C^* \)-algebras always satisfy \( \| \varphi(x) \| \leq \| x \| \), see Lemma 3.8, the only way to obtain a “universal” norm on a \( * \)-algebra is by taking the supremum of all \( C^* \)-seminorms.

**Example 6.8.** Let us revisit Exm. 6.3 and consider the universal \( C^* \)-algebra generated by \( E = \{ x \} \) and the relations \( R = \{ x^2, xx^*x - x \} \). We write shorthand \( C^*(x | x^2 = 0, xx^*x = x) \). Since \( p(x)^2 = p(x^*x) = p(xx^*x) = p(x^*x) = p(x)^4 \) for any \( C^* \)-seminorm \( p \) on \( A(x | x^2 = 0, xx^*x = x) \) – and hence \( p(x) \in \{ 0, 1 \} \) – we infer that \( C^*(x | x^2 = 0, xx^*x = x) \) exists, by Lemma 6.6(b).

Now, consider \( y := E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{C}) \). Since \( y^2 = 0 \) and \( yy^*y = y \), there is a \( * \)-homomorphism \( \varphi : C^*(x | x^2 = 0, xx^*x = x) \to M_2(\mathbb{C}) \) mapping \( x \) to \( y \), by the universal property (Prop. 6.7). Moreover, \( \varphi \) is surjective, as it hits all matrix units (see Exc 2.5): \( \varphi(y) = E_{12}, \varphi(y^*) = E_{21}, \varphi(yy^*) = E_{11} \) and \( \varphi(yy^*) = E_{11} \).
But as $C^*(x \mid x^2 = 0, xx^*x = x)$ is four-dimensional, see Exm. 6.3, we conclude that $\varphi$ is an isomorphism, i.e. $M_2(\mathbb{C})$ may be written as the universal $C^*$-algebra $C^*(x \mid x^2 = 0, xx^*x = x)$. In particular, this shows $C^*(x \mid x^2 = 0, xx^*x = x) \neq 0$.

Let us interpret this example a bit. Suppose we are interested in nilpotent partial isometries for some reason. We compute the universal $C^*$-algebra of some reason. We compute the universal $C^*$-algebra of a single nilpotent partial isometry: it is $M_2(\mathbb{C})$. What does this tell us? Well, note that $M_2(\mathbb{C})$ is simple (Exc. 2.5) – so, the universal $C^*$-algebra generated by a nilpotent partial isometry is simple. This means: We may not add any relations to a nilpotent partial isometry – whenever we have a nilpotent partial isometry satisfying further relations (which are not implied by the relations $x^2 = 0$ and $xx^*x = x$), it must be trivial, see also Exm. 6.10. We conclude: the knowledge of the universal $C^*$-algebra tells us which additional relations may (or may not) hold, encoded in the ideal structure of the $C^*$-algebra; if the $C^*$-algebra is simple, we reached the end – no further relations may be added.

We now come to two non-examples.

**Example 6.9.** The $C^*$-algebra $C^*(x \mid x = x^*)$ does not exist: For any $\lambda > 0$, we find a $C^*$-algebra $B$ and a selfadjoint element $y \in B$ with $\|y\|_B = \lambda$. Then $p(\sum_{k=1}^N \alpha_k x^k) := \sum_{k=1}^N \alpha_k y^k \|_B$, $N \in \mathbb{N}$, $\alpha_k \in \mathbb{C}$, defines a $C^*$-seminorm on $A(x \mid x = x^*)$ with $p(x) = \lambda$. Hence $\|x\| = \infty$ in Def. 6.5.

**Example 6.10.** The $C^*$-algebra $C^*(x \mid x^2 = 0, xx^*x = x, x = x^*)$ exists (all $C^*$-seminorms are bounded, see Exm. 6.8) – but $C^*(x \mid x^2 = 0, xx^*x = x, x = x^*) = 0$, since $\|x\|^2 = \|x^*x\| = \|x^2\| = 0$.

6.3. **Example: matrix algebras.** The remainder of this lecture is devoted to the study of further examples of universal $C^*$-algebras. Our goal is to write well-known $C^*$-algebras as universal $C^*$-algebras. We have seen that $M_2(\mathbb{C})$ can be written as a universal $C^*$-algebra, see Exm. 6.8. How about $M_N(\mathbb{C})$ in general?

As before, denote by $E_{ij} \in M_N(\mathbb{C})$, $i, j = 1, \ldots, N$ the matrix units, i.e. the $i$-$j$-th entry of $E_{ij}$ is one and it is zero otherwise.

**Proposition 6.11.** Let $N \geq 2$. The following $C^*$-algebras are isomorphic.

1. $M_N(\mathbb{C})$
2. $C^*(e_{ij}, i, j = 1, \ldots, N \mid e_{ij}^* = e_{ji}, e_{ij}e_{kl} = \delta_{jk}e_{il}$ for all $i, j, k, l$)
3. $C^*(x_i, i = 1, \ldots, N \mid x_i^*x_j = \delta_{ij}x_j$ for all $i, j$)

**Proof.** Denote the $C^*$-algebra in (ii) by $A_1$ and the one in (iii) by $A_2$. We first check that $A_1$ exists: We have $p(e_{jj})^2 = p(e_{jj}^*e_{jj}) = p(e_{jj}) \in \{0, 1\}$ for all $j$ and thus $p(e_{jj})^2 = p(e_{jj}^*e_{jj}) = p(e_{jj}) \leq 1$ for all $i, j$ and all $C^*$-seminorms $p$. Let us now show $M_N(\mathbb{C}) \cong A_1$. It is easy to check that the matrix units $E_{ij} \in M_N(\mathbb{C})$ satisfy the relations of $A_1$; by the universal property (Prop. 6.7), we thus obtain a surjective $\ast$-homomorphism $\varphi : A_1 \rightarrow M_N(\mathbb{C})$ sending $e_{ij}$ to $E_{ij}$, for all $i, j$. The monomials in $A_1$ are exactly the elements $e_{ij}$, hence $A_1$ is $N^2$-dimensional and we conclude that $\varphi$ is also injective.
As for $A_1 \cong A_2$, see Exc. 6.1 for details: the universal property of $A_1$ yields a *-homomorphism $\varphi : A_1 \to A_2$ sending $e_{ij}$ to $x_i x_j^*$ while the universal property of $A_2$ yields a *-homomorphism $\psi : A_2 \to A_1$ sending $x_i$ to $e_i$. The homomorphisms are inverse to each other, which shows the isomorphism. In particular, the elements $x_i$ correspond to $E_{i1} \in M_N(\mathbb{C})$.

**Corollary 6.12.** Let $B$ be any $C^*$-algebra with $f_{ij} \in B$, $i, j = 1, \ldots, N$ satisfying $f_{ij} = f_{ji} \neq 0$ and $f_{ij} f_{kl} = \delta_{jk} f_{il}$ for all $i, j, k, l$. Let $B' := C^*(f_{ij}, i, j = 1, \ldots, N) \subseteq B$ be the $C^*$-subalgebra generated by the elements $f_{ij}$. Then $B' \cong M_N(\mathbb{C})$.

**Proof.** In Exc. 2.5, we have seen that $M_N(\mathbb{C})$ is simple, i.e. the only closed ideals in $M_N(\mathbb{C})$ are $\{0\}$ and $M_N(\mathbb{C})$ itself. Now, given a $C^*$-algebra $B$ with the asserted properties, we find a *-homomorphism $\varphi : M_N(\mathbb{C}) \to B$ sending $E_{ij}$ to $f_{ij}$, for all $i, j$, by Prop. 6.11 and Prop. 6.7. The kernel $\ker \varphi$ is an ideal in $M_N(\mathbb{C})$, hence $\ker \varphi = \{0\}$. Likewise for the statement on $D$.

6.4. Example: algebra of compact operators on a separable Hilbert space.

There is an infinite analog of Prop. 6.11.

**Proposition 6.13.** The following $C^*$-algebras are isomorphic.

(i) The algebra of compact operators $K(H)$ on a separable Hilbert space $H$.
(ii) $C^*(e_{ij}, i, j \in \mathbb{N} \mid e_{ij}^* = e_{ji}, e_{ij} e_{kl} = \delta_{jk} e_{il}$ for all $i, j, k, l$)
(iii) $C^*(x_i, i \in \mathbb{N} \mid x_i^* x_j = \delta_{ij} x_1$ for all $i, j$)

**Proof.** We leave the verification of the existence of the universal $C^*$-algebras as an exercise, and the isomorphism with the $C^*$-algebra in item (iii) as well, see Exc. 6.1. Denote the $C^*$-algebra in (ii) by $A$. We prove $K(H) \cong A$.

Let $(e_n)_{n \in \mathbb{N}}$ be an orthonormal basis of $H$. For $i, j \in \mathbb{N}$, let $f_{ij} \in B(H)$ be the operator given by $f_{ij} e_n := \delta_{jn} e_i$, for $n \in \mathbb{N}$. (Observe that $f_{ij}$ is an infinite analog of a matrix unit: while $E_{ij}$ maps the $j$-th basis vector of $\mathbb{C}^N$ to the $i$-th basis vector, $f_{ij}$ does the same in $H$.) Then $f_{ij} \in K(H)$, since its range is one-dimensional. By the universal property, there is a *-homomorphism $\varphi : A \to K(H)$ mapping $e_{ij}$ to $f_{ij}$, for all $i, j$. Let us show that $\varphi$ is an isomorphism.

The image of $\varphi$ contains all linear combinations of the maps $f_{ij}$ – in fact, even all limits of such linear combinations, by Prop. 4.24. Now, any compact operator may be approximated by limits of linear combinations of the maps $f_{ij}$, see Exc. 6.2. We conclude that $\varphi$ is surjective.

As for injectivity, put $M_N := C^*(e_{ij}, i, j = 1, \ldots, N) \subseteq A$, for $N \in \mathbb{N}$. Then, $M_N \cong M_N(\mathbb{C})$, by Cor. 6.12. Let $\varphi_N$ be the restriction of $\varphi$ to $M_N \subseteq A$. Then $\varphi_N \neq 0$ is injective, since its kernel is an ideal in $M_N \cong M_N(\mathbb{C})$ and $M_N(\mathbb{C})$ is simple. Thus, $\varphi_N$ is isometric, by Prop. 4.15. Hence, $\varphi$ is isometric on the dense subset $\bigcup_{N \in \mathbb{N}} M_N \subseteq A$. This implies that $\varphi$ is also isometric on all of $A$, i.e. $\varphi$ is injective. 

\[]
Similarly to $M_N(\mathbb{C})$, one can show that $\mathcal{K}(H)$ is simple, given $H$ is separable, see Exc. 6.3. Thus, an analog of Cor. 6.12 holds true, as follows. Note that we allow ourselves to consider any countable, infinite set $I$ as an indexing set, since such a set is in bijection with the indexing set $\mathbb{N}$ of Prop. 6.13.

**Corollary 6.14.** If $B$ is a $C^*$-algebra with $f_{ij} \in B$, $i, j \in I$, where $I$ is a countable and infinite set, and if $f_{ij} = f_{ji} \neq 0$ and $f_{ij}f_{kl} = \delta_{jk}f_{il}$ for all $i, j, k, l$, then $C^*(f_{ij}, i, j \in I) \subseteq B$ is isomorphic to $\mathcal{K}(H)$.

**Proof.** Similar to the proof of Cor. 6.12. □

6.5. **Example: algebra of functions on the circle.** In Exm. 6.8 we were wondering about the universal $C^*$-algebra generated by a single nilpotent partial isometry, so about a “universal single nilpotent partial isometry”, if you want. In the same sense we now ask: what is the “universal unitary” (in the sense of Def. 1.33)? Before thinking about this question in precise terms, let us think intuitively.

We know that the spectrum of a unitary is a subset of the circle $S^1$, see Prop. 3.30. So, the universal unitary shall allow for all possible spectra of unitaries. Hence, the universal unitary shall have full spectrum: it shall be all of $S^1$. A unitary with spectrum $S^1$ is also called a Haar unitary, by the way. As the $C^*$-algebra generated by a unitary is commutative, our guess is that the universal $C^*$-algebra generated by a unitary is isomorphic to the functions on its spectrum, i.e. to $C(S^1)$. This is indeed the case, as we will see soon.

Coming back to precise maths, we consider $C^*(u, 1 | u^*u = uu^* = 1)$, the universal $C^*$-algebra generated by a unitary. In fact, it has two generators: $u$ and $1$. The relations are $u^*u = uu^* = 1$, but also the relations that turn $1$ into the unit, so $1u = u1 = u$ and $1^2 = 1^* = 1$. We usually omit to write down these relations regarding the unit, and we sometimes even omit to write down $1$ as a generator.

Checking that $C^*(u, 1 | u^*u = uu^* = 1)$ exists is easy: we have $p(1)^2 = p(1^*)^1 = p(1) \in \{0, 1\}$ and $p(u)^2 = p(u^*u) = p(1) \in \{0, 1\}$ for any $C^*$-seminorm $p$; so $C^*(u, 1 | u^*u = uu^* = 1)$ exists by Lemma 6.6.

**Proposition 6.15.** Let $A$ be a unital $C^*$-algebra and $z \in A$ be a unitary with $\text{sp}(z) = S^1$. Then $C^*(u, 1 | u^*u = uu^* = 1) \cong C^*(z) \subseteq A$.

**Proof.** We denote $C^*(u) := C^*(u, 1 | u^*u = uu^* = 1)$. Let $A$ be a unital $C^*$-algebra, $z \in A$ a unitary and $\text{sp}(z) = S^1$. By the universal property (Prop. 6.7), there is a *-homomorphism $\varphi : C^*(u) \to A$ sending $u$ to $z$; it is surjective onto $C^*(z)$.

On the other hand, $C^*(z) \subseteq A$ is a commutative $C^*$-algebra and we have the isomorphism $\Psi_z : C(\text{sp}(z)) \to C^*(z) \subseteq A$ from functional calculus, see Thm. 3.28. By the same argument, there is an isomorphism $\Psi_u : C(\text{sp}(u)) \to C^*(u)$. Let $\Phi : C(S^1) \to C(\text{sp}(u))$ be the restriction map $f \mapsto f_{\text{sp}(u)}$, $f \in C(S^1)$. Observe that $C(\text{sp}(z)) = C(S^1)$, by assumption. Putting everything together, we obtain a map $\psi : C^*(z) \to C^*(u)$ as the composition $\psi := \Psi_u \circ \Phi \circ \Psi_z^{-1}$. It maps $z$ to $u$. Hence $\psi \circ \varphi$ maps $u$ to $u$, so $\psi \circ \varphi$ coincides with the identity homomorphism $id : A \to A$, by Lemma 3.26. This shows that $\varphi$ is injective. □
Corollary 6.16. We have \( C^*(u, 1 \mid u^*u = uu^* = 1) \cong C(S^1) \).

Proof. Denote the identity map on \( S^1 \) by \( z \), so \( z(t) = t \) for all \( t \in S^1 \) and \( z \in C(S^1) \). Then \( z \) is a unitary: \( z^*z = zz^* = 1 \). Here, \( 1 \) denotes the constant function \( 1(t) = 1 \), \( t \in S^1 \) on \( S^1 \). We have \( \text{sp}(z) = S^1 \). Moreover, \( C^*(z) = C(S^1) \) by the Stone-Weierstrass Theorem (Thm. 3.3). \( \square \)

6.6. The bilateral shift. We learned from Cor. 6.16 that the identity function on \( S^1 \) is a “universal unitary”. Another such unitary is the bilateral shift. Recall it from Exc. 1.7: let \( H \) be a separable Hilbert space with orthonormal basis \( (e_n)_{n \in \mathbb{Z}} \). The bilateral shift \( \tilde{S} \in B(H) \), given by \( \tilde{S}e_n = e_{n+1}, n \in \mathbb{Z} \), is a unitary. We want to compute its spectrum. In order to do so, let \( \lambda \in S^1 \) and denote by \( d(\lambda) \in B(H) \) the diagonal operator given by \( d(\lambda)e_n = \lambda^n e_n, n \in \mathbb{Z} \).

Lemma 6.17. On the Hilbert space \( H \) with orthonormal basis \( (e_n)_{n \in \mathbb{Z}} \), we have:

(a) \( d(\lambda)d(\lambda') = d(\lambda\lambda') \) and \( d(\lambda)^* = d(\overline{\lambda}) \), for all \( \lambda, \lambda' \in S^1 \).

(b) \( d(\lambda) \) is a unitary and we have \( d(\lambda)\tilde{S} = \lambda\tilde{S}d(\lambda) \), for all \( \lambda \in S^1 \).

(c) The map \( \beta_\lambda : C^*(\tilde{S}) \to C^*(\tilde{S}) \) given by \( T \mapsto d(\lambda)Td(\lambda)^* \) is a *-isomorphism.

(d) The spectrum of the bilateral shift is \( \text{sp}(\tilde{S}) = S^1 \).

