ISEM24
C^*-ALGEBRAS AND DYNAMICS
LECTURE NOTES

ABSTRACT. In these lectures, we aim at providing an introduction to the general theory of C^*-algebras (first two thirds of the lectures) as well as to the more particular area of C^*-dynamical systems as a tool to deal with dynamics (last third of the lectures).

CONTENTS

Date: February 20, 2021.

Introduction and Motivation

Let us briefly motivate the lectures on C^*-algebras and dynamics and the main results we want to learn. This introduction is meant to serve as a teaser for the lectures omitting any technical details – the mathematical background for the following will be developed throughout the upcoming lectures. So, sit back and enjoy a short overview and motivation for the future lectures.

Recall that matrices $T \in M_N(\mathbb{C})$ may be seen as linear maps $T : \mathbb{C}^N \to \mathbb{C}^N$. In functional analysis, we deal with infinite dimensional versions of these and we consider linear maps

$$T : H \to H$$

between possibly infinite dimensional Hilbert spaces H. In contrast to linear algebra – i.e. the finite dimensional setting – these maps do not need to be continuous (which is equivalent to being bounded), so this comes as an extra assumption making life easier. So, let us consider

$$B(H) := \{T : H \to H \mid T \text{ is linear and bounded}\},$$

where H is some Hilbert space H. If $\dim(H) = N$, then $B(H) = M_N(\mathbb{C})$.

A main feature of bounded, linear operators on a Hilbert space is noncommutativity: We have $ST \neq TS$ in general, where $S, T \in B(H)$ and the multiplication is defined via composition of maps. We know such a feature already from the matrix multiplication in linear algebra. This noncommutativity appears in quantum physics, in linear algebra, in the representation theory of groups and in many further areas of mathematics and science.

The theory of operator algebras captures this noncommutativity turning it into a powerful tool in mathematics. The pioneers Francis Murray and John von Neumann wrote in their very first article [35] on von Neumann algebras in 1936 that

> "various aspects of the quantum mechanical formalism suggest strongly the elucidation of this subject."

In addition, they claim that their work may be viewed as part of

> "attempts to generalise the theory of unitary group-representations [sic!] essentially beyond their classical frame [...]."

Representing groups as unitary operators in $B(H)$ has also been in the scope of Israel Gelfand and Mark Naimark, when they wrote their seminal article [19] in 1943 introducing C^*-algebras. In 1993, Richard Kadison commented [25] on this article

> "from the vantage point of a fifty year history, it is safe to say that that paper changed the face of modern analysis. Together with the monumental 'Rings of operators' series [...] authored by F. J. Murray and J. von Neumann, it introduced 'non-commutative analysis', the

> vast area of mathematics that provides the mathematical model for quantum physics."

Nowadays, the following areas may be counted to such a "non-commutative analysis" or "quantum mathematics":

Classical theory	Quantum/noncomm. version	Founders and pioneers
Topology	C^*-Algebras	Gelfand-Naimark 1940s
Measure Theory	von Neumann Algebras	Murray-vonNeumann 1930s
Probability Theory	Free Probability Theory,	Voiculescu 1980s
	Quantum Probability Theory	Accardi,
		Hudson-Parthasarathy 1970s
Differential Geometry	Noncommutative Geometry	Connes 1980s
(Compact) Groups	(Compact) Quantum Groups	Woronowicz 1980s
Information Theory	Quantum Information Theory	Feynman, Deutsch 1980s
Complex Analysis	Free Analysis	J. L. Taylor 1970s

The main reason why C^*-algebras may be seen as a "quantum version" of topology comes from the famous Gelfand-Naimark Theorem, which we allow ourselves to call the 1st Fundamental Theorem of C^*-Algebras in these lectures.

1st Fundamental Theorem of C^*-Algebras (Gelfand-Naimark 1940s)**.** *Let A be a unital C^*-algebra. We have the following equivalence.*

$$A \text{ is commutative} \quad \Longleftrightarrow \quad \exists X \text{ compact}\colon A \cong C(X) := \{f : X \to \mathbb{C} \text{ is continuous}\}$$

Hence, any compact topological space gives rise to a commutative unital C^*-algebra – on the other hand *any* commutative C^*-algebra is exactly of this form. In this sense, commutative C^*-algebras "correspond" to topology and we may view the theory of noncommutative C^*-algebras as a kind of "noncommutative topology".

This Gelfand duality is also the basis for other quantum theories (namely von Neumann algebras, Free probability, noncommutative geometry and quantum groups).

Besides proving the above first fundamental theorem, our goal is to prove that any (abstractly defined) C^*-algebra may be represented concretely on a Hilbert space:

2nd Fundamental Theorem of C^*-Algebras (Gelfand-Naimark, Segal 1940s)**.** *Any C^*-algebra is isomorphic to a norm closed *-subalgebra of $B(H)$, for some H.*

From these fundamental theorems, we should keep in mind, that the algebra $C(X)$ of continuous functions on a compact space X as well as closed (in the operator norm topology) *-subalgebras of $B(H)$ are our main examples of C^*-algebras.

We will spend about two thirds of the lecture (October – December 2020) in order to develop the above basic knowledge on C^*-algebras including also a treatment of universal C^*-algebras. Afterwards (January – February 2021), we turn to dynamical systems. Let us sketch some basic ideas of the latter, referring to [45] for a nice survey on dynamical systems and operator algebras.

Our starting point is a group G and a compact space X. Assume that G acts on this space, i.e. we have a map $\alpha : G \times X \to X$. This is a topological dynamical system. See [45] for a motivation how to derive this setting from more physically motivated dynamical systems or from differential equations.

Now, let us define $\alpha_g : C(X) \to C(X)$ via $\alpha_g(f)(x) := f(\alpha(g^{-1}, x))$. This induces a group homomorphism from G to the automorphism group of $C(X)$ by $g \mapsto \alpha_g$. We may then construct a C^*-algebra $C(X) \rtimes_\alpha G$ containing the information of X, of G and of the action of G on X (in terms of conjugation with unitaries) – hence, $C(X) \rtimes_\alpha G$ encodes the whole dynamical system!

Surprisingly, although $C(X)$ is commutative, the crossed product C^*-algebra $C(X) \rtimes_\alpha G$ may fail to be commutative. In fact, this is the generic situation: Unless the action is trivial, $C(X) \rtimes_\alpha G$ is always noncommutative (as conjugation with unitaries is trivial in commutative C^*-algebras). Hence, although our input X and G is classical data, we might want to enter the "nonclassical" or "quantum" world of noncommutative C^*-algebras in order to study this dynamical system. The philosophy is, that the theory of C^*-algebras provides a number of tools whith which we may investigate $C(X) \rtimes_\alpha G$ – in order to learn something about the classical dynamical system.

More generally, we will treat C^*-dynamical systems, i.e. actions α of compact groups G on possibly noncommutative C^*-algebras A, leading to crossed products $A \rtimes_\alpha G$.

We wish you a pleasant reading of the lecture notes and we hope you will enjoy the theory of C^*-algebras as much as we do!

1. Reminder on bounded operators on Hilbert spaces

Abstract. We recall some basic notions from Hilbert space theory, such as Hilbert spaces, Cauchy-Schwarz inequality, orthogonality, decomposition of Hilbert spaces, Riesz Representation Theorem, orthonormal bases and isomorphisms of Hilbert spaces. We then turn to bounded linear operators on Hilbert spaces, their operator norms and the existence of adjoints. We define the notion of a C^*-algebra and verify that $B(H)$ is a unital C^*-algebra. We finish this lecture with a number of algebraic reformulations of properties of operators on Hilbert spaces (such as unitaries, isometries, orthogonal projections, etc.), and we give a brief survey on compact operators. As Lecture 1 is seen as a reminder to lay the foundations for the upcoming lectures, it does not contain many complete proofs, but we give at least some ideas. You may take [3, 11, 36] as general references for Lecture 1.

1.1. **Hilbert spaces.** Informally speaking, Hilbert spaces are normed vector spaces equipped with a tool to measure "angles" between vectors, see also Exc. 1.5.