Proof. Item (a) is straightforward. Thus, \( d(\lambda) \) is a unitary in the sense of Def. 1.33
Moreover, we check for \( n \in \mathbb{Z} \):

\[
d(\lambda)\tilde{S}e_n = d(\lambda)e_{n+1} = \lambda^{n+1}e_{n+1} = \lambda\tilde{S}d(\lambda)e_n
\]

Regarding (c), consider the map \( \alpha_\lambda : B(H) \to B(H) \) given by \( T \mapsto d(\lambda)Td(\lambda)^* \). It can be verified directly that it is a *-homomorphism. As it maps \( \tilde{S} \) to \( \lambda\tilde{S} \), by (b), we infer that its restriction \( \beta_\lambda \) to \( C^*(\tilde{S}) \) yields a *-homomorphism from \( C^*(\tilde{S}) \) to itself. Its inverse is given by \( \beta_\lambda^{-1} \), so \( \beta_\lambda \) is a *-isomorphism.

For (d), we use (c) and Lemma 3.8(a):

\[
\text{sp}(\tilde{S}) = \text{sp}(\beta_\lambda(\tilde{S})) = \lambda\text{sp}(\tilde{S})
\]

holds for all \( \lambda \in S^1 \). This shows \( \text{sp}(\tilde{S}) = S^1 \). \( \square \)

Proposition 6.18. We have \( C^*(u, 1 \mid u^*u = uu^* = 1) \cong C^*(\tilde{S}) \subseteq B(H) \), where \( \tilde{S} \in B(\ell^2(\mathbb{Z})) \) is the bilateral shift operator as in Exc. 1.7

Proof. This follows from Prop. 6.15 and Lemma 6.17 \( \square \)

6.7. The unilateral shift. We consider an analog of the preceding subsection for the unilateral shift, see Exc. 1.7. Let \( H \) be a separable Hilbert space with orthonormal basis \( (e_n)_{n \in \mathbb{N}} \). The unilateral shift \( S \in B(H) \), given by \( Se_n = e_{n+1}, n \in \mathbb{N} \), is an isometry \( (S^*S = 1) \) which is not a unitary \( (SS^* \neq 1) \). Denote again by \( d(\lambda) \in B(H) \) the diagonal operator given by \( d(\lambda)e_n = \lambda^n e_n, n \in \mathbb{N} \), given some \( \lambda \in S^1 \).

Recall from Prop. 1.38 that \( K(H) \) is a closed ideal in \( B(H) \). It can be written as the closure of the span of all rank one operators \( f_{ij}, i, j \in \mathbb{N} \), see Exc. 6.2.
Lemma 6.19. We have $f_{ij} = S^{i-1}(1 - SS^*)(S^*)^{j-1}$ for all $i,j \in \mathbb{N}$. Hence, the compact operators $\mathcal{K}(H)$ form an ideal in $C^*(S)$.

Proof. The formula $f_{ij} = S^{i-1}(1 - SS^*)(S^*)^{j-1}$ is a direct verification. Hence $f_{ij} \in C^*(S)$ for all $i,j \in \mathbb{N}$. Thus $\mathcal{K}(H) \subseteq C^*(S)$ by Exc. 6.2. Since $\mathcal{K}(H)$ is an ideal in $B(H)$, so it is in $C^*(S)$.

A slight modification of Lemma 6.17 holds true. Denote the quotient map by $\sigma : B(H) \to B(H)/\mathcal{K}(H)$.

Lemma 6.20. On the Hilbert space $H$ with orthonormal basis $(e_n)_{n \in \mathbb{N}}$, we have:

(a) $d(\lambda)d(\lambda') = d(\lambda\lambda')$ and $d(\lambda)^* = d(\bar{\lambda})$, for all $\lambda, \lambda' \in S^1$.
(b) $d(\lambda)$ is a unitary and we have $d(\lambda)S = \lambda Sd(\lambda)$, for all $\lambda \in S^1$.
(c) The map $\beta_\lambda : C^*(S) \to C^*(S)$ given by $T \mapsto d(\lambda)Td(\lambda)^*$ is a *-isomorphism with $\beta_\lambda(\mathcal{K}(H)) \subseteq \mathcal{K}(H)$.
(d) $\sigma(S) \in B(H)/\mathcal{K}(H)$ is a unitary and its spectrum is $\text{sp}(\sigma(S)) = S^1$.
(e) The quotient of $C^*(S)$ by $\mathcal{K}(H)$ is isomorphic to $C(S^1)$. Hence, we have the following short exact sequence (see also Rem. 4.25 and Exc. 4.6):

$$0 \to \mathcal{K}(H) \to C^*(S) \to C(S^1) \to 0$$

Proof. Items (a) and (b) are as in Lemma 6.17. Also, the fact that $\beta_\lambda$ is a *-isomorphism is analogous. It maps $S$ to $\lambda S$; thus it maps $f_{ij}$ to $\lambda^{i-j}f_{ij}$, by Lemma 6.19. We infer $\beta_\lambda(\mathcal{K}(H)) \subseteq \mathcal{K}(H)$. Hence, $\beta_\lambda$ induces a *-isomorphism $\hat{\beta}_\lambda : C^*(S)/\mathcal{K}(H) \to C^*(S)/\mathcal{K}(H)$, $\sigma(T) \mapsto \sigma(\beta_\lambda(T))$.

As in Lemma 6.17, we conclude $\text{sp}(\sigma(S)) = S^1$. Note that $\sigma(1 - SS^*) = \sigma(f_{11}) = 0$, i.e. $\sigma(S)\sigma(S)^* = 1$ and hence the isometry $\sigma(S)$ is actually a unitary.

For (e), note that $\mathcal{K}(H)$ is an ideal in $C^*(S) \subseteq B(H)$, by Lemma 6.19. By (d) and Prop. 6.15, the quotient $C^*(\sigma(S)) = C^*(S)/\mathcal{K}(H)$ is isomorphic to $C(S^1)$. □

6.8. The Toeplitz algebra. In Sect. 6.5, we considered the universal $C^*$-algebra generated by a unitary. How about an isometry?

Definition 6.21. The Toeplitz algebra $\mathcal{T}$ is the universal $C^*$-algebra generated by an isometry:

$$\mathcal{T} := C^*(v, 1 \mid vv^* = 1)$$

Reflecting upon our discussion in Sect. 6.2 on universal $C^*$-algebras in general, we may ask: what is the difference between an isometry $v$ and a unitary in terms of relations? It is the relation $vv^* = 1$. So, if we add the relation $vv^* = 1$ to the Toeplitz algebra, we obtain the universal $C^*$-algebra generated by a unitary. In other words: taking the quotient by the ideal $\langle 1 - vv^* \rangle \triangleleft \mathcal{T}$ generated by $1 - vv^*$, we obtain $C(S^1)$; we write $\langle 1 - vv^* \rangle \triangleleft \mathcal{T}$ for the ideal generated by $1 - vv^* \in \mathcal{T}$, i.e. $\langle 1 - vv^* \rangle$ is the smallest closed (two-sided) ideal in $\mathcal{T}$ containing $1 - vv^*$. How to describe this ideal, can we name a well-known $C^*$-algebra to which this ideal is isomorphic? We prepare an answer. We use the notation $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. 
Lemma 6.22. Consider the generator \( v \in \mathcal{T} \).

(a) The element \( 1 - vv^* \in \mathcal{T} \) is a projection in the sense of Def. 1.33. It is called the defect projection of \( v \).

(b) We have \( v^*(1 - vv^*) = (1 - vv^*)v = 0 \).

(c) The ideal \( \langle 1 - vv^* \rangle \) coincides with the closed linear span \( I \) of all elements \( g_{ij} := v^i(1 - vv^*)(v^*)^j \), for \( i, j \in \mathbb{N}_0 \).

Proof. From the relation \( v^*v = 1 \), the assertions in (a) and (b) follow immediately. As for (c), clearly, \( I \subseteq \langle 1 - vv^* \rangle \), since all elements \( g_{ij} \) lie in \( \langle 1 - vv^* \rangle \).

In order to show \( I \supseteq \langle 1 - vv^* \rangle \), we need to convince ourselves that \( I \) is an ideal. Firstly, note that \( v^jg_{ij} = g_{i+1,j} \) and \( v^jg_{i,j+1} = g_{ij} \) for all \( i, j \in \mathbb{N}_0 \); moreover, \( v^*g_{ij} = 0 \) for all \( j \in \mathbb{N}_0 \), by (b). We conclude \( vI, v^*I \subseteq I \). Thus, \( v^k(v^*)(v^*)^lI \subseteq I \) for all \( k, l \in \mathbb{N}_0 \).

Since elements in \( \mathcal{T} \) may be approximated by linear combinations of element \( v^k(v^*)^l \), this shows \( xI \subseteq I \) for all \( x \in \mathcal{T} \). Now, as \( I \) is closed under taking adjoints, we infer \( Ix = (x^*)^*I I \subseteq I \) for all \( x \in \mathcal{T} \). This proves that \( I \) is a two-sided ideal. Of course, it is also a closed linear subspace, so we conclude \( I = \langle 1 - vv^* \rangle \). \( \square \)

Proposition 6.23. The ideal \( \langle 1 - vv^* \rangle \triangleleft \mathcal{T} \) is isomorphic to \( \mathcal{K}(H) \), where \( H \) is a separable Hilbert space. The quotient by this ideal is isomorphic to \( C(S^1) \). Hence, we have the following short exact sequence:

\[
0 \to \mathcal{K}(H) \to \mathcal{T} \to C(S^1) \to 0
\]

Proof. We first describe the ideal \( \langle 1 - vv^* \rangle \). Consider the elements \( g_{ij} \) from Lemma 6.22. It is clear that \( g_{ij}^* = g_{ji} \) holds. Furthermore, for \( j > k \), we have \( (v^*)^j v^k(1 - vv^*) = (v^*)^j v^k (1 - vv^*) = 0 \) by Lemma 6.22(b). Likewise, \( j < k \) implies \( (1 - vv^*)(v^*)^j v^k = 0 \). Hence, for \( j \neq k \),

\[
g_{ij}g_{kl} = v^j(1 - vv^*)(v^*)^j v^k(1 - vv^*)(v^*)^l = 0,
\]

whereas \( j = k \) implies \( g_{ij}g_{kl} = g_{il} \). Thus, \( g_{ij}g_{kl} = \delta_{jk}g_{il} \) for all \( i, j, k, l \). By Cor. 6.14, \( C^*(g_{ij}, i, j \in \mathbb{N}_0) \cong \mathcal{K}(H) \). Note that \( C^*(g_{ij}, i, j \in \mathbb{N}_0) = I \), where \( I \) is as in Lemma 6.22(c), so \( \langle 1 - vv^* \rangle \cong \mathcal{K}(H) \) by Lemma 6.22(c).

Finally, recall that \( C(S^1) \) is isomorphic to the universal \( C^* \)-algebra \( C^* (u, 1 | u^*u = uu^* = 1) \), by Cor. 6.16. As \( u \) is in particular an isometry, there is a *-homomorphism \( \varphi : \mathcal{T} \to C^* (u) \) sending \( v \) to \( u \), by the universal property (Prop. 6.7). As \( \varphi (1 - vv^*) = 0 \), we have \( \varphi (\langle 1 - vv^* \rangle) = 0 \). Thus, there is a *-homomorphism \( \hat{\varphi} : \mathcal{T} / \langle 1 - vv^* \rangle \to C^* (u) \) sending \( \hat{v} \) to \( u \), where \( \hat{v} \in \mathcal{T} / \langle 1 - vv^* \rangle \) denotes the image of \( v \) under the quotient map.

On the other hand, \( \hat{v} \) is a unitary. Thus \( \psi : C^* (u) \to \mathcal{T} / \langle 1 - vv^* \rangle \) mapping \( u \) to \( \hat{v} \) exists by the universal property. We conclude that \( \varphi \) and \( \psi \) are inverse to another finishing the proof. For the short exact sequence, see Rem. 4.25 and Exc. 4.6. \( \square \)

We can interpret the above proposition by saying that the Toeplitz algebra is very close to \( C(S^1) \) – or rather that the universal isometry is almost a universal unitary – up to a small defect: the compacts.
Finally, let us consider an analog of Prop. 6.18. There, we saw that the bilateral shift is a model of a universal unitary. Shouldn’t the unilateral shift be a model of a universal isometry then? Yes, that is the case.

**Corollary 6.24.** The canonical \( \varphi: \mathcal{T} \to C^*(S) \subseteq B(H) \) mapping \( v \) to \( S \), where \( S \) is the unilateral shift on a separable Hilbert space \( H \), is an isomorphism.

**Proof.** This is a classical diagram chase (Lemma 6.27) given the exact sequences

\[
0 \to \mathcal{K}(H) \to \mathcal{T} \to C(S^1) \to 0
\]

from Prop. 6.23 and

\[
0 \to \mathcal{K}(H) \to C^*(S) \to C(S^1) \to 0
\]

from Lemma 6.20; the short exact sequences are linked by the identity maps on \( \mathcal{K}(H) \) and \( C(S^1) \) respectively, as well as \( \varphi: \mathcal{T} \to C^*(S) \) producing a commutative diagram. See Lemma 6.27 for finishing the proof. \( \square \)

**Remark 6.25.** The Toeplitz algebra is often introduced in a different form: as the algebra of Toeplitz operators, see for instance [10, Sect. V.1]. The idea is as follows. Consider the Hilbert space \( L^2(S^1) \) with orthonormal basis \( (e_n)_{n \in \mathbb{Z}} \) given by \( e_n = z^n \), where \( z \) is the identity function on \( S^1 \). Let \( H^2 \subseteq L^2(S^1) \) be the space spanned by \( (e_n)_{n \geq 0} \), called the Hardy space, and let \( P_{H^2} \) be the projection onto this space. For \( g \in L^\infty(S^1) \), let \( M_g \in B(L^2(S^1)) \) be the multiplication operator given by \( M_g(f) := fg, f \in L^2(S^1) \). The Toeplitz operator \( T_g \in B(H^2) \) is defined as \( T_g := P_{H^2} M_g \), for \( g \in L^\infty(S^1) \).

Observe that \( T_z e_n = T_z z^n = z^{n+1} = e_{n+1} \) for \( n \geq 0 \), so \( T_z \) is the unilateral shift on the Hardy space. One can show:

\[ \mathcal{T} \cong C^*(T_z) = \{ T_g + K \mid g \in C(S^1), K \in \mathcal{K}(H^2) \} \subseteq B(H^2) \]

In a way, this is just a reformulation of Cor. 6.24 and Prop. 6.23: the Toeplitz algebra is an extension of \( C(S^1) \) by the compacts.

**Remark 6.26.** There is the famous Wold decomposition of isometries [10, Sect. V.2]: let \( w \) be an isometry on a Hilbert space \( H \). Then, \( w \) is unitarily equivalent to \( (S \otimes 1) \oplus u \), where \( S \) is the unilateral shift, \( S \otimes 1 \) is an amplification of the shift and \( u \) is a unitary. In other words: the unilateral shift is basically the only isometry!

Coburn then showed [10, Sect. V.2]: if \( w \) is a proper isometry on some Hilbert space \( H \), i.e. \( w^* w = 1 \) and \( ww^* \neq 1 \), then \( C^*(w) \subseteq B(H) \) is isomorphic to the Toeplitz algebra \( \mathcal{T} \). This is a generalization of Cor. 6.24.
6.9. **Diagram chase.** We finish this lecture by mentioning a classic in homological algebra and category theory: the (short) five lemma. We formulate it for C*-algebras, but it holds in much wider generality.

Assume we have the following commutative diagram of two short exact sequences.

\[
\begin{array}{ccc}
0 & \rightarrow & I_1 \xrightarrow{\iota_1} A_1 \xrightarrow{\pi_1} B_1 \rightarrow 0 \\
\downarrow{\alpha} & & \downarrow{\beta} \\
0 & \rightarrow & I_2 \xrightarrow{\iota_2} A_2 \xrightarrow{\pi_2} B_2 \rightarrow 0
\end{array}
\]

Explicitly, this means (see also Rem. 4.25 and Exc. 4.6): for \( j = 1, 2 \), let \( I_j, A_j \) and \( B_j \) be C*-algebras. Let \( \iota_j : I_j \rightarrow A_j \) be injective maps, \( \pi_j : A_j \rightarrow B_j \) be surjective maps, and assume \( \ker \pi_j = \operatorname{ran} \iota_j \). In particular, \( \iota_j(I_j) \) is a closed ideal in \( A_j \), and \( A_j/\iota_j(I_j) \cong B_j \), for \( j = 1, 2 \). For convenience, we may think of \( I_j \triangleleft A_j \) and \( \iota_j \) simply being the embeddings.

Moreover, assume that there are \( * \)-homomorphisms \( \alpha : I_1 \rightarrow I_2 \) and \( \varphi : A_1 \rightarrow A_2 \) and \( \beta : B_1 \rightarrow B_2 \) such that \( \pi_2 \circ \varphi = \beta \circ \pi_1 \) and \( \iota_2 \circ \alpha = \varphi \circ \iota_1 \), i.e. the diagram is commutative.

**Lemma 6.27 (Five Lemma).** Assume we have the following commutative diagram of two short exact sequences.

\[
\begin{array}{ccc}
0 & \rightarrow & I_1 \xrightarrow{\iota_1} A_1 \xrightarrow{\pi_1} B_1 \rightarrow 0 \\
\downarrow{\alpha} & & \downarrow{\beta} \\
0 & \rightarrow & I_2 \xrightarrow{\iota_2} A_2 \xrightarrow{\pi_2} B_2 \rightarrow 0
\end{array}
\]

If \( \alpha \) and \( \beta \) are \( * \)-isomorphisms, then also \( \varphi \) is a \( * \)-isomorphism.

**Proof.** The proof is fun and you should do it yourself once in your life. Here it goes.

\( \varphi \) is injective: Let \( x \in A_1 \) and \( \varphi(x) = 0 \). Then \( \beta \circ \pi_1(x) = \pi_2 \circ \varphi(x) = 0 \). Then \( \pi_1(x) = 0 \) as \( \beta \) is injective. Then \( x \in \ker \pi_1 = \operatorname{ran} \iota_1 \), i.e. \( x = \iota_1(y) \) for some \( y \in I_1 \).

Then \( \iota_2 \circ \alpha(y) = \varphi \circ \iota_1(y) = \varphi(x) = 0 \). Then \( y = 0 \) as \( \iota_2 \) and \( \alpha \) are injective. Then \( x = \iota_1(y) = 0 \) and \( \varphi \) is injective.

\( \varphi \) is surjective: Let \( y \in A_2 \). Then \( \pi_2(y) \in B_2 \). There is an \( x_0 \in A_1 \) with \( \beta \circ \pi_1(x_0) = \pi_2(y) \) since \( \beta \) and \( \pi_1 \) are surjective. Then \( \pi_2(y - \varphi(x_0)) = \pi_2(y) - \beta \circ \pi_1(x_0) = 0 \). Then \( y - \varphi(x_0) \in \ker \pi_2 = \operatorname{ran} \iota_2 \), i.e. \( y - \varphi(x_0) = \iota_2(z) \) for some \( z \in I_2 \).

Then \( z = \alpha(w) \) for some \( w \in I_1 \). Put \( x := \iota_1(w) + x_0 \in A_1 \). Then

\[
\varphi(x) = \varphi \circ \iota_1(w) + \varphi(x_0) = \iota_2 \circ \alpha(w) + \varphi(x_0) = (y - \varphi(x_0)) + \varphi(x_0) = y
\]

and \( \varphi \) is surjective. \[\square\]
6.10. Exercises.

Exercise 6.1. Let \( N \geq 2 \). As in Prop. 6.11, consider the universal \( C^* \)-algebras
\[
A_1 := C^*(e_{ij}, i, j = 1, \ldots, N \mid e_{ij} = e_{ji}, e_{ij}e_{kl} = \delta_{jk}e_{il} \text{ for all } i, j, k, l),
\]
\[
A_2 := C^*(x_i, i = 1, \ldots, N \mid x_i^*x_j = \delta_{ij}x_1 \text{ for all } i, j).
\]
(a) Show that \( x_1 \in A_2 \) is a projection in the sense of Def. 1.33. Moreover, verify
\((x_i x_1 - x_i)^* (x_i x_1 - x_i) = 0 \) and conclude \( x_i x_1 = x_i \) for all \( i \).
(b) Show that \( A_1 \) and \( A_2 \) exist, using Lemma 6.6(b).
(c) Show that there is a \( * \)-homomorphism \( \varphi : A_1 \rightarrow A_2 \) sending \( e_{ij} \) to \( x_i x_j^* \), for
all \( i, j = 1, \ldots, N \).
(d) Show that there is a \( * \)-homomorphism \( \psi : A_2 \rightarrow A_1 \) sending \( x_i \) to \( e_{ii} \), for all
\( i = 1, \ldots, N \).
(e) Show that \( \varphi \circ \psi = \text{id}_{A_2} \) and \( \psi \circ \varphi = \text{id}_{A_1} \). Hint: You only need to check this
(f) Conclude \( A_1 \cong A_2 \).
(g) Perform (a) to (d) also in the case of \( N = \infty \).

Exercise 6.2. Let \( (e_n)_{n \in \mathbb{N}} \) be an orthonormal basis of a separable Hilbert space \( H \).
For \( i, j \in \mathbb{N} \), let \( f_{ij} \in B(H) \) be the operator given by \( f_{ij} e_n := \delta_{jn} e_i \), for \( n \in \mathbb{N} \), as in
the proof of Prop. 6.13.
(a) Convince yourself that \( f_{ij} \) is a rank one operator and \( f_{ij} \in \mathcal{K}(H) \).
(b) Let \( F \) be the set of all linear combinations of the maps \( f_{ij}, i, j \in \mathbb{N} \). Show
that \( F \) is dense in \( \mathcal{K}(H) \). Hint: Use Prop. 1.38 and the approximate unit
given by \( p_n = \sum_{i=1}^{n} f_{ii}, n \in \mathbb{N} \).

Exercise 6.3. Let \( H \) be a separable Hilbert space and let \( 0 \neq I \triangleleft B(H) \) be a closed
ideal.
(a) Show that all rank one operators \( f_{ij} \) from Exc. 6.2 are contained in \( I \).
(b) Use (a) and Exc. 6.2 to show that \( \mathcal{K}(H) \subseteq I \).
(c) Deduce that \( \mathcal{K}(H) \) is simple. (Use Lemma 4.22.)

Exercise 6.4. Let \( N \in \mathbb{N} \). We view \( \mathbb{C}^N \) as a \( C^* \)-algebra with pointwise operations.
In other words, given the finite set \( X_N = \{1, \ldots, N\} \), we view \( \mathbb{C}^N = C(X_N) \) in a
natural way.
(a) Show that \( C^*(p, 1 \mid p = p^2 = p^*) \cong \mathbb{C}^2 \).
(b) Show that \( C^*(p_1, \ldots, p_N, 1 \mid \sum_{k=1}^{N} p_k = 1) \cong \mathbb{C}^N \).
(c) Show that \( C^*(u, 1 \mid u^*u = uu^* = 1, u = u^*) \cong \mathbb{C}^2 \).
(d) Show that \( C^*(u, 1 \mid u^*u = uu^* = 1, u^N = 1) \cong \mathbb{C}^N \).
(e) Write down an explicit isomorphism between \( C^*(p, 1 \mid p = p^2 = p^*) \) and
\( C^*(u, 1 \mid u^*u = uu^* = 1, u = u^*) \).
7. Universal $C^*$-algebras II: rotation algebra and Cuntz algebra

Abstract. We continue our investigation of examples of universal $C^*$-algebras. We introduce the famous irrational rotation algebra $A_\vartheta$ (also called the noncommutative torus), we construct two faithful conditional expectations of it and composing them we obtain a faithful tracial state. This enables us to show that $A_\vartheta$ is simple. We then turn to another famous $C^*$-algebra: the Cuntz algebra $O_n$. Again, we find a conditional expectation and we show that $O_n$ is purely infinite – which implies that it is simple, too. The proofs of simplicity for $A_\vartheta$ and $O_n$ have some similarities which we will point out.

7.1. Definition and existence of the rotation algebra $A_\vartheta$.

Definition 7.1. Let $\vartheta \in \mathbb{R}$. The rotation algebra (also called the noncommutative torus) is defined as the universal $C^*$-algebra

$$A_\vartheta := C^*(u,v \mid u,v \text{ are unitaries, } uv = e^{2\pi i \vartheta} vu).$$

We often abbreviate $\lambda := e^{2\pi i \vartheta} \in S^1$. If $\vartheta \notin \mathbb{Q}$, then $A_\vartheta$ is called the irrational rotation algebra.

Later, we will see that the case $\vartheta \notin \mathbb{Q}$ behaves much nicer than $\vartheta \in \mathbb{Q}$. This is why the adjective “irrational” is sometimes dropped in the literature and “the rotation algebra” then refers to the irrational rotation algebra only.

As the generators $u$ and $v$ are unitaries, we have $p(u), p(v) \in \{0, 1\}$ for all $C^*$-seminorms $p$ on the $*$-algebra generated by $u$ and $v$. Hence, $A_\vartheta$ exists by Lemma 6.6. Note that we omit to write down the generator 1 corresponding to the unit; it is implicitly mentioned by the term “unitaries” – recall that $u$ is a unitary, if $u^* u = uu^* = 1$.

So, $A_\vartheta$ exists in the sense that the underlying universal $*$-algebra admits a $C^*$-norm, by Lemma 6.6. How about non-triviality?

Lemma 7.2. The rotation algebra $A_\vartheta$ may be represented on $\ell^2(\mathbb{Z})$ as follows. Let $\lambda := e^{2\pi i \vartheta}$. Consider the bilateral shift $\tilde{S}$ and the diagonal operator $d(\lambda)$ given by

$$\tilde{S}e_n = e_{n+1}, \quad d(\lambda)e_n = \lambda^n e_n, \quad \text{for all } n \in \mathbb{Z}.$$

Then $\pi : A_\vartheta \rightarrow B(\ell^2(\mathbb{Z}))$ mapping $u \mapsto d(\lambda)$ and $v \mapsto \tilde{S}$ is a representation of $A_\vartheta$.

Proof. This follows immediately from Lemma 6.17(b): we have $d(\lambda)\tilde{S} = \lambda \tilde{S} d(\lambda)$. \qed

An alternative representation is given in Exc. 7.1. We conclude that $A_\vartheta \neq 0$. Let us take a look at the elements in $A_\vartheta$. The following is a technical lemma. Denote by $S$ the set consisting of elements $\sum_{k,l \in \mathbb{Z}} a_{kl} u^k v^l$ with $a_{kl} \in \mathbb{C}$ such that only finitely many coefficients $a_{kl}$ are non-zero. Here, $u^{-k} := (u^*)^k$ and $v^{-k} := (v^*)^k$ for $k > 0$.

Lemma 7.3. In $A_\vartheta$, we have $u^* v = \lambda v^* u^*$, and more generally $u^k v^l = \lambda^{kl} v^l u^k$ for all $k,l \in \mathbb{Z}$. Moreover, the set $S$ is a $*$-algebra which is dense in $A_\vartheta$. 

Proof. Multiplying \(uv = \lambda vu\) with \(u^*\) from both sides implies \(u^*v = \bar{\lambda}vu^*\). Inductively we deduce \(u^kv^l = \lambda^{kl}u^kv^l\) for all \(k, l \in \mathbb{Z}\). It is then easy to check that \(S\) is a \(*\)-algebra containing \(u\) and \(v\). By the construction of universal \(C^*\)-algebras, it is thus dense in \(A_\vartheta\). \(\square\)

Let us justify the name “noncommutative torus” by looking at the case \(\vartheta = 0\).

Lemma 7.4. Let \(\vartheta = 0\). Then \(A_\vartheta \cong C(T^2)\), where \(T^2 \subseteq \mathbb{C}^2\) is the 2-torus.

Proof. Observe that \(A_\vartheta\) is commutative in case \(\vartheta = 0\). Indeed, it is the universal \(C^*\)-algebra generated by two commuting unitaries. Hence, by the Gelfand-Naimark Theorem (Thm. 3.23), it must be isomorphic to \(C(\text{Spec}(A_\vartheta))\). The spectrum \(\text{Spec}(A_\vartheta)\) is homeomorphic to the 2-torus in the case \(\vartheta = 0\). Indeed, given a character \(\varphi \in \text{Spec}(A_\vartheta)\), it is uniquely determined by the values \((\varphi(u), \varphi(v)) \in S^1 \times S^1 = T^2\), by Lemma 3.26 Conversely, every value in \((\mu_1, \mu_2) \in T^2\) gives rise to a character in \(\text{Spec}(A_\vartheta)\), simply because \(\mu_1\) and \(\mu_2\) are commuting unitaries in \(\mathbb{C}\); then use the universal property (Prop. 6.7). This shows the assertion.

Working out the isomorphism \(A_\vartheta \cong C(T^2)\), we infer that it is given as follows. Consider the functions \(\tilde{u}, \tilde{v} \in C(T^2)\) defined by \(\tilde{u}(\mu_1, \mu_2) := \mu_1\) and \(\tilde{v}(\mu_1, \mu_2) := \mu_2\), for \(\mu_1, \mu_2 \in S^1\). The \(*\)-homomorphism \(\varphi : A_\vartheta \rightarrow C(T^2)\) sending \(u \mapsto \tilde{u}\) and \(v \mapsto \tilde{v}\) is the above isomorphism. \(\square\)

So, if \(\vartheta = 0\), the rotation algebra \(A_\vartheta\) corresponds to the (algebra of functions on) the torus \(T^2\). Hence, \(A_\vartheta\) can be viewed as a kind of “algebra of functions on the noncommutative torus \(T^2_\vartheta\)” for general \(\vartheta \in \mathbb{R}\). Let us be clear: the noncommutative torus \(T^2_\vartheta\) does not exist as such! However, its “algebra of functions” does exist: it is \(A_\vartheta\). So, in the philosophy of Gelfand duality (Sect. 3.12), we study \(C(T^2)\) instead of its underlying space \(T^2\) — and we study \(A_\vartheta\) as if there was some underlying noncommutative space \(T^2_\vartheta\).

This way of thinking may appear to be very weird when being confronted with it for the first time. However, it is very instructive when working in the field: imagining an underlying object, we may develop questions about this object, we may get inspiration from related classical objects and we may express structural theorems in an intuitive way. We elaborate more on the noncommutative torus and its role in quantization in Sect. 7.9.

7.2. Conditional expectations and a tracial state on \(A_\vartheta\). We are now preparing the proof of an important property of \(A_\vartheta\): this \(C^*\)-algebra is simple. Our tools to prove this will be conditional expectations and tracial states.

Definition 7.5. Let \(A\) be a unital \(C^*\)-algebra, \(B \subseteq A\) a \(C^*\)-subalgebra and \(1 \in B\).

(a) A (conditional) expectation of \(A\) on \(B\) is a positive, linear, unital map \(\varphi : A \rightarrow B\) with \(\varphi \circ \varphi = \varphi\) (which is equivalent to \(\varphi(b) = b\) for all \(b \in \varphi(A)\)).

(b) A positive, linear map is faithful, if \(\varphi(a) = 0\) and \(a \geq 0\) imply \(a = 0\).
If \( B = \mathbb{C} \subseteq A \), we see that any state is an expectation. So, expectations can be seen as \( B \)-valued states. We prepare the construction of two conditional expectations of \( A_\vartheta \). Given \( \zeta, \mu \in S^1 \), we define

\[
\rho_{\zeta, \mu} : A_\vartheta \to A_\vartheta, \quad u \mapsto \zeta u, \; v \mapsto \mu v.
\]

This map exists by the universal property (Prop. 6.7), since \( u' := \zeta u \) and \( v' := \mu v \) are unitaries with \( u'v' = \lambda v'u' \). Note that \( \rho_{\zeta, \bar{\mu}} \) is inverse to \( \rho_{\zeta, \mu} \), so \( \rho_{\zeta, \mu} \) is in fact a *-isomorphism.

**Lemma 7.6.** Let \( \vartheta \in \mathbb{R} \) and let \( x \in A_\vartheta \).

(a) The maps \( f_x : [0, 1] \to A_\vartheta, \; t \mapsto f_x(1, e^{2\pi it}) \) and \( h_x : [0, 1] \to A_\vartheta, \; t \mapsto f_x(e^{2\pi it}, 1) \) are continuous in norm.

(b) The maps \( g_x : T^2 \to A_\vartheta, \; (\zeta, \mu) \mapsto \rho_{\zeta, \mu}(x) \) is continuous in norm.

(c) The Riemannian sums \( \frac{1}{n} \sum_{j=1}^n \rho_{1, e^{2\pi it}}(x) \) and \( \frac{1}{n} \sum_{j=1}^n \rho_{e^{2\pi it}, 1}(x) \) converge, for \( 0 = t_0 < t_1 < \ldots < t_n = 1 \) of the unit normal interval. We denote their limits by \( \int_0^1 \rho_{1, e^{2\pi it}}(x)dt \) and \( \int_0^1 \rho_{e^{2\pi it}, 1}(x)dt \) respectively.

**Proof.** Given \( x = \sum_{k,l \in \mathbb{Z}} a_{kl} u^k v^l \in S \subseteq A_\vartheta \), we compute

\[
\| f_x(\zeta_1, \mu_1) - f_x(\zeta_2, \mu_2) \| = \left\| \sum_{k,l \in \mathbb{Z}} a_{kl} (\zeta_1^k \mu_1^l - \zeta_2^k \mu_2^l) u^k v^l \right\| \leq \sum_{k,l \in \mathbb{Z}} |a_{kl}| |\zeta_1^k \mu_1^l - \zeta_2^k \mu_2^l|.
\]

This expression tends to zero as \( (\zeta_1, \mu_1) \) tends to \( (\zeta_2, \mu_2) \), which proves (a) for \( x \in S \).