Definition 1.1. Let H be a complex vector space. An *inner product* is a map $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{C}$ satisfying for all $x, y, z \in H$ and all $\lambda, \mu \in \mathbb{C}$:

(1) $\langle \lambda x + \mu y, z \rangle = \lambda \langle x, z \rangle + \mu \langle y, z \rangle$
(2) $\langle z, \lambda x + \mu y \rangle = \bar{\lambda} \langle z, x \rangle + \bar{\mu} \langle z, y \rangle$
(3) $\overline{\langle x, y \rangle} = \langle y, x \rangle$
(4) $\langle x, x \rangle \geq 0$
(5) If $\langle x, x \rangle = 0$, then $x = 0$.

A space equipped with an inner product is called a *pre-Hilbert space*. An inner product induces a norm $\|x\| := \sqrt{\langle x, x \rangle}$. A *(complex) Hilbert space* is a pre-Hilbert space, which is complete with respect to the induced norm.

Example 1.2. The following spaces are examples of Hilbert spaces.

(a) Given $n \in \mathbb{N}$, the vector space $\mathbb{C}^n$ endowed with $\langle x, y \rangle := \sum_{i=1}^n x_i \bar{y}_i$, $x, y \in \mathbb{C}^n$ is a Hilbert space. The induced norm is the well-known Euclidean norm.
(b) The space $\ell^2(\mathbb{N})$ of complex-valued sequences $(a_n)_{n\in\mathbb{N}}$ with $\sum_{n\in\mathbb{N}} |a_n|^2 < \infty$ endowed with $\langle (a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \rangle := \sum_{n\in\mathbb{N}} a_n \bar{b}_n$, $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$ is a Hilbert space.
(c) More generally, recall that we may define $L^2(X, \mu)$ where (X, μ) is a measure space. The inner product is then given by:

$$\langle f, g \rangle := \int_X f(x) \bar{g}(x) \, \mathrm{d}\mu(x), \qquad f, g \in L^2(X, \mu)$$

Note that for $X = [0, 1]$ the unit interval and $\mu = \lambda$ the Lebesgue measure, this defines an inner product on the space $\mathcal{C}([0, 1])$ of continuous complex-valued functions. However, $\mathcal{C}([0, 1])$ is not complete with respect to the induced norm (which is the so called L^2-norm), i.e. it is only a pre-Hilbert space but no Hilbert space.

Choosing $X = I$ a set and $\mu = \zeta$ the counting measure, we obtain $\ell^2(I)$, with the above examples $\ell^2(\mathbb{N})$ and $\mathbb{C}^n$ as special cases.

(d) Any closed subspace of a Hilbert space is a Hilbert space (closed with respect to the norm topology, subspace in the sense of a linear subspace).

The most important inequality for inner products is the following one.

Proposition 1.3 (Cauchy-Schwarz inequality)**.** *If H is a Hilbert space (or a pre-Hilbert space), we have for all $x, y \in H$:*

$$|\langle x, y\rangle| \leq \|x\|\|y\|$$

Here, equality holds if and only if x and y are linearly dependent.

Proof (idea): Use $0 \leq \langle x + \lambda y, x + \lambda y\rangle = \langle x, x\rangle - \frac{|\langle x,y\rangle|^2}{\langle y,y\rangle}$ with $\lambda = -\frac{\langle x,y\rangle}{\langle y,y\rangle}$. □

Actually, one needs the Cauchy-Schwarz inequality for proving that the norm in Def. 1.1 is a norm indeed; moreover we may derive continuity of the inner product. There are two further important properties of the inner product and its induced norm.

Proposition 1.4. *Let H be a Hilbert space (or a pre-Hilbert space) and let $x, y \in H$.*

(a) *The* parallelogram identity *holds: $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$.*
(b) *The* polarisation identity *holds: $\langle x, y\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k\|x + i^k y\|^2$.*

Proof (idea): (a): Direct computation. (b): Use $\|x + i^k y\|^2 = \langle x + i^k y, x + i^k y\rangle$. □

The first of the above identities characterizes pre-Hilbert spaces: A normed space is a pre-Hilbert space if and only if the parallelogram identity holds. The second identity shows that the inner product is completely determined by its induced norm.

1.2. **Orthogonality and decomposition of Hilbert spaces.** As mentioned before, an inner product is the abstract information of an angle between vectors, see also Exc. 1.5. The notion of orthogonality plays the role of right angles.