For general \( x \in A_\vartheta \), we use Lemma 7.3. Now, also (b) follows immediately. Using \( g_x(t) = \rho_{1, e^{2\pi it}}(x) \) and \( h_x(t) = \rho_{e^{2\pi it}, 1}(x) \), we derive (c) just like in the classical case of Riemannian sums and integrals for complex valued functions.

We consider \( \varphi_1, \varphi_2 : A_\vartheta \to A_\vartheta \) given by

\[
\varphi_1(x) := \int_0^1 \rho_{1, e^{2\pi it}}(x)dt, \quad \varphi_2(x) := \int_0^1 \rho_{e^{2\pi it}, 1}(x)dt, \quad x \in A_\vartheta.
\]

**Lemma 7.7.** Let \( \vartheta \in \mathbb{R} \).

(a) For all \( \sum_{k,l \in \mathbb{Z}} a_{kl} u^k v^l \in S \) we have \( \varphi_1(\sum_{k,l \in \mathbb{Z}} a_{kl} u^k v^l) = \sum_{k \in \mathbb{Z}} a_{k0} u^k \) and \( \varphi_2(\sum_{k,l \in \mathbb{Z}} a_{kl} u^k v^l) = \sum_{l \in \mathbb{Z}} a_{0l} v^l \).

(b) We have \( \varphi_1|_{C^*(u)} = \text{id}|_{C^*(u)} \) and \( \varphi_2|_{C^*(v)} = \text{id}|_{C^*(v)} \).

(c) We have \( \varphi_1(A_\vartheta) = C^*(u) \subseteq A_\vartheta \) and \( \varphi_2(A_\vartheta) = C^*(v) \subseteq A_\vartheta \).

(d) The maps \( \varphi_j, j = 1, 2 \), are faithful conditional expectations with \( \| \varphi_j \| \leq 1 \).

**Proof.** For (a), we first consider \( l \in \mathbb{Z} \) and check, using Lemma 7.6 (c):

\[
\varphi_1(v^l) = \int_0^1 \rho_{1, e^{2\pi it}}(v^l)dt = \int_0^1 e^{2\pi it} v^l dt = \left( \int_0^1 e^{2\pi it} dt \right) v^l = \delta_{0l}.
\]

Secondly, let \( k, l \in \mathbb{Z} \) and \( x = u^k v^l \). Note that \( \rho_{1, e^{2\pi it}}(u^k v^l) = u^k \rho_{1, e^{2\pi it}}(v^l) \) for any \( t \in [0, 1] \). Hence, by Lemma 7.6 (c):

\[
\varphi_1(u^k v^l) = \int_0^1 \rho_{1, e^{2\pi it}}(u^k v^l)dt = u^k \left( \int_0^1 \rho_{1, e^{2\pi it}}(v^l)dt \right) = u^k \varphi_1(v^l) = \delta_{0l} u^k.
\]
Passing to linear combinations, we finish the proof of (a) for $\varphi_1$; similarly for $\varphi_2$.

As elements $\sum_{k \in \mathbb{Z}} a_k u^k$ with finitely many nonzero coefficients $a_k \in \mathbb{C}$ are dense in $C^*(u) \subseteq A_\theta$, item (b) follows from (a). For (c), we use Lemma 7.3.

As for (d), let $x \in A_\theta$. Note that $\|\rho_{1,e^{2x\pi i t}}(x)\| = \|x\|$ since $\rho_{1,e^{2x\pi i t}}$ is a *-isomorphism. Then $\|\frac{1}{n} \sum_{j=1}^n \rho_{1,e^{2x\pi it_j}}(x)\| \leq \|x\|$ for any $t \in [0, 1]$ and hence $\|\varphi_1(x)\| \leq \|x\|$ by Lemma 7.6(c). This proves $\|\varphi_1\| \leq 1$.

Furthermore, since $\rho_{1,e^{2x\pi it}}$ is linear and unital, it is clear that $\varphi_1$ is linear and unital, by Lemma 7.6(c). Moreover, given $x \geq 0$, we have $\rho_{1,e^{2x\pi it}}(x) \geq 0$ for any $t \in [0, 1]$. Thus, $\varphi_1(x) \geq 0$ by Lemma 7.6(c) and Prop. 4.9, since the cone of positive elements is closed.

For showing that $\varphi_1$ is faithful, let us assume $x \neq 0$ in addition to $x \geq 0$. Hence, $\rho_{1,e^{2x\pi it}}(x) \neq 0$ for any $t \in [0, 1]$, since $\rho_{1,e^{2x\pi it}}$ is a *-isomorphism. Now, let $\psi : A_\theta \to \mathbb{C}$ be a state with $\psi(\rho_{1,e^{2x\pi it_0}}(x)) \neq 0$ for some fixed $t_0 \in (0, 1)$. It exists by the Hahn-Banach Theorem (Thm. 5.13). Define $f : [0, 1] \to \mathbb{C}$ by $f(t) := \psi(\rho_{1,e^{2x\pi it}}(x))$, $t \in [0, 1]$. Then $f$ is continuous by Lemma 7.6, positive and non-zero. Thus,

$$
\psi(\varphi_1(x)) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^n \psi(\rho_{1,e^{2x\pi it_j}}(x)) = \frac{1}{n} \sum_{j=1}^n f(t_j) \longrightarrow \int_0^1 f(t) dt \neq 0,
$$

where we used Lemma 7.6(c) again. This shows $\varphi_1(x) \neq 0$.

Finally, $\varphi_1^2 = \varphi_1$ follows from (b) and (c), and the same for $\varphi_2^2 = \varphi_2$. □

The next lemma will be a key ingredient for the proof of simplicity of $A_\theta$: we have an algebraic description of $\varphi_1$ and $\varphi_2$. It allows us to deduce that $\varphi_1$ and $\varphi_2$ map ideals to themselves.

**Lemma 7.8.** Let $\vartheta \notin \mathbb{Q}$ and let $\lambda := e^{2\pi i \vartheta} \in S^1 \subseteq \mathbb{C}$ as before.

(a) Given $l \in \mathbb{Z}$, the sequence $\left(\frac{1}{2n+1} \sum_{j=-n}^n \lambda^{jl}\right)_{n \in \mathbb{N}}$ converges to $\delta_0$.

(b) For all $x \in A_\theta$, we have:

$$
\varphi_1(x) = \lim_{n \to \infty} \frac{1}{2n+1} \sum_{j=-n}^n u^j xu^{-j}, \quad \varphi_2(x) = \lim_{n \to \infty} \frac{1}{2n+1} \sum_{j=-n}^n v^j xv^{-j}
$$

(c) If $I \triangleleft A_\theta$ is a closed ideal, then $\varphi_1(I) \subseteq I$ and $\varphi_2(I) \subseteq I$. 


Proof. The statement in (a) is easy analysis, see Exc. 7.2. As for (b), let $k, l \in \mathbb{Z}$ and $x = u^kv^l$. We check:

\[
\frac{1}{2n+1} \sum_{j=-n}^{n} u^j(u^k v^l)u^{-j} = \frac{1}{2n+1} \sum_{j=-n}^{n} u^{j+k}v^l u^{-j} = \frac{1}{2n+1} \sum_{j=-n}^{n} \lambda^j u^{j+k}u^{-j}v^l = \left( \frac{1}{2n+1} \sum_{j=-n}^{n} \lambda^j \right) u^k v^l
\]

Since \( \left( \frac{1}{2n+1} \sum_{j=-n}^{n} \lambda^j \right)_{n \in \mathbb{N}} \) converges to \( \delta_{l0} \) by (a), this proves the assertion for \( x = u^k v^l \) and \( \varphi_1 \), using also Lemma 7.7(a). By linearity, it holds true for elements \( x \in S \); by continuity and Lemma 7.3 also for all \( x \in A_\theta \). Likewise for \( \varphi_2 \).

Finally, let \( I \triangleleft A_\theta \) be a closed ideal and \( x \in I \). Then \( \frac{1}{2n+1} \sum_{j=-n}^{n} u^j x u^{-j} \in I \) and hence \( \varphi_1(x) \in I \), by (b). Similarly, \( \varphi_2(I) \subseteq I \).

Note that (a) of the above lemma fails to be true, if \( \vartheta \notin \mathbb{Q} \), see Exc. 7.2. We see already here in Lemma 7.8, that the irrational case is nicer than the rational case.

7.3. Simplicity of \( A_\theta \). We are now going to prove the main result on \( A_\theta \) in this lecture: simplicity. We use the technique of faithful traces in order to do so.

Definition 7.9. Let \( A \) be a unital \( C^* \)-algebra. A \textit{(normalized) trace} (or a \textit{tracial state}) on \( A \) is a state \( \tau : A \to \mathbb{C} \) with \( \tau(xy) = \tau(yx) \) for all \( x, y \in A \).

We constructed the expectations \( \varphi_1 \) and \( \varphi_2 \) in the last section. The map \( \varphi_1 \) reads out the part generated by \( u \), whereas \( \varphi_2 \) reads out the one generated by \( v \), see Lemma 7.7(c). Composing these two maps, we shall obtain: \( \mathbb{C} \). This is how we construct our faithful trace.

Proposition 7.10. Putting \( \tau := \varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1 : A_\theta \to \mathbb{C} \subseteq A_\theta \) we obtain a unital faithful trace on \( A_\theta \). It satisfies

\[
\tau \left( \sum_{k,l \in \mathbb{Z}} a_{kl} u^k v^l \right) = a_{00}.
\]

If \( \vartheta \notin \mathbb{Q} \), it is the unique (normalized) trace on \( A_\theta \).

Proof. We first check \( \varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1 \). Let \( k, l \in \mathbb{Z} \). By Lemma 7.7(a):

\[
\varphi_1(\varphi_2(u^k v^l)) = \delta_{k0} \varphi_1(v^l) = \delta_{k0} \delta_{l0} = \delta_{l0} \varphi_2(u^k) = \varphi_2(\varphi_1(u^k v^l))
\]

Now, putting \( \tau := \varphi_1 \circ \varphi_2 \), we have \( \tau \left( \sum_{k,l \in \mathbb{Z}} a_{kl} u^k v^l \right) = a_{00} \in \mathbb{C} \subseteq A_\theta \) by Lemma 7.7(a). Moreover, \( \tau \) is positive, linear, and unital by Lemma 7.7(d), hence it is a
state; it is faithful, again by Lemma 7.7(d). For showing traciality, let $x = u^k v^l$ and $y = u^m v^n$, for $k, l, m, n \in \mathbb{Z}$. Then:

$$
\tau(xy) = \tau(u^k v^l u^m v^n) = \lambda^{-lm} \tau(u^{k+m} v^{l+n}) = \delta_{k+m,0} \delta_{l+n,0} \lambda^{-lm}
$$

$$
\tau(yx) = \tau(u^m v^n u^k v^l) = \lambda^{-nk} \tau(u^{k+m} v^{l+n}) = \delta_{k+m,0} \delta_{l+n,0} \lambda^{-nk}
$$

Since $k + m = 0$ and $l + n = 0$ imply $(k + m)n = 0$ and $l = -n$, we infer $nk = -mn = lm$. Thus, $\tau(xy) = \tau(yx)$ for $x = u^k v^l$ and $y = u^m v^n$. Passing to linear combinations and using the continuity of $\tau$, we derive $\tau(xy) = \tau(yx)$ for all $x, y \in A_\vartheta$; see also Lemma 7.3.

Finally, assume $\vartheta \notin \mathbb{Q}$ and let $\tau'$ be another normalized trace. Let $x \in A_\vartheta$. Note that $\tau'(u^j x u^{-j}) = \tau'(x)$ for all $j$, by traciality. Thus, by Lemma 7.8(b):

$$
\tau'(x) = \lim_{n \to \infty} \frac{1}{2n + 1} \sum_{j=-n}^{n} \tau'(u^j x u^{-j}) = \tau'(\varphi_1(x))
$$

$$
\tau'(x) = \lim_{n \to \infty} \frac{1}{2n + 1} \sum_{j=-n}^{n} \tau'(v^j x v^{-j}) = \tau'(\varphi_2(x))
$$

As $\tau'$ is unital, we conclude:

$$
\tau'(x) = \tau'(\varphi_1(x)) = \tau'(\varphi_2(\varphi_1(x))) = \tau'(\tau(x)) = \tau(x)
$$

We observe that the trace $\tau$ is given by Fourier analysis on $A_\vartheta$: it is the Fourier coefficient $a_{00}$.

**Theorem 7.11.** The irrational rotation algebra $A_\vartheta$ is simple.

**Proof.** Let $I$ be a non-zero closed ideal in $A_\vartheta$. We thus find an element $x \neq 0$ in $I$. Hence, we have $0 \neq x^* x \in I$. Since $\tau$ is faithful, we infer $\tau(x^* x) \neq 0$. On the other hand, $\tau(x^* x) = \varphi_1(\varphi_2(x^* x)) \in I$, by Lemma 7.8(c). Since $\tau(x^* x)$ is a nonzero multiple of $1 \in A_\vartheta$, this shows $1 \in I$ and hence $I = A_\vartheta$. 

As mentioned in the last lecture (Sect. 6.2), simplicity for the universal C*-algebra $A_\vartheta$ means, that we may add no further relations to those of $A_\vartheta$, if $\vartheta$ is irrational. Indeed, let $p$ be a polynomial in $u$, $u^*$, $v$ and $v^*$ (i.e. $p$ is a relation); by Thm. 7.11, the closed ideal $I$ generated by $p$ is either 0 (in which case the relation $p$ is already implied by the relations of $A_\vartheta$), or it is all of $A_\vartheta$ (in which case adding the relation $p$ to those of $A_\vartheta$ – i.e. taking the quotient of $A_\vartheta$ by $I$ – would yield a trivial C*-algebra). This is kind of surprising, because we do may add additional relations to the universal C*-algebra generated by two commuting unitaries (which is $A_\vartheta$ for $\vartheta = 0$, see Lemma 7.4), see also Exc. 6.4.

We end our discussion on $A_\vartheta$ here although a lot more could be said about it; it might be one of the best studied examples of a C*-algebra. See also Sect. 7.9.
Hence, the Cuntz algebra is generated by $n$ copies of the space.

**Definition 7.12.** Let $n \in \mathbb{N}$, $n \geq 2$. The Cuntz algebra is the universal $C^*$-algebra

$$O_n := C^*(S_1, \ldots, S_n \mid S_i \text{ is an isometry, for all } i = 1, \ldots, n, \sum_{i=1}^{n} S_i S_i^* = 1).$$

We may easily check that the Cuntz algebra exists in the sense of Lemma 6.6 and we check $O_n \neq 0$ by finding some nontrivial representation on a Hilbert space.

For instance, if $(e_k)_{k \in \mathbb{N}}$ is an orthonormal basis of a separable Hilbert space $H$, choose injective functions $f_1, \ldots, f_n : \mathbb{N} \to \mathbb{N}$ such that $f_i(\mathbb{N}) \cap f_j(\mathbb{N}) = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^{n} f_i(\mathbb{N}) = \mathbb{N}$ for their ranges. Putting $T_i e_k := e_{f_i(k)}$, for $i = 1, \ldots, n$ and $k \in \mathbb{N}$, we obtain isometries $T_1, \ldots, T_n \in B(H)$ with $\sum_i T_i T_i^* = 1$.

Hence, there is a non-trivial representation of $O_n$ proving $O_n \neq 0$. Note that the elements $T_i T_i^*$ are projections onto subspaces $K_i \subseteq H$ such that $H = K_1 \oplus \cdots \oplus K_n$.

Hence, the Cuntz algebra is generated by $n$ isometries decomposing the space into $n$ copies of the space.

**7.5. Words in $O_n$.** The main technical tool for studying Cuntz algebras is a detailed investigation of words in the generators.

**Definition 7.13.** Let $n \in \mathbb{N}$. A multi index is a tuple $\mu = (\mu_1, \ldots, \mu_k) \in \{1, \ldots, n\}^k$ and $|\mu| = k$ is its length. Denote by $\mathcal{M}(k)$ the set of all multi indices of length $k$. A word in $O_n$ is an element $S_{\mu} := S_{\mu_1} \cdots S_{\mu_k} \in O_n$.

Let us take a look at the arithmetics of such words.

**Lemma 7.14.** The words in $O_n$, $n \geq 2$ satisfy the following relations.

(a) $S_i^* S_j = \delta_{ij}$ for all $i, j = 1, \ldots, n$.

(b) Given multi indices $\mu$ and $\nu$, we have:

- If $|\mu| = |\nu|$, then $S_{\mu}^* S_{\nu} = \delta_{\mu \nu}$.

- If $|\mu| < |\nu|$, then $S_{\mu}^* S_{\nu} = \begin{cases} S_{\nu} & \text{if } \nu = \mu \nu' \\ 0 & \text{otherwise} \end{cases}$.

- If $|\mu| > |\nu|$, then $S_{\mu}^* S_{\nu} = \begin{cases} S_{\nu}^* & \text{if } \mu = \nu \mu' \\ 0 & \text{otherwise} \end{cases}$.

(c) Let $k \in \mathbb{N}$. Then $\sum_{\alpha \in \mathcal{M}(k)} S_{\alpha} S_{\alpha}^* = 1$.

(d) Let $\mu, \nu$ be multi indices with $|\mu| \neq |\nu|$ and $|\mu|, |\nu| \leq k$. Let $\alpha, \beta \in \mathcal{M}(k)$ and $\gamma \in \mathcal{M}(k+1)$. Then $S_{\gamma}^* S_{\alpha}^* (S_{\mu}^* S_{\nu}) S_{\beta} S_{\gamma} = 0$.

**Proof.** For (a) and $i = j$, the relation $S_i^* S_i = 1$ holds by definition, since $S_i$ is an isometry. For $i \neq j$, note that $S_i S_i^* + S_j S_j^* \leq \sum_k S_k S_k^* = 1$. Hence:

$$1 + S_i^* S_j S_j^* S_i = S_i^* (S_i S_i^* + S_j S_j^*) S_i \leq S_i^* S_i = 1$$
This shows $0 \leq -S^*_i S_j S_j^* S_i = -(S^*_i S_j)^*(S^*_j S_i)$. Thus $S^*_i S_i = 0$ by Lemma 4.7. The other relations in (b), (c) and (d) follow by direct algebraic manipulations and are left as an exercise, see Exc. 7.4.

As a consequence, we find matrix units in $\mathcal{O}_n$ – and hence copies of matrix algebras. This is a crucial technical observation on the structure of $\mathcal{O}_n$. We define:

$$\mathcal{F}^k_n := \text{span}\{S_\mu S^*_\nu \mid |\mu| = |\nu| = k\} \subseteq \mathcal{O}_n$$

$$\mathcal{F}_n := \text{span}\{S_\mu S^*_\nu \mid |\mu| = |\nu|\} \subseteq \mathcal{O}_n$$

$$\mathcal{S} := \text{span}\{S_\mu S^*_\nu \mid \mu, \nu \text{ arbitrary multi indices}\} \subseteq \mathcal{O}_n$$

**Lemma 7.15.** For $n \geq 2$, we have the following.

(a) For $l \leq k$, we have $\mathcal{F}^l_n \subseteq \mathcal{F}^k_n$. Hence, $\mathcal{F}^k_n = \cup_{l \leq k} \mathcal{F}^l_n$.

(b) $\mathcal{S}$ is dense in $\mathcal{O}_n$.

(c) $\mathcal{F}^k_n$ is isomorphic to $M_{n^k}(\mathbb{C})$. Under this isomorphism, $S^k_1(S^*_1)^k$ corresponds to the matrix unit $E_{11}$.

**Proof.** For (a), let $\mu, \nu \in \mathcal{M}(l)$. Then $S_\mu S^*_\nu = \sum_{\delta \in \mathcal{M}(k-l)} S_\mu S_\delta S^*_\delta S^*_\nu \in \mathcal{F}^k_n$.

Item (b) follows directly from Lemma 7.14(b), since all monomials in $\mathcal{O}_n$ are of the form $S_\mu S^*_\nu$ for $\mu, \nu$ arbitrary multi indices (cf. also the proof of Lemma 7.3).

As for (c), put $e_{\mu\nu} := S_\mu S^*_\nu \in \mathcal{F}^k_n$. Using the relations from Lemma 7.14, we infer $e_{\mu\nu} = e_{\nu\mu}$ and $e_{\mu\nu} e_{\rho\sigma} = \delta_{\nu\sigma} e_{\mu\rho}$ for all $\mu, \nu, \rho, \sigma \in \mathcal{M}(k)$. Thus, the elements $e_{\mu\nu}$ are matrix units indexed by $|\mathcal{M}(k)| = n^k$ indices. By Cor. 6.12, $\mathcal{F}^k_n \cong M_{n^k}(\mathbb{C})$. We may arrange that the isomorphism maps $S^k_1(S^*_1)^k \in \mathcal{F}^k_n$ to the matrix unit $E_{11} \in M_{n^k}(\mathbb{C})$. \hfill $\Box$

**7.6. Conditional expectation of $\mathcal{O}_n$.** In analogy to Lemma 7.7 and Lemma 7.8 we will now construct a conditional expectation of $\mathcal{O}_n$. Given $\zeta \in S^1$, consider

$$\rho_\zeta : \mathcal{O}_n \to \mathcal{O}_n, \quad S_i \mapsto \zeta S_i, \text{ for all } i = 1, \ldots, n.$$  

This map exists by the universal property of $\mathcal{O}_n$ and it is a $^*$-isomorphism with inverse $\rho_\zeta$. We prove an analogue of Lemma 7.6.

**Lemma 7.16.** Let $n \geq 2$ and $x \in \mathcal{O}_n$.

(a) The map $f_\zeta : S^1 \to S^1, \zeta \mapsto \rho_\zeta(x)$ is continuous in norm.

(b) The Riemannian sums $\frac{1}{n} \sum_{j=1}^n \rho_{e^{2\pi i j/n}}(x)$ converge, for partitions $0 = t_0 < t_1 < \ldots < t_n = 1$ of the unit interval. We denote their limit by $\int_0^1 \rho_{e^{2\pi i t}}(x)dt$.

**Proof.** The proof is similar to the one for Lemma 7.6. We use $\rho_\zeta(S_\mu S^*_\nu) = \zeta^{|\nu|-|\mu|} S_\mu S^*_\nu$ and Lemma 7.15(b) for (a) and we argue as for classical Riemannian sums for (b). \hfill $\Box$

We then put

$$\varphi : \mathcal{O}_n \to \mathcal{O}_n, \quad x \mapsto \int_0^1 \rho_{e^{2\pi i t}}(x)dt.$$
Lemma 7.18. The map $\varphi : \mathcal{O}_n \to \mathcal{O}_n$ is a faithful conditional expectation with $\|\varphi\| \leq 1$, $\varphi(S_\mu S_\nu^*) = \delta_{|\mu|,|\nu|} S_\mu S_\nu^*$ for all multi indices $\mu$ and $\nu$, and $\varphi(\mathcal{O}_n) = \mathcal{F}_n$.

Proof. As in the proof of Lemma 7.7(d), we derive that $\varphi$ is a faithful conditional expectation with $\|\varphi\| \leq 1$. Moreover, let $\mu$ and $\nu$ be multi indices. Then:

$$\varphi(S_\mu S_\nu^*) = \int_0^1 e^{\pi \text{i} t (|\mu| - |\nu|)} dt S_\mu S_\nu^* = \delta_{|\mu|,|\nu|} S_\mu S_\nu^*$$

As in the case of $A_\vartheta$, we have an “algebraic description” of $\varphi$ – at least locally.

Lemma 7.17. Let $n \geq 2$ and $k \in \mathbb{N}$.

(a) There is an isometry $w \in \mathcal{O}_n$ with $wx = xw$ for all $x \in \mathcal{F}_n^k$.

(b) We have $w^* S_\mu S_\nu^* w = \delta_{|\mu|,|\nu|} S_\mu S_\nu^*$ for all $\mu, \nu \in \cup_{l \leq k} \mathcal{M}(l)$.

(c) We have $\varphi(x) = w^* x w \in \mathcal{F}_n^k$ for all $x \in \text{span}\{S_\mu S_\nu^* : \mu, \nu \in \cup_{l \leq k} \mathcal{M}(l)\}$.

Proof. Let $S_\gamma := S_1^2 S_2$ and put

$$w := \sum_{\alpha \in \mathcal{M}(k)} S_\alpha S_\gamma S_\alpha^*.$$

For (a), we use Lemma 7.14 and check:

$$w^* w = \sum_{\alpha, \beta \in \mathcal{M}(k)} S_\alpha S_\gamma S_\alpha^* S_\beta S_\beta^* S_\beta S_\gamma S_\alpha = \sum_{\alpha \in \mathcal{M}(k)} S_\alpha S_\gamma S_\alpha^* S_\gamma S_\alpha = \sum_{\alpha \in \mathcal{M}(k)} S_\alpha S_\alpha^* = 1$$

Moreover, for $x = S_\mu S_\nu^* \in \mathcal{F}_n^k$ (i.e. $|\mu| = |\nu| = k$) we have

$$w S_\mu = \sum_{\alpha \in \mathcal{M}(k)} S_\alpha S_\gamma S_\alpha^* S_\mu = S_\mu S_\gamma$$

and hence $w S_\mu S_\nu^* = S_\mu S_\gamma S_\nu^* = S_\mu S_\nu^* w$. Passing to linear combinations, we infer $wx = xw$ for all $x \in \mathcal{F}_n^k$.

For (b), let $\mu, \nu \in \cup_{l \leq k} \mathcal{M}(l)$. In the first case, assume $|\mu| = |\nu| = k$. Then $w S_\mu S_\nu^* = S_\mu S_\nu^* w$ by (a) and hence $w^* S_\mu S_\nu^* w = S_\mu S_\nu^*$.

In the second case, assume $|\mu| = |\nu| = l < k$. Then $S_\mu S_\nu^* = \sum_{\alpha \in \mathcal{M}(k-l)} S_\mu S_\alpha S_\alpha^* S_\nu^*$ by Lemma 7.16(c). Then, $w^* S_\mu S_\alpha S_\alpha^* S_\nu^* w = S_\mu S_\alpha S_\alpha^* S_\nu^*$ by the first case and hence $w^* S_\mu S_\nu^* w = S_\mu S_\nu^*$.

In the third case, assume $|\mu| \neq |\nu|$. Let $\alpha, \beta \in \mathcal{M}(k)$. By Lemma 7.14

$$w^* S_\mu S_\nu^* w = \sum_{\alpha, \beta \in \mathcal{M}(k)} S_\alpha S_\gamma S_\alpha^* S_\mu S_\beta S_\beta^* S_\gamma S_\beta = 0$$

Finally, for (c), let $x = S_\mu S_\nu^*$ with $\mu, \nu \in \cup_{l \leq k} \mathcal{M}(l)$. If $|\mu| = |\nu|$, then $\varphi(S_\mu S_\nu^*) = S_\mu S_\nu^* w^* S_\mu S_\nu^* w$ by Lemma 7.17 and (b). On the other hand, if $|\mu| \neq |\nu|$, then $\varphi(S_\mu S_\nu^*) = 0$ by Lemma 7.17 and (b). Passing to linear combinations, we conclude $\varphi(x) = w^* x w$ for $x \in \text{span}\{S_\mu S_\nu^* : \mu, \nu \in \cup_{l \leq k} \mathcal{M}(l)\}$; furthermore, $\varphi(x) \in \mathcal{F}_n^k$, by Lemma 7.15(a).
7.7. Pure infiniteness of $O_n$. We have all ingredients to prove that $O_n$ is simple. We may even show a stronger property: it is purely infinite.

Definition 7.19. A unital $C^*$-algebra is purely infinite, if for any non-zero element $x \in A$, there are $a, b \in A$ with $axb = 1$.

Note that any unital, purely infinite $C^*$-algebra $A$ is simple. Indeed, let $I \lhd A$ be a nonzero closed ideal and choose $0 \neq x \in I$. We then find $a, b \in A$ such that $1 = axb \in I$, which shows $I = A$.

Theorem 7.20. The Cuntz algebra $O_n$ is purely infinite, for all $n \geq 2$.

Proof. The proof is really cool: given $0 \neq x \in O_n$, we shift it (or rather an element which is close to it) via the expectation $\varphi$ to the matrix algebras $F_n^k \cong M_{nk}(C)$; we view it as a matrix and we project onto the eigenspace of its largest eigenvalue; we then rescale this small space to 1 which we can do since $\varphi$ is locally algebraic in the sense of Lemma 7.18. Let us make this more precise. Let $0 \neq x \in O_n$.

(1) We find some $y \in S$ with $y = y^*$, $\|x^*x - y\| < \frac{1}{4}$ and $\|\varphi(y)\| > \frac{3}{4}$.

Since $x^*x \geq 0$ and $x^*x \neq 0$, we infer $\varphi(x^*x) \neq 0$, since the expectation $\varphi$ is faithful by Lemma 7.17. Hence, we may assume $\|\varphi(x^*x)\| = 1$ after possibly normalizing. Since $S$ is dense in $O_n$, see Lemma 7.15, we find $y_0 \in S$ with $\|x^*x - y_0\| < \frac{1}{4}$. Putting $y := \frac{1}{2}(y_0 + y_0^*) \in S$, we have $\|x^*x - y\| \leq \frac{1}{2}\|x^*x - y_0\| + \frac{1}{2}\|x^*x - y_0\| < \frac{1}{4}$. Moreover:

$$1 = \|\varphi(x^*x)\| \leq \|\varphi(x^*x - y)\| + \|\varphi(y)\| < \frac{1}{4} + \|\varphi(y)\|$$

This shows $\|\varphi(y)\| > \frac{3}{4}$.

(2) There is an isometry $w \in O_n$ with $w^*yw = \varphi(y)$.

Since $y \in S$, it is of the form $y = \sum_{i=1}^m \alpha_i S_{\mu(i)}^* S_{\nu(i)}$ for some $m \in \mathbb{N}$, $\alpha_i \in \mathbb{C}$ and multi indices $\mu(i)$ and $\nu(i)$, $i = 1, \ldots, m$. Denote by $k \in \mathbb{N}$ the maximal length of these multi indices. By Lemma 7.18 there is an isometry $w \in O_n$ with $w^*yw = \varphi(y)$.

(\beta) There is a unitary $u \in F_n^k$ and $\beta \in \{-1, 1\}$ with $(S_1^k)^*u\varphi(y)u^* = \beta\|\varphi(y)\|((S_1^k)^k$.

This is the crucial step in the proof. We have $\varphi(y) \in F_n^k$ by Lemma 7.18. By Lemma 7.15 $F_n^k \cong M_{nk}(C)$. Hence, $\varphi(y)$ corresponds to a matrix $Y \in M_{nk}(C)$. It satisfies $r(Y) = r(\varphi(y)) = \|\varphi(y)\|$, by Cor. 2.14. Moreover, $Y$ is selfadjoint, since $\varphi(y)$ is selfadjoint. Hence, all eigenvalues of $Y$ are real, and $\beta\|\varphi(y)\|$ is an eigenvalue with $\beta = -1$ or $\beta = 1$.

Next, we firstly work in $M_{nk}(C)$. Since $Y$ is selfadjoint, we may diagonalize it. Thus, there is a unitary $U \in M_{nk}(C)$ such that $UYU^*$ is a diagonal matrix with the upper left entry being $\beta\|\varphi(y)\|$. Recall that $E_{11} \in M_{nk}(C)$ is the projection with 1 in the upper left entry and zero elsewhere. We thus have:

$$E_{11}UYU^* = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} \beta\|\varphi(y)\| & 0 & \cdots & 0 \\ 0 & * & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & * \end{pmatrix} = \beta\|\varphi(y)\|E_{11}$$
Secondly, we transfer this equation to $\mathcal{F}_h^k$ using the isomorphism $\mathcal{F}_h^k \cong M_{nk}(\mathbb{C})$. As mentioned in Lemma 7.15, the projection $E_{11} \in M_{nk}(\mathbb{C})$ corresponds to the projection $S_1^k(S_1^*)^k \in \mathcal{F}_h^k$; the unitary $U \in M_{nk}(\mathbb{C})$ corresponds to some unitary $u \in \mathcal{F}_h^k$; and $Y \in M_{nk}(\mathbb{C})$ corresponds to $\varphi(y) \in \mathcal{F}_h^k$. We thus have:

$$S_1^k(S_1^*)^k u \varphi(y) u^* = \beta \| \varphi(y) \| S_1^k(S_1^*)^k$$

Multiplying with $(S_1^*)^k$ from the left yields $(S_1^*)^k u \varphi(y) u^* = \beta \| \varphi(y) \|(S_1^*)^k$.

(4) For $z := \| \varphi(y) \|^{-\frac{1}{2}} (S_1^*)^k u \varphi(y) u^*$, we have $zyz^* = \beta 1$ and $zz^*x^z^*$ is invertible.

Building on (2) and (3), we compute:

$$yz y z^* = \| \varphi(y) \|^{-\frac{1}{2}} (S_1^*)^k u \varphi(y) u^* S_1^k$$

$$= \| \varphi(y) \|^{-1} (S_1^*)^k u \varphi(y) u^* S_1^k$$

$$= \| \varphi(y) \|^{-1} \beta \| \varphi(y) \|(S_1^k)^k S_1^k$$

$$= \beta 1$$

Next, let us compute the norm of $z$. Using (1), we have:

$$\| z \| \leq \| \varphi(y) \|^{-\frac{1}{2}} (S_1^*)^k \| u \| \| w \| \leq \| \varphi(y) \|^{-\frac{1}{2}} < \frac{2}{\sqrt{3}}$$

Again, using (1), we have:

$$\| 1 - \beta zz^* x^z^* \| = \| \beta (y - x^* x) z^* \| \leq \| z \|^2 \| y - x^* x \| \leq \frac{4}{3} \cdot \frac{1}{4} = \frac{1}{3} < 1$$

This shows that $\beta zz^* x^z^*$ is invertible, by Lemma 2.6 and so is $zz^* x^z^*$.