Definition 1.5. Let H be a Hilbert space and $K, K_1, K_2 \subseteq H$ be subsets.

(a) Two vectors $x, y \in H$ are *orthogonal* ($x \perp y$), if $\langle x, y\rangle = 0$.
(b) We write $K_1 \perp K_2$, if $x \perp y$ for all $x \in K_1$ and $y \in K_2$.
(c) The *orthogonal complement* of K is $K^\perp := \{x \in H \mid x \perp y \text{ for all } y \in K\}$.

Even when K is just a subset without any further structure, its orthogonal complement will be of a nice form.

Lemma 1.6. *Given a subset $K \subseteq H$, its orthogonal complement $K^\perp \subseteq H$ is a closed subspace of H and we have $(\overline{K})^\perp = K^\perp$, where $\overline{K}$ is the closure of K.*

Proof (idea): Due to the continuity of the inner product (Exc. 1.4). □

The following is a version of the antique Greek theorem by Pythagoras verifying that orthogonality corresponds to right angles indeed, see also Exc. 1.5.

Proposition 1.7 (Pythagoras' Theorem)**.** *If H is a Hilbert space and $x, y \in H$ are orthogonal, then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.*

Proof. Direct computation. □

One of the most important features of Hilbert spaces is that we may decompose them into direct sums.

Definition 1.8. Let $K_1, K_2 \subseteq H$ be two closed subspaces of a Hilbert space, such that $K_1 \perp K_2$. We then write $K_1 \oplus K_2 \subseteq H$ for the subspace given by elements $x + y \in H$, where $x \in K_1$ and $y \in K_2$.

Proposition 1.9. *Given a closed subspace $K \subseteq H$, we may decompose the Hilbert space H as a direct sum:*

$$H = K \oplus K^\perp$$

Then, every vector $x \in H$ has a unique decomposition $x = x_1 + x_2$ with $x_1 \in K$ and $x_2 \in K^\perp$.

Proof (idea): By Lemma 1.6, $K^\perp$ is closed. Trivially, $K \perp K^\perp$. We need to do some hard work to show that given $x \in H$, there is a unique "best approximation" $x_1 \in K$ such that $\|x - x_1\| = \inf\{\|x - y\| \mid y \in K\}$. With some further efforts, we then show $x_2 := x - x_1 \in K^\perp$. That this decomposition of x is unique easily follows from $K \cap K^\perp = \{0\}$. □

Corollary 1.10. *Given a subspace $K \subseteq H$, the double complement $(K^\perp)^\perp$ coincides with the closure $\overline{K}$ of K.*

Proof. By the previous proposition and using Lemma 1.6, we may decompose H in two ways, $H = \overline{K} \oplus K^\perp$ and $H = (K^\perp)^\perp \oplus K^\perp$, which shows $\overline{K} = (K^\perp)^\perp$. □

1.3. **Dual space and the Representation Theorem of Riesz.** Another nice feature of Hilbert spaces is that they have nice dual spaces – themselves! Given $y \in H$, we denote by $f_y : H \to \mathbb{C}$ the linear map given by $f_y(x) := \langle x, y \rangle$. In Exc. 1.4, it is shown that f_y is linear and continuous.

Proposition 1.11 (Riesz Representation Theorem)**.** *Let H be a Hilbert space and denote by H' its dual space, i.e. the space consisting in all linear, continuous maps $f : H \to \mathbb{C}$. The map $j : H \to H'$ given by $j(y) := f_y$ is an antilinear isometric isomorphism.*

Proof (idea): By Exc. 1.4, j maps to H' and it is isometric, i.e. $\|f_y\| = \|y\|$ (and hence injective); antilinearity follows from Def. 1.1(2). As for surjectivity, let $f \in H'$ be non-zero and decompose $H = K \oplus K^\perp$, where $K := \ker f$. You will find out that $K^\perp$ is one-dimensional and $j(y) = f_y = f$ for some $y \in K^\perp$. □

This has some nice consequences when working with Hilbert spaces. For instance, given a linear, continuous functional $f : L^2(X, \mu) \to \mathbb{C}$, then it must come from a function $g \in L^2(X, \mu)$, i.e. $f(h) = \int_X h\bar{g}\, \mathrm{d}\mu$ for all $h \in L^2(X, \mu)$.