(5) Putting $b := z^* (zz^* x^z^*)^{-\frac{1}{2}}$ and $a := b^* x^*$, we have $axb = 1$.

Finally, compute:

$$axb = (zz^* x^z^*)^{-\frac{1}{2}} zzz^*(zz^* x^z^*)^{-\frac{1}{2}} = 1 \quad \Box$$

7.8. Exercises.

Exercise 7.1. Let $\varrho \in \mathbb{R}$, $\lambda := e^{2\pi i \varrho} \in S^1$ and $L^2(S^1)$ be the Hilbert space of $L^2$-integrable functions on the circle $S^1$. We define $\tilde{u} : L^2(S^1) \rightarrow L^2(S^1)$ and $\tilde{v} : L^2(S^1) \rightarrow L^2(S^1)$ by:

$$(\tilde{u} f)(z) := f(\lambda z), \quad (\tilde{v} f)(z) := z f(z), \quad f \in L^2(S^1), z \in S^1$$

Show that $\pi : A_{\varrho} \rightarrow B(L^2(S^1))$ mapping $u \mapsto \tilde{u}$ and $v \mapsto \tilde{v}$ is a representation of $A_{\varrho}$. Considering the orthonormal basis $(e_n)_{n \in \mathbb{Z}}$, with $e_n(z) := z^n$ for $z \in S^1$, $n \in \mathbb{Z}$ – how does this representation relate to the one given in Lemma 7.2?

Exercise 7.2. Let $\zeta \in S^1 \subseteq \mathbb{C}$ and put:

$$a_n(\zeta) := \frac{1}{2n+1} \sum_{j=-n}^{n} \zeta^j, \quad n \in \mathbb{N}$$

Let $\varrho \in \mathbb{R}$ and put $\lambda := e^{2\pi i \varrho} \in S^1$. Let $l \in \mathbb{Z}$.
(a) Show that \( (a_n(\zeta))_{n \in \mathbb{N}} \) converges to zero, if \( \zeta \neq 1 \). \textit{Hint:} Show that \( \sum_{j=-n}^{n} \zeta^j \) is bounded using the formulas from geometric progression.

(b) Assume \( \vartheta \notin \mathbb{Q} \). Show that \( (a_n(\lambda^l))_{n \in \mathbb{N}} \) converges to \( \delta_{l0} \).

(c) Assume \( \vartheta = \frac{p}{q} \in \mathbb{Q} \). Show that \( (a_n(\lambda^l))_{n \in \mathbb{N}} \) converges to \( \delta_{lq\mathbb{Z}} \). In particular, the sequence \( (a_n(\lambda^l))_{n \in \mathbb{N}} \) converges to 1 for infinitely many powers \( \lambda^l, l \in \mathbb{Z} \).

Thus, in the rational case, the map \( x \mapsto \lim_{n \to \infty} \frac{1}{2n+1} \sum_{j=-n}^{n} u^j x u^{-j} \) is way different from \( \varphi_1 \), see Lemma \( 7.8 \).

**Exercise 7.3.** Let \( \vartheta = \frac{p}{q} \in \mathbb{Q} \).

(a) Find a representation \( \pi : A_{\vartheta} \to M_q(\mathbb{C}) \).

(b) Find unital \( C^* \)-algebras \( B \) and \( D \) as well as unital *-homomorphisms \( \varphi : A_{\vartheta} \to B \) and \( \psi : A_{\vartheta} \to D \) such that \( \varphi(v^q) = 1 \) and \( \psi(v^q) \neq 1 \).

(c) Conclude that \( A_{\vartheta} \) is not simple.

(d) Show that there is a *-homomorphism \( \sigma : C(\mathbb{T}^2) \to C(u^q, v^q) \subseteq A_{\vartheta} \) mapping the generators \( \tilde{u} \) and \( \tilde{v} \) of \( C(\mathbb{T}^2) \) (see Lemma \( 7.4 \)) to \( u^q \) and \( v^q \). (In fact, it is even a *-isomorphism.)

(e) Convince yourself that none of these statements holds true for \( A_{\vartheta} \) if \( \vartheta \notin \mathbb{Q} \).

**Exercise 7.4.** Show the relations in Lemma \( 7.14 \) for \( n \geq 2 \).

(a) Convince yourself that \( S_i S^*_i \) are projections in the sense of Def. \( 1.33 \).

(b) Given multi indices \( \mu \) and \( \nu \), show:

\[
\begin{align*}
\text{If } |\mu| = |\nu|, & \quad S^*_\mu S_\nu = \delta_{\mu\nu}, \\
\text{If } |\mu| < |\nu|, & \quad S^*_\mu S_\nu = \begin{cases} S_\nu & \text{if } \nu = \mu' \\
0 & \text{otherwise} \end{cases}, \\
\text{If } |\mu| > |\nu|, & \quad S^*_\mu S_\nu = \begin{cases} S^*_\nu & \text{if } \mu = \nu \mu' \\
0 & \text{otherwise} \end{cases}.
\end{align*}
\]

(c) Let \( k \in \mathbb{N} \). Show \( \sum_{\alpha \in \mathcal{M}(k)} S_\alpha S^*_\alpha = 1 \).

(d) Let \( \mu, \nu \) be multi indices with \( |\mu| \neq |\nu| \) and \( |\mu|, |\nu| \leq k \). Let \( \alpha, \beta \in \mathcal{M}(k) \) and \( \gamma \in \mathcal{M}(k+1) \). Show \( S_\gamma S^*_\alpha (S_\mu S^*_\nu) S_\beta S_\gamma = 0 \).

7.9. **Comments on** \( A_{\vartheta} \). The rotation algebra – in particular the irrational one – is one of the most studied examples of \( C^* \)-algebras. See for instance \[10\] or \[14\] Ch. 12] for more on \( A_{\vartheta} \). It is a key example of a noncommutative manifold in Connes’s noncommutative geometry \[7\] and this is also a place where we study the noncommutative torus \( \mathbb{T}^2_{\vartheta} \) in close analogy to the classical torus \( \mathbb{T}^2 \). This noncommutative geometry is very much inspired from physics and we quote from \[14\] Ch. 12]:

At a deep and perhaps fundamental level, quantum field theory and noncommutative geometry are made of the same stuff.

See \[14\] Ch. 12] and \[7\] for more on this link to physics. Also, the noncommutative torus has been used to describe some mathematics behind the famous quantum Hall effect, see \[2\].
Back to mathematics, we may wonder whether the rotation algebra $A_\vartheta$ depends on the parameter $\vartheta \in \mathbb{R}$: do we obtain distinct $C^*$-algebras given distinct parameters? The answer is: $A_\vartheta$ and $A_{\vartheta'}$ are isomorphic if and only if $\vartheta' = \pm \vartheta \mod \mathbb{Z}$. For the proof, we need the so called Powers-Rieffel projections and methods from $K$-theory, which is a homological theory of invariants for $C^*$-algebras. See [10] for this characterization of parameter dependence.

Along the lines of deforming the torus $\mathbb{T}^2$ to $\mathbb{T}^2_\vartheta$, Rieffel developed a program of the so called Rieffel deformation [27]. The basic idea is to use cocycle twists in order to deform the Cartesian product $S^1 \times S^1 = \mathbb{T}^2$. The Cartesian product corresponds to the tensor product of $C^*$-algebras on the $C^*$-algebraic side (note that there is no unique way to equip an algebraic tensor product of $C^*$-algebras with a $C^*$-norm in general [6]), so we may view $A_\vartheta$ as a deformation of the tensor product $C(S^1) \otimes C(S^1)$.

And in the upcoming lectures, we will see that the rotation algebra is actually an example of a crossed product: we will show that $A_\vartheta = C(S^1) \rtimes \vartheta \mathbb{Z}$. So, $A_\vartheta$ arises from a dynamical system on the circle $S^1$, which is a rotation.

Besides, in case you are wondering about an “isometry version” of $A_\vartheta$: we studied one “universal unitary” and one “universal isometry” in Lecture 6, and we studied two “universal commuting unitaries” as well as a scalar deformation of this commutation relation in the present lecture; how about two “universal commuting isometries” as well as scalar deformations of the commutation relation? This has actually been the content of the lecturer’s PhD thesis; you may take a look at [35].

7.10. Comments on $O_n$. The Cuntz algebra $O_n$ is a very important example of a $C^*$-algebra, too. It has been studied extensively since its introduction by Cuntz in 1977 [9]. A remarkable feature is, that it is not only an example, but also a building block in the theory of $C^*$-algebras. For instance, there are statements of the form: “a separable $C^*$-algebra is exact if and only if it embeds into $O_2$” or “a $C^*$-algebra $A$ is unital, separable, simple and nuclear if and only if the tensor product $A \otimes O_2$ is isomorphic to $O_2$”; here, exactness and nuclearity are important approximation properties of $C^*$-algebras (actually, nuclearity corresponds to amenability of groups) [6]. So, amazingly, we may characterize certain properties with the help of the Cuntz algebra.

As for the term purely infinite, this comes from the theory of von Neumann algebras. A von Neumann algebra is of type III (or purely infinite), if it does not possess any finite projections. One can show that a similar feature holds for purely infinite $C^*$-algebras. For instance, any $S_\mu S_\mu^*$ is a projection; it can be decomposed into arbitrarily many subprojections, since

$$S_\mu S_\mu^* = \sum_{\alpha \in \mathcal{M}(k)} S_\alpha S_\mu S_\mu^* S_\alpha^*$$

for all $k$. Thus, $S_\mu S_\mu^*$ cannot be finite. Let us also mention that purely infiniteness plays an important role in the classification of $C^*$-algebras. There is the concept
of a Kirchberg algebra, also called pi-sun algebras, where pi-sun stands for purely infinite, separable, unital, nuclear. Such $C^*$-algebras are classifiable via $K$-theory. See [33] for more on the classification of $C^*$-algebras.

Actually, $K$-theory (or Ext-theory) also plays a role when distinguishing the Cuntz algebras: again, we may ask whether $\mathcal{O}_n$ depends on the parameter $n \in \mathbb{N}$. The answer is yes: we have $\mathcal{O}_n \not\cong \mathcal{O}_m$ for $n \neq m$, see [10].

Let us mention a couple of generalizations of Cuntz algebras. Firstly, what is $\mathcal{O}_1$ for $n = 1$, by the way? Well, you could say it is the $C^*$-algebra generated by one single isometry $S_1$ such that $S_1S_1^* = 1$. Hence, $S_1$ is in fact a unitary and we conclude $\mathcal{O}_1 = C(S^1)$, by Cor. 6.16.

How about $n = \infty$? Yes, this $C^*$-algebra exists. But, $\sum_{i=1}^{\infty} S_iS_i^* = 1$ would not be a reasonable relation, since $a_n := \sum_{i=1}^{n} S_iS_i^*$ cannot converge in norm (check that!). The way out is to use the seemingly weaker relation $S_i^*S_j = \delta_{ij}$ from Lemma 7.14. The Cuntz algebra $\mathcal{O}_\infty$ is defined as the universal $C^*$-algebra generated by isometries $S_k$, $k \in \mathbb{N}$ with $S_i^*S_j = \delta_{ij}$ for all $i, j \in \mathbb{N}$. One can show that also $\mathcal{O}_\infty$ is purely infinite, so it is simple in particular. Hence, the relations $S_i^*S_j = \delta_{ij}$ are already the best we can do – we may not add further relations.

Now, we could wonder: okay, if $S_i^*S_j = \delta_{ij}$ is good enough for defining a simple $C^*$-algebra in the case $n = \infty$ – how about for $n < \infty$? Do the relations $S_i^*S_j = \delta_{ij}$ imply the relations $\sum_{i=1}^{n} S_iS_i^* = 1$? The answer is no, obviously – otherwise we would have $\sum_{i=1}^{n} S_iS_i^* = 1$ and $\sum_{i=1}^{n+1} S_iS_i^* = 1$ in $\mathcal{O}_{n+1}$, so $S_{n+1}$ would be zero. In fact, the $C^*$-algebra $\mathcal{E}_n$ generated by isometries $S_1, \ldots, S_n$ and relations $S_i^*S_j = \delta_{ij}$ is called the extended Cuntz algebra. It contains the algebra of compact operators as an ideal and $\mathcal{O}_n$ as the corresponding quotient. So, in some sense, the difference between the relations $S_i^*S_j = \delta_{ij}$ and $\sum_{i=1}^{n} S_iS_i^*$ is $K(H)$, see also Prop. 6.23.

Finally, let us briefly mention that Cuntz algebras have been generalized to Cuntz-Krieger algebras. They are given by partial isometries $S_1, \ldots, S_n$ (not necessarily isometries) with mutually orthogonal ranges and relations $S_i^*S_i = \sum_{j} a_{ij} S_jS_j^*$, where $a_{ij} \in \{0, 1\}$. Cuntz-Krieger algebras in turn have been generalized to graph $C^*$-algebras, where we assign a $C^*$-algebra $C^*(\Gamma)$ to a graph $\Gamma$. Graph $C^*$-algebras include the matrix algebras $M_N(\mathbb{C})$, the algebra of compact operators $K(H)$ on a separable Hilbert space, the function algebra $C(S^1)$, the Toeplitz algebra $\mathcal{T}$, the Cuntz algebras $\mathcal{O}_n$ and many more. See [26] for more.
Abstract. We introduce the concept of inductive limits of $C^*$-algebras. We then show that finite-dimensional $C^*$-algebras are exactly given by direct sums of matrix algebras. This is followed by a brief introduction to AF algebras (approximately finite-dimensional $C^*$-algebras) and the main tool to study them: Bratteli diagrams. We discuss how to read off the ideal structure of an AF algebra from its Bratteli diagram.

8. Inductive limits and AF algebras

8.1. Inductive limits. In mathematics, the concept of approximation is everywhere: we want to understand a complicated object by approximating it with simpler ones. In the language of category theory, we may formulate this idea in terms of inductive limits. We present this concept adapted to the theory of $C^*$-algebras, but it is a principle of much wider generality.

**Definition 8.1.** An inductive system (of $C^*$-algebras) $(A_n, \varphi_n)_{n \in \mathbb{N}}$ is given by $C^*$-algebras $A_n$ and $*$-homomorphisms $\varphi_n : A_n \to A_{n+1}$, for all $n \in \mathbb{N}$, i.e. we have:

\[
A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \ldots \quad \xrightarrow{\varphi_{n-1}} A_n \xrightarrow{\varphi_n} A_{n+1} \xrightarrow{\varphi_{n+1}} \ldots
\]

More generally, in category theory, an inductive system is given by objects $(A_n)_{n \in \mathbb{N}}$ and morphisms $(\varphi_n)_{n \in \mathbb{N}}$ between them, in a diagram as above. We may then ask for the existence of a limit object for such a sequence.

**Proposition 8.2.** Let $(A_n, \varphi_n)_{n \in \mathbb{N}}$ be an inductive system of $C^*$-algebras. There are a $C^*$-algebra $\lim_{\longrightarrow} A_n$ and $*$-homomorphisms $\varphi_n : A_n \to \lim_{\longrightarrow} A_n$ with $\varphi_{n+1} \circ \varphi_n = \varphi_n$ satisfying the following universal property: Given any $C^*$-algebra $B$ and $*$-homomorphisms $\beta_n : A_n \to B$, $n \in \mathbb{N}$ with $\beta_{n+1} \circ \varphi_n = \beta_n$ there is a unique $*$-homomorphism $\beta : \lim_{\longrightarrow} A_n \to B$ such that the following diagram commutes.

\[
\begin{array}{ccc}
\ldots & A_n & \ldots \\
\varphi_n & \downarrow & \varphi_n \\
A_n & \xrightarrow{\beta_n} & A_{n+1} \\
\varphi_{n+1} & \downarrow & \varphi_{n+1} \\
\beta_{n+1} & \downarrow & \beta \\
B & \xrightarrow{\beta} & \ldots
\end{array}
\]

The $C^*$-algebra $\lim_{\longrightarrow} A_n$ is unique up to isomorphism; it is called the inductive limit.

**Proof.** The idea is to define the inductive limit $C^*$-algebra as the set of all eventually stationary sequences. We are going to develop this algebra step by step.
(1) Construction of $A$.

Let $A_0$ be the set of all sequences $(x_n)_{n \in \mathbb{N}}$ with $x_n \in A_n$ for $n \in \mathbb{N}$ and the additional requirement: there exists some $N \in \mathbb{N}$ such that we have $x_{n+1} = \varphi_n(x_n)$ for all $n \geq N$. We say that $(x_n), (y_n) \in A_0$ are equivalent, if they coincide on their tails, i.e. if there is some $N \in \mathbb{N}$ such that $x_n = y_n$ for all $n \geq N$. It is easy to check that this is an equivalence relation indeed; we denote by $[(x_n)_{n \in \mathbb{N}}]$ the equivalence classes. Let $A$ be the quotient of $A_0$ by this equivalence relation.