1.4. **Orthonormal basis for a Hilbert space.** In finite dimensions, we usually understand vector spaces with respect to certain coordinates. We may transport this concept to the infinite-dimensional setting within the framework of Hilbert spaces.

Lemma 1.12. *Let H be a Hilbert space and let $(e_i)_{i\in I}$ be an* orthonormal system*, i.e. $\langle e_i, e_j\rangle = \delta_{ij}$. The following are equivalent:*

1. $\|x\|^2 = \sum_{i\in I}|\langle x, e_i\rangle|^2$ *for all* $x \in H$
2. $x = \sum_{i\in I}\langle x, e_i\rangle e_i$ *for all* $x \in H$
3. $\operatorname{span}\{e_i \mid i \in I\} \subseteq H$ *is dense.*
4. *If $z \in H$ is orthogonal to all e_i, $i \in I$, then $z = 0$.*
5. $(e_i)_{i\in I}$ *is a maximal orthonormal family (with respect to inclusion).*

Proof (idea): First note that the sums over a possibly uncountable index set I are defined as the limits of nets of finite sums; in particular, only countably many summands are non-zero. As for the proof of the lemma, the easy parts are the equivalences of (2) and (3) (just a reformulation), of (4) and (5) (just a reformulation) as well as of (2) and (4) (use $z := x-\sum_{i\in I}\langle x, e_i\rangle e_i$). The hard part is the equivalence of (1) and (2), where we use Pythagoras (Prop. 1.7) on finite subsets $F \subseteq I$ proving that $\|x - \sum_{i\in F}\langle x, e_i\rangle e_i\|^2$ tends to zero. The key words are Bessel's Inequality and Parseval's Identity. □

Definition 1.13. An orthonormal system is called an *orthonormal basis* of a Hilbert space, if one of the equivalent conditions in Lemma 1.12 is satisfied.

We should not be misled by the word "basis" here: The elements of an orthonormal basis are linearly independent, but they do *not* necessarily form a basis in the sense of linear algebra (Hamel basis) – we may *not* represent any vector in H by a *finite* linear combination of the e_i. However, passing to *infinite* linear combination, we may do so. This is the content of Lemma 1.12(2) – and we even know the coefficients thanks to our inner product. See also Schauder bases for the general Banach space setting.

Example 1.14. For $\mathbb{C}^n$, the vectors e_i having 1 at the i-th entry and zero otherwise form an orthonormal basis – in fact, in finite dimensions any orthonormal basis is also a (Hamel) basis.

More generally, for $\ell^2(I)$, the sequence having 1 at the i-th entry and zero otherwise form an orthonormal basis. If I is infinite, then this is not a basis.

Lemma 1.15. *Any Hilbert space possesses an orthonormal basis $(e_i)_{i\in I}$ and the cardinality of I is independent of the choice of the vectors.*

Proof (idea): Use Zorn's Lemma for the existence and Cantor-Schröder-Bernstein for the uniqueness of the cardinality. □

Definition 1.16. Given a Hilbert space H with orthonormal basis $(e_i)_{i\in I}$, its *(Hilbert space) dimension* is defined as the cardinality of I. If I is countable, we call H *separable.*

Thanks to the above lemma, the dimension is well-defined.

1.5. **Isomorphisms of Hilbert spaces.** Let us think about isomorphisms of Hilbert spaces – which structure are they supposed to preserve? Well, the vector space and the inner product!

Definition 1.17. Let H and K be Hilbert spaces. An *isomorphism* between H and K is a surjective linear map $U : H \to K$ which is *isometric* (or *preserves the inner product*), i.e. it satisfies $\langle Ux, Uy\rangle_K = \langle x, y\rangle_H$ for all $x, y \in H$.

The preservation of the inner product implies that U is injective, which means that it is an isomorphism of the level of vector spaces, in particular. One can show that Hilbert spaces are isomorphic if and only if they have the same Hilbert space dimension in the sense of Def. 1.16. Hence, any Hilbert space is isomorphic to some $\ell^2(I)$. In particular, $\ell^2(\mathbb{N})$ is *the* separable Hilbert space.