(2) Construction of a $C^*$-seminorm on $A$ and definition of $\lim_{\varphi_n} A_n$.

Given a sequence $(x_n)_{n \in \mathbb{N}} \in A_0$, there is some $N \in \mathbb{N}$ with $x_{n+1} = \varphi_n(x_n)$ for all $n \geq N$, by definition. We then have for all $n \geq N$:

$$\|x_{n+1}\|_{A_{n+1}} = \|\varphi_n(x_n)\|_{A_{n+1}} \leq \|x_n\|_{A_n}$$

Thus, the sequence $(\|x_n\|_{A_n})_{n \in \mathbb{N}} \in \mathbb{C}$ converges and we may define

$$\|[(x_n)_{n \in \mathbb{N}}]\| := \lim_{n \to \infty} \|x_n\|_{A_n}$$

for $[(x_n)_{n \in \mathbb{N}}] \in A$; one may check that this is a $C^*$-seminorm on $A$. We mod out its null space and define $\lim_{\varphi_n} A_n$ as the completion:

$$\lim_{\varphi_n} A_n := \overline{A/\{[(x_n)_{n \in \mathbb{N}}] \mid \|[(x_n)_{n \in \mathbb{N}}]\| = 0\}}$$

One may then check that $\lim_{\varphi_n} A_n$ is a $C^*$-algebra with the canonical entrywise operations. The elements in $\lim_{\varphi_n} A_n$ are again denoted by $[(x_n)_{n \in \mathbb{N}}]$.

(3) Construction of the maps $\overline{\varphi_n} : A_n \to \lim_{\varphi_n} A_n$ with $\lim_{\varphi_n} A_n = \bigcup_{n \in \mathbb{N}} \overline{\varphi_n}(A_n)$.

For $x \in A_n$ we put:

$$\overline{\varphi_n}(x) := [(0, \ldots, 0, x, \varphi_n(x), \varphi_{n+1}(\varphi_n(x)), \ldots)]$$

Here, the entry $x$ is at the $n$-th position. We may check that this is a $^*$-homomorphism. Moreover, it satisfies:

$$\overline{\varphi_{n+1}}(\varphi_n(x)) = [(0, \ldots, 0, 0, \varphi_n(x), \varphi_{n+1}(\varphi_n(x)), \ldots)]$$

$$= [(0, \ldots, 0, x, \varphi_n(x), \varphi_{n+1}(\varphi_n(x)), \ldots)]$$

$$= \overline{\varphi_n}(x)$$

Here, we used the fact that two sequences are in the same equivalence class, if they coincide on their tails. Let us now prove that $\lim_{\varphi_n} A_n = \bigcup_{n \in \mathbb{N}} \overline{\varphi_n}(A_n)$ holds. Indeed, given $[(x_n)_{n \in \mathbb{N}}] \in A$, we may find some $N \in \mathbb{N}$ with $x_{n+1} = \varphi_n(x_n)$ for all $n \geq N$. Thus, $[(x_n)_{n \in \mathbb{N}}] = \overline{\varphi_N}(x_N) \in \bigcup_{n \in \mathbb{N}} \overline{\varphi_n}(A_n)$.

(4) The universal property of $\lim_{\varphi_n} A_n$.

Let $B$ be a $C^*$-algebra and let $\beta_n : A_n \to B$ be $^*$-homomorphisms with $\beta_{n+1} \circ \varphi_n = \beta_n$, for all $n \in \mathbb{N}$. We construct $\beta : \lim_{\varphi_n} A_n \to B$. Given $[(x_n)_{n \in \mathbb{N}}] \in \lim_{\varphi_n} A_n$, we find some $N \in \mathbb{N}$ such that $x_{n+1} = \varphi_n(x)$ for all $n \geq N$. Thus, $\beta_n(x_n) = \beta_{n+1}(\varphi_n(x_n)) = \beta_{n+1}(x_{n+1})$ for all $n \geq N$. We may thus define

$$\beta([(x_n)_{n \in \mathbb{N}}]) := \lim_{n \to \infty} \beta_n(x_n)$$
and check that it is a *-homomorphism. Moreover:

\[ \beta(\varphi_n(x)) = \beta([0, \ldots, 0, x, \varphi_n(x), \varphi_{n+1}(\varphi_n(x)), \ldots]) = \beta_n(x) \]

Hence, the diagram of the assertion is commutative. Moreover, \( \beta \) is unique, since any other map \( \beta' : \lim_{\rightarrow} A_n \rightarrow B \) with \( \beta' \circ \varphi_n = \beta_n \), for all \( n \in \mathbb{N} \), coincides with \( \beta \) on \( \cup_{n \in \mathbb{N}} \varphi_n(A_n) \) and hence also on \( \lim \varphi_n A_n = \bigcup_{n \in \mathbb{N}} \varphi_n(A_n) \).

As a side note, observe that putting \( B = \lim_{\rightarrow} A_n \), the map \( \text{id}_{\lim A_n} : \lim_{\rightarrow} A_n \rightarrow \lim_{\rightarrow} A_n \) is the unique map with the property \( \text{id}_{\lim A_n} \circ \varphi_n = \varphi_n \), for all \( n \in \mathbb{N} \), by the universal property of \( \lim_{\rightarrow} A_n \).

(5) **Uniqueness of** \( \lim_{\rightarrow} A_n \).

Finally, let us show that \( \lim_{\rightarrow} A_n \) is unique up to isomorphism. So, let \( A' \) be another \( C^* \)-algebra with *-homomorphisms \( \varphi_n : A_n \rightarrow A' \) and \( \varphi_{n+1} \circ \varphi_n = \varphi'_n \), for all \( n \in \mathbb{N} \), and assume that \( A' \) satisfies the above universal property. Then, there is a map \( \beta : \lim_{\rightarrow} A_n \rightarrow A' \) by the universal property of \( \lim_{\rightarrow} A_n \) satisfying \( \beta \circ \varphi_n = \varphi'_n \) for all \( n \in \mathbb{N} \); likewise, we have a map \( \beta' : A' \rightarrow \lim_{\rightarrow} A_n \) by the universal property of \( A' \) satisfying \( \beta' \circ \varphi_n = \varphi_n \) for all \( n \in \mathbb{N} \). This implies \( \beta' \circ \beta = \beta \circ \beta' = \text{id}_{A'} \).

We conclude that \( \lim_{\rightarrow} A_n \) and \( A' \) are isomorphic. \( \square \)

**Remark 8.3.** We comment on some facts, omitting the proofs.

(a) In step (3) of the above proof, we have seen that \( \lim_{\rightarrow} A_n = \bigcup_{n \in \mathbb{N}} \varphi_n(A_n) \) holds. In the special case of a sequence of \( C^* \)-subalgebras \( A_1 \subseteq A_2 \subseteq \ldots \subseteq A \), we have \( \lim_{\rightarrow} A_n = \bigcup_{n \in \mathbb{N}} A_n \subseteq A \), where \( \iota_n : A_n \rightarrow A_{n+1} \) is the identity map.

(b) If all maps \( \beta_n : A_n \rightarrow B \) in the above proposition are injective, then so is \( \beta : \lim \rightarrow A_n \rightarrow B \). Likewise, if \( \cup \beta_n(A_n) \) is dense in \( B \), then \( \beta \) is surjective.

(c) If all \( C^* \)-algebras \( A_n \) are simple, then so is \( \lim \rightarrow A_n \).

We will come to examples of inductive limits soon.

8.2. **Finite-dimensional \( C^* \)-algebras.** We briefly discuss finite-dimensional \( C^* \)-algebras. In fact, the topology does not play any role in finite dimensions, so we are actually dealing with purely algebraic objects.

We have seen examples of finite-dimensional \( C^* \)-algebras before: there are the matrix algebras \( M_N(\mathbb{C}) \), and there is \( \mathbb{C}^N \), see Exc. 6.4. Actually, we can write the latter one also as \( \mathbb{C} \oplus \cdots \oplus \mathbb{C} \), where we used the direct sum of \( C^* \)-algebras as defined in Def. 1.8. Generalizing these considerations, we infer that we know many examples of finite-dimensional \( C^* \)-algebras, namely \( \bigoplus_{i=1}^m M_{N_i}(\mathbb{C}) \) with \( N_1, \ldots, N_m \in \mathbb{N} \) and \( m \in \mathbb{N} \). And in fact: these are already all finite-dimensional \( C^* \)-algebras! This is known as Wedderburn’s Theorem, or Artin-Wedderburn Theorem, which holds true more generally, for semisimple rings. It predates the theory of \( C^* \)-algebras by a number of decades.
Lemma 8.4. Any simple finite-dimensional C*-algebra $A$ is isomorphic to some $M_N(\mathbb{C})$.

Proof. We do not give a proof here, but the idea is as follows. We represent $A$ on some $B(H)$ by an irreducible representation $\pi$. Irreducibility will help us to deduce that $H$ is finite-dimensional, so $B(H) \cong M_N(\mathbb{C})$ for some $N \in \mathbb{N}$. Since $A$ is simple, we obtain $A \cong \pi(A) \subseteq M_N(\mathbb{C})$. Further work shows $\pi(A) = M_N(\mathbb{C})$. \hfill \Box

Proposition 8.5 (Wedderburn’s Theorem). Let $A$ be a finite-dimensional C*-algebra. There are $m \in \mathbb{N}$ and $N_1, \ldots, N_m \in \mathbb{N}$ such that:

$$A \cong \bigoplus_{k=1}^m M_{N_k}(\mathbb{C})$$

Proof. We give the proof in the language of C*-algebras.

(1) $A$ is unital.

Let $(u_\lambda)$ be an approximate unit of $A$. Since $A$ is finite-dimensional, the set $\{x \in A \mid \|x\| \leq 1\}$ is compact; hence $(u_\lambda)$ possesses a convergent subsequence. Denoting its limit by 1, we may check that 1 is a unit of $A$ indeed.

(2) For any ideal $I \triangleleft A$, there is a central projection $p \in A$ with $I = pA$.

Since $A$ is finite-dimensional, $I$ is closed. Hence, $I$ is a C*-algebra. By (1) it is unital, i.e. there is some $p \in I$ with $p = p^* = p^2$ and $pa = a$ for all $a \in I$. This shows $I \subseteq pA$. The converse, $I \supseteq pA$ follows from $p \in I$. Finally, $p$ is central, i.e. $ap = pa$ for all $a \in A$. Indeed, $ap = pap$ for all $a \in A$, since $ap \in I$ and $p$ is a unit for $I$. Thus, $p = (a^*p)^* = (pa^*p)^* = pap = ap$ for all $a \in A$.

(3) There exist $m \in \mathbb{N}$ and central projections $p_1, \ldots, p_m \in A$ with $p_ip_j = 0$ for $i \not= j$, $\sum_k p_k = 1$ and $A = \sum_k p_k A$, where all $p_k A$ are simple.

The center $Z(A) := \{a \in A \mid ab = ba \text{ for all } b \in A\} \subseteq A$ is a commutative C*-algebra. Hence $Z(A) \cong C(X)$ for some compact space $X$. As $A$ is finite-dimensional, $X$ is finite, so $X = \{1, \ldots, m\}$ for some $m \in \mathbb{N}$. The characteristic functions $p_k := \chi_{\{k\}} \in C(X)$ are continuous and they are projections with $p_i^2 = 1$ for $i \not= j$ and $\sum_k p_k = 1$; see also Exc. 6.4. We thus find corresponding projections $p_1, \ldots, p_m \in Z(A)$ with $p_ip_j = 0$ for $i \not= j$. Since the unit of $A$ is in $Z(A)$, the unit of $Z(A)$ and the one of $A$ coincide and we obtain $\sum_k p_k = 1$ and $A = \sum_k p_k A$. Checking $\mathbb{C}p_k \subseteq Z(p_k A) \subseteq Z(A)p_k = \mathbb{C}p_k$, we may easily derive that $p_k A$ is simple, for $k = 1, \ldots, m$, by (2).

(4) We have $A \cong \bigoplus_{k=1}^m M_{N_k}(\mathbb{C})$.

By Lemma 8.4 and (3), we obtain (4). \hfill \Box

We are now going to study homomorphisms between matrix algebras. Given a finite-dimensional C*-algebra $A = \bigoplus_{k=1}^m M_{N_k}(\mathbb{C})$ with $k \in \{1, \ldots, m\}$, denote by $e_{ij}^{(k)} \in M_{N_k}(\mathbb{C})$, $i, j = 1, \ldots, N_k$ the matrix units in $M_{N_k}(\mathbb{C})$; let us use small letters for matrices from now on. Note that the elements $e_{ii}^{(k)}$ are projections in the sense of Def. 1.33, i.e. $e_{ii}^{(k)} = (e_{ii}^{(k)})^* = (e_{ii}^{(k)})^2$. They form a partition of unity: $\sum_{k=1}^m \sum_{i=1}^{N_k} e_{ii}^{(k)} = 1 \in A$. 
Recall that $\text{Tr} : M_N(\mathbb{C}) \to \mathbb{C}$ denotes the (unnormalized) trace, i.e. $\text{Tr}((a_{ij})_{i,j}) = \sum_{i=1}^N a_{ii}$, see Exm. 5.2. If $p \in M_N(\mathbb{C})$ is a projection, then $\text{Tr}(p) \in \mathbb{N}$ is the dimension of the subspace onto which $p$ projects.

Given $A$ as above, let $B = \bigoplus_{l=1}^n M_{K_l}(\mathbb{C})$ be another finite-dimensional $C^*$-algebra and let $\varphi : A \to B$ be a $*$-homomorphism. We denote by $\varphi_1, \ldots, \varphi_n$ its components, i.e. we have $\varphi_l : A \to M_{K_l}(\mathbb{C})$, for $l = 1, \ldots, n$. We put

$$\Phi_{lk} := \text{Tr}_{M_{K_l}}(\varphi_l(e_{11})) \in \mathbb{N}_0$$

for $l = 1, \ldots, n$ and $k = 1, \ldots, m$. Let us call the matrix $\Phi = (\Phi_{lk})_{l,k} \in M_{n\times m}(\mathbb{N}_0)$ the coefficient matrix of $\varphi$. By the way, note that

$$\text{Tr}_{M_{K_l}}(\varphi_l(e_{11})) = \text{Tr}_{M_{K_l}}(\varphi_l(e_{11}e_{11})) = \text{Tr}_{M_{K_l}}(\varphi_l(e_{11}e_{11})).$$

So, $\Phi$ contains the information about the whole partition of unity $\sum_k \sum_i e_{ii}^{(k)} = 1$.

**Lemma 8.6.** The coefficient matrix determines a map between finite-dimensional $C^*$-algebras completely. More precisely, let $A$ and $B$ be finite-dimensional $C^*$-algebras and let $\varphi, \psi : A \to B$ be $*$-homomorphisms. If $\Phi = \Psi$ for their coefficient matrices, then there is some unitary $u \in B$ such that $\varphi(x) = u\psi(x)u^*$ for all $x \in A$.

**Proof.** The details of the proof are shifted to Exc. 8.1. By comparison of the coefficients $\Phi$ and $\Psi$, we know that given $k \in \{1, \ldots, m\}$ and $l \in \{1, \ldots, n\}$, the subspaces $\varphi_l(e_{11}^{(k)})\mathbb{C}^{K_k}$ and $\psi_l(e_{11}^{(k)})\mathbb{C}^{K_k}$ have the same dimensions. We may thus find a partial isometry $v_{lk} \in M_{K_l}(\mathbb{C})$ mapping $\varphi_l(e_{11}^{(k)})\mathbb{C}^{K_k}$ to $\psi_l(e_{11}^{(k)})\mathbb{C}^{K_k}$. Putting these partial isometries $v_{lk}$ together in a clever way, we obtain the unitary $u$. \qed

As a consequence, we may present a $*$-homomorphism $\varphi : A \to B$ between finite-dimensional $C^*$-algebras $A = \bigoplus_{k=1}^m M_{N_k}(\mathbb{C})$ and $B = \bigoplus_{l=1}^n M_{K_l}(\mathbb{C})$ by a diagram of the following form, where we write $\Phi_{lk}$ many arrows between $N_k$ and $K_l$. 

$$
\begin{array}{c}
N_1 \\
| \uparrow \\
N_2 \\
| \uparrow \\
\vdots \\
| \uparrow \\
N_m \\
| \uparrow \\
K_n \\
\end{array}
$$
Example 8.7. Let $n \in \mathbb{N}$, $A = M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ and $B = M_{2n}(\mathbb{C}) \oplus M_{2n}(\mathbb{C})$. Consider $\varphi : A \rightarrow B$ given by:

$$\varphi(x, y) = \left( \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \right) \in M_{2n}(\mathbb{C}) \oplus M_{2n}(\mathbb{C}), \quad (x, y) \in M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$$

Then, the corresponding diagram is of the form:

$$\begin{array}{ccc}
n & \rightarrow & 2n \\
\downarrow & \searrow & \\
n & \rightarrow & 2n
\end{array}$$

Note that the map

$$\varphi(x, y) = \left( \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix} \right) \in M_{2n}(\mathbb{C}) \oplus M_{2n}(\mathbb{C}), \quad (x, y) \in M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$$

yields the same diagram as above.

8.3. AF algebras and Bratteli diagrams.

Definition 8.8. A $C^*$-algebra $A$ is an AF-algebra (or approximately finite-dimensional $C^*$-algebra) if it is the inductive limit $\varinjlim A_n$ of an inductive system $(A_n, \varphi_n)_{n \in \mathbb{N}}$, where all $C^*$-algebras $A_n$ are finite-dimensional.

Let $(A_n, \varphi_n)_{n \in \mathbb{N}}$ be an inductive system underlying an AF algebra $A$. According to Lemma 8.6 and the subsequent discussion, we may represent the maps $\varphi_n : A_n \rightarrow A_{n+1}$ by a diagram. So, the whole inductive system yields an infinite diagram, where the vertices represent the matrix algebras of the finite-dimensional $C^*$-algebras $A_n$ and the arrows represent the maps $\varphi_n$ in terms of their coefficient matrices. Such a diagram is called a Bratteli diagram, named after the Finnish mathematician Ola Bratteli\footnote{Don’t pronounce it like an Italian name – the stress is on the first syllable.}. It completely determines an AF algebra.

Lemma 8.9. Two AF algebras with the same Bratteli diagrams are isomorphic.

Proof. The details of the proof are contained in Exc. 8.2. Let $A$ and $B$ be AF algebras arising from inductive systems $(A_n, \varphi_n)_{n \in \mathbb{N}}$ and $(B_n, \psi_n)_{n \in \mathbb{N}}$ respectively. Assume that they have the same Bratteli diagrams.

From the numbers in the Bratteli diagrams, we infer that $A_n = B_n$ for all $n \in \mathbb{N}$ (up to reordering of direct sums). By Lemma 8.6, for any $n \in \mathbb{N}$, we may find unitaries $v_n \in B_n$ with $\varphi_n(x) = v_n \psi_n(x) v_n^*$ for all $x \in A_n$.

Putting $u_1 := 1$ and $u_{n+1} := v_n \psi_n(u_n) \in B_{n+1}$ in case $\psi_n(1) = 1$ (and some extension to a unitary otherwise), we obtain

$$u_{n+1}^* \varphi_n(x) u_{n+1} = \psi_n(u_n^*) v_n^* v_n \psi_n(x) v_n^* v_n \psi_n(u_n) = \psi_n(u_n^* x u_n).$$

One then checks that the maps $\alpha_n : A_n \rightarrow B_n, x \mapsto u_n^* x u_n$ induce $A \cong B$. \qed
Example 8.10. Let us take a look at a number of examples of AF algebras.

(a) The Bratteli diagram

\[
\begin{array}{c}
1 \\
\downarrow
\end{array}
\begin{array}{c}
2 \\
\downarrow
\end{array}
\begin{array}{c}
3 \\
\downarrow
\end{array}
\begin{array}{c}
\vdots
\end{array}
\]

yields the algebra of compact operators \( \mathcal{K}(H) \) on a separable Hilbert space \( H \). Indeed, this diagram may be translated to

\[
\begin{array}{c}
\mathbb{C} \\
\xrightarrow{\iota_1}
\end{array}
\begin{array}{c}
M_2(\mathbb{C}) \\
\xrightarrow{\iota_2}
\end{array}
\begin{array}{c}
M_3(\mathbb{C}) \\
\xrightarrow{\iota_3}
\end{array}
\begin{array}{c}
\vdots
\end{array}
\]

where the homomorphisms \( \iota_n : M_n(\mathbb{C}) \to M_{n+1}(\mathbb{C}) \) are given by

\[
x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}.
\]

So, we are in the situation of Rem. 8.3(a) with \( \mathbb{C} \subseteq M_2(\mathbb{C}) \subseteq \ldots \subseteq \mathcal{K}(H) \) (see also the proof of Prop. 6.13) and \( \lim_{n \to \infty} M_n(\mathbb{C}) = \bigcup_{n \in \mathbb{N}} M_n(\mathbb{C}) = \mathcal{K}(H) \).

(b) The Bratteli diagram

\[
\begin{array}{c}
1 \\
\downarrow
\end{array}
\begin{array}{c}
1 \\
\downarrow
\end{array}
\begin{array}{c}
1 \\
\downarrow
\end{array}
\begin{array}{c}
\vdots
\end{array}
\]

yields the unitization \( \tilde{\mathcal{K}}(H) \) as in Exm. 2.23.

(c) The Bratteli diagram

\[
\begin{array}{c}
1 \\
\uparrow
\end{array}
\begin{array}{c}
2 \\
\downarrow
\end{array}
\begin{array}{c}
4 \\
\downarrow
\end{array}
\begin{array}{c}
8 \\
\downarrow
\end{array}
\begin{array}{c}
\vdots
\end{array}
\]

yields the so called CAR (canonical anticommutation relations) algebra denoted by \( M_{2^\infty} \), the arrows being interpreted as

\[
x \mapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}.
\]

In fact, the following Bratteli diagram also yields the CAR algebra:

\[
\begin{array}{c}
1 \\
\downarrow
\end{array}
\begin{array}{c}
2 \\
\downarrow
\end{array}
\begin{array}{c}
4 \\
\downarrow
\end{array}
\begin{array}{c}
\vdots
\end{array}
\]

Hence, the converse of Lemma 8.9 is not true: Different Bratteli diagrams may produce the same AF algebra. In the present case, consider a third diagram.

---

\(^6\)By the way, the CAR algebra may also be represented by creation and annihilation operators on the antisymmetric Fock space.
Bratteli diagram

Check that \((A_{2n+1}, \varphi_{2n+1})_{n \in \mathbb{N}}\) yields the first diagram of (c), whereas \((A_{2n}, \varphi_{2n})_{n \in \mathbb{N}}\) produces the second one. Since

\[
\lim(A_n, \varphi_n) = \lim(A_{2n}, \varphi_{2n}) = \lim(A_{2n+1}, \varphi_{2n+1}),
\]

all these three Bratteli diagrams yield the same AF algebra.

(d) Given a sequence \((n_k)_{k \in \mathbb{N}}\) of natural numbers \(n_k \geq 2\), the Bratteli diagram

yields a so called UHF algebra generalizing \(M_{2^\infty}\).

8.4. **Ideals in AF algebras.** One of the main advantages of Bratteli diagrams is that we can read the ideal structure of an AF algebra from its diagram as follows. We view a Bratteli diagram as a directed graph in the canonical way.

**Definition 8.11.** Let a Bratteli diagram be given.

(a) If \(\zeta\) is a vertex in the diagram and there is an arrow pointing to another vertex \(\xi\), then \(\xi\) is called a \textit{successor} of \(\zeta\).

(b) A subdiagram of a Bratteli diagram is called \textit{directed}, if it is closed under all successors of its vertices, i.e. if \(\zeta\) is a vertex in the subdiagram and \(\xi\) is its successor, then also \(\xi\) must be in the subdiagram.

(c) A subdiagram is called \textit{hereditary}, if it is closed under predecessors in the following way: if \(\zeta\) is a vertex in the Bratteli diagram such that all of its successors lie in the subdiagram, then also \(\zeta\) must be in the subdiagram.

**Theorem 8.12.** Given an AF algebra \(A\) with some Bratteli diagram, there is a bijection between the closed ideals in \(A\) and the directed, hereditary subdiagrams of this Bratteli diagram. Such a subdiagram gives rise to a Bratteli diagram of the corresponding ideal \(I\); the complement of this subdiagram in turn is a Bratteli diagram of the quotient \(A/I\). In particular, ideals and quotients of AF algebras are AF algebras.

**Proof.** We omit the proof; see [10, Sect. III.4].
Example 8.13. Let us take a look at the AF algebras in Exm. 8.10 and their ideals.

(a) The Bratteli diagram in Exm. 8.10(a) has no directed, hereditary subdiagram apart from itself. Hence, the corresponding AF algebra $K(H)$ is simple. We knew this already from Cor. 6.14.

(b) The lower line of the diagram in Exm. 8.10(b) is the only non-trivial directed, hereditary subdiagram. By Exm. 8.10(a), it corresponds to the ideal $K(H)$ (cf. also Prop. 2.20). The quotient of $K(H)$ by $K(H)$ has the upper line of the diagram in Exm. 8.10(a) as its Bratteli diagram. It is thus $\mathbb{C}$.

(c) UHF algebras are simple, in particular the CAR algebra is simple.

One may give a converse of Thm. 8.12 as follows, combining results by Bratteli [4], Brown and Elliott [5, 12].

Theorem 8.14. Let $A$ be a separable $C^*$-algebra and $I \triangleleft A$ be a closed ideal. The algebra $A$ is an AF algebra, if and only if $I$ and $A/I$ are AF algebras.

Proof. We omit the proof; see [10, Sect. III.6].

8.5. Some remarks on AF algebras. AF algebras were the first class of $C^*$-algebras that was classified (by Elliott) via $K$-theory, the latter being a homological theory of invariants. This was the starting point of Elliott’s classification program for $C^*$-algebras which found a climax in [33].

In 1980, Pimsner and Voiculescu [24] constructed an AF algebra into which they embedded the rotation algebra $A_\vartheta$. This was a key step towards the classification of $A_\vartheta$ by its parameter $\vartheta$, see Sect. 7.9. Their AF algebra is constructed as follows. Let $\vartheta \in \mathbb{R}\setminus \mathbb{Q}$. We may write $\vartheta$ as a continued fraction

$$\vartheta = \lim_{n \to \infty} \frac{p_n}{q_n},$$

where $\frac{p_n}{q_n}$ is given as

$$\frac{p_n}{q_n} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + a_n}}}.$$
There is a recursion formula for $p_n$ and $q_n$ given by

$\begin{pmatrix} p_{n+1} & q_{n+1} \\ p_n & q_n \end{pmatrix} = \begin{pmatrix} a_n+1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_n & q_n \\ p_{n-1} & q_{n-1} \end{pmatrix}$

The AF algebra constructed in [24] is then given by $A_n := M_{q_n}(\mathbb{C}) \oplus M_{q_{n-1}}(\mathbb{C})$ and $\varphi_n : A_n \to A_{n+1}$ with the diagram

\[
\begin{array}{ccc}
q_n & \xrightarrow{(a_n+1 \text{ many})} & q_{n+1} \\
\downarrow & & \downarrow \\
q_{n-1} & \rightarrow & q_n
\end{array}
\]

One may then show that there is an injective $^*$-homomorphism from $A_0$ to $\lim_{n \to \infty} A_n$.

### 8.6. Exercises.

**Exercise 8.1.** We prove some details needed in the proof of Lemma 8.6. Let $A = \bigoplus_{k=1}^m M_{N_k}(\mathbb{C})$ and $B = \bigoplus_{l=1}^n M_{K_l}(\mathbb{C})$ and let $\varphi, \psi : A \to B$ be two $^*$-homomorphisms. Assume $\Phi = \Psi$, i.e. assume for all $k = 1, \ldots, m$ and $l = 1, \ldots, n$:

$\text{Tr}_{M_{K_l}}(\varphi(e_{11}^{(k)})) = \Phi_{lk} = \Psi_{lk} = \text{Tr}_{M_{K_l}}(\psi(e_{11}^{(k)})) \in \mathbb{N}_0$

(a) Let $k \in \{1, \ldots, m\}$ and $l \in \{1, \ldots, n\}$. Show that there is a partial isometry $v_{lk} \in M_{K_l}(\mathbb{C})$ such that $v_{lk}^*v_{lk} = \varphi(e_{11}^{(k)})$ and $v_{lk}v_{lk}^* = \psi(e_{11}^{(k)})$. (cf. Exc. 1.8)

(b) Put $v_l := \sum_{k=1}^m \sum_{i=1}^{N_k} \psi_i(e_{11}^{(k)})v_{ki}^*v_{ki} \in M_{K_l}(\mathbb{C})$. Show that we have $v_l^*v_l = \sum_{k=1}^m \sum_{i=1}^{N_k} \varphi_i(e_{ii}^{(k)})$.

(c) Deduce from (b) that $v := \sum_{l=1}^n v_l \in B$ is a partial isometry with $v^*v = \varphi(1)$ and $vv^* = \psi(1)$.

(d) Use $\Phi = \Psi$ to find a partial isometry $w \in B$ with $w^*w = 1 - \varphi(1)$ and $ww^* = 1 - \psi(1)$.

(e) Put $u := v + w \in B$. Show that $u$ is a unitary with $u^*\psi_i(e_{ij}^{(k)})u = \varphi_i(e_{ij}^{(k)})$ for all $i, j = 1, \ldots, N_k$ and $k = 1, \ldots, m$.

(f) Deduce $u^*\psi(x)u = \varphi(x)$ for all $x \in A$.

**Exercise 8.2.** We investigate the details of the proof of Lemma 8.9. Let $A$ and $B$ be AF algebras arising from inductive systems $(A_n, \varphi_n)_{n \in \mathbb{N}}$ and $(B_n, \psi_n)_{n \in \mathbb{N}}$ respectively. Assume that they have the same Bratteli diagrams.

(a) Convince yourself: $A_n = B_n$ for all $n \in \mathbb{N}$, up to reordering of direct sums.

(b) Let $n \in \mathbb{N}$. Convince yourself that there is a unitary $v_n \in B_n$ with $\varphi_n(x) = v_n^*\psi_n(x)v_n^*$ for all $x \in A_n$, by Lemma 8.6.

(c) Put $u_1 := 1 \in A_1$. For $n \geq 1$, put $u_{n+1} := v_n^*\psi_n(u_n) + w_n \in B_{n+1}$, where $w_n$ is a partial isometry with $w_n^*w_n = 1 - (u_{n+1}^*u_{n+1}^*)$ and $w_nw_n^* = 1 - u_{n+1}^*(u_{n+1})^*$. Show that such a partial isometry $w_n$ exists. Show that $u_{n+1}$ is a unitary.
(d) Define $\alpha_n : A_n \to B_n$ via $\alpha_n(x) := u_n^* xu_n$ for $x \in A_n$. Show that the following diagram is commutative.

\[
\begin{array}{cccccc}
A_1 & \phi_1 & A_2 & \phi_2 & A_3 & \phi_3 & \cdots \\
\downarrow \alpha_1 & \downarrow & \downarrow \alpha_2 & \downarrow & \downarrow \alpha_3 & \\
B_1 & \psi_1 & B_2 & \psi_2 & B_3 & \psi_3 & \cdots \\
\end{array}
\]

(e) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence with $x_n \in A_n$ for $n \in \mathbb{N}$ and assume that we have some $N \in \mathbb{N}$ with $x_{n+1} = \phi_n(x_n)$ for all $n \geq N$. Use (d) to show that $\alpha_{n+1}(x_{n+1}) = \psi_n(\alpha_n(x_n))$ for all $n \geq N$. Deduce that we may define $\alpha : A \to B$ by $[(x_n)_{n\in\mathbb{N}}] \mapsto [(\alpha_n(x_n))_{n\in\mathbb{N}}]$. Show that it is a $^*$-isomorphism.

\textbf{Exercise 8.3.} Consider the Bratteli diagram:

\[
\begin{array}{cccccc}
1 & \to & 1 & \to & 1 & \to & 1 & \to & 1 & \to & \cdots \\
& & & & & & & & & & \\
& & & & & & & & & & 1 \\
& & & & & & & & & & \\
& & & & & & & & & & 1 \\
& & & & & & & & & & \\
& & & & & & & & & & 1 \\
& & & & & & & & & & \\
& & & & & & & & & & 1 \\
& & & & & & & & & & \\
1 & \to & 1 & \to & 1 & \to & 1 & \to & 1 & \\
& & & & & & & & & & 1 \\
& & & & & & & & & & \\
& & & & & & & & & & 1 \\
& & & & & & & & & & \\
& & & & & & & & & & 1 \\
& & & & & & & & & & \\
& & & & & & & & & & 1 \\
& & & & & & & & & & \\
& & & & & & & & & & 1 \\
& & & & & & & & & & \\
& & & & & & & & & & 1 \\
& & & & & & & & & & \\
& & & & & & & & & & 1 \\
& & & & & & & & & & \\
& & & & & & & & & & 1 \\
& & & & & & & & & & \\
& & & & & & & & & & 1 \\
& & & & & & & & & & 1 \\
& & & & & & & & & 1 \\
& & & & & & & & 1 \\
\end{array}
\]

(a) Determine all ideals of the corresponding AF algebra $A$.
(b) Show that $A$ is isomorphic to $C(X)$, where $X$ is the Cantor set.
References