1.6. **Bounded linear operators on Hilbert spaces.** In the subsequent lectures, we are not so much interested in the theory of Hilbert spaces as such but rather in the theory of bounded linear operators on Hilbert spaces. Let us first prove that "bounded" and "continuous" means the same for linear operators.

Lemma 1.18. *Let H, K be Hilbert spaces and let $T : H \to K$ be linear. The following are equivalent:*

(a) *T is continuous everywhere.*
(b) *T is continuous in zero.*
(c) *T is bounded, i.e. there is a $C > 0$ such that $\|Tx\| \leq C\|x\|$ for all $x \in H$.*

Proof (idea): The step from (a) to (b) is trivial. Assuming (b) with $\varepsilon = 1$, there is a $\delta > 0$ such that $\|x\| \leq \delta$ implies $\|Tx\| \leq 1$; put $C := \delta^{-1}$ to derive (c). Passing from (c) to (a) is straightforward. □

Definition 1.19. Given a Hilbert space H, we denote by $B(H)$ the space of all bounded, linear operators $T : H \to H$.

Example 1.20. If $\dim(H) = N$, i.e. if $H = \mathbb{C}^N$, then $B(H) = M_N(\mathbb{C})$, the algebra of $N \times N$ matrices with complex entries. Indeed, in this case, *any* linear map is automatically bounded.

Definition 1.21. Given $T \in B(H)$, we denote by

$$\|T\| := \inf\{C > 0 \mid \|Tx\| \leq C\|x\| \text{ for all } x \in H\}$$

the *operator norm* of T.

One can check that the operator norm is a norm indeed.

Lemma 1.22. *Given $T \in B(H)$, we have $\|Tx\| \leq \|T\|\|x\|$ for all $x \in H$.*

Proof. Choosing $C_n > \|T\|$ with $C_n \to \|T\|$ yields $\|Tx\| \leq C_n\|x\| \to \|T\|\|x\|$. □

Let us express the operator norm in an alternative way.

Lemma 1.23. *The norm $\|T\|$ may be written as*

$$\|T\| = \sup\{\|Tx\| \mid \|x\| = 1\}.$$

You may replace $\|x\| = 1$ by $\|x\| \leq 1$, if you prefer.

Proof. By Lemma 1.22, we have $\|Tx\| \leq \|T\|$, if $\|x\| \leq 1$. Thus the supremum s over all $\|Tx\|$ with $\|x\| = 1$ is less or equal to $\|T\|$. Conversely,

$$\|Tx\| = \|T\left(\frac{x}{\|x\|}\right)\|\|x\| \leq s\|x\|$$

whenever $x \neq 0$, so $\|T\| \leq s$ by Def. 1.21, which yields $\|T\| = s$ in total. The same proof works if s is the supremum over $\|Tx\|$ with $\|x\| \leq 1$. $\square$

1.7. **Existence of adjoints.** How does a bounded, linear operator T behave with respect to evaluations under the inner product? Here, the existence of adjoints is a useful fact.

Proposition 1.24. *Let H be a Hilbert space and $T \in B(H)$. There exists a unique operator $T^* \in B(H)$ (the* adjoint *of T) such that*

$$\langle Tx, y\rangle = \langle x, T^*y\rangle$$

for all $x, y \in H$.

Proof (idea): Let $y \in H$. We define $g^y : H \to \mathbb{C}$ by $g^y(x) := \langle Tx, y\rangle$. Then $g^y \in H'$ and by the Riesz Representation Theorem 1.11, there is a $z \in H$ such that $g^y = f_z$. Thus $\langle Tx, y\rangle = \langle x, z\rangle$ and we put $T^*y := z$. Check $T^* \in B(H)$. $\square$

Example 1.25. If $H = \mathbb{C}^N$ and $T \in B(H) = M_N(\mathbb{C})$, we may express T by $Te_i = \sum_j t_{ji}e_j$ for the canonical basis $e_1, \ldots, e_N$ of $\mathbb{C}^N$. Thus, $T \in M_N(\mathbb{C})$ has coefficients t_{ij} and $T^* \in M_N(\mathbb{C})$ has coefficients $\bar{t}_{ji}$.

Some operators coincide with their adjoints; they will play a special role.

Definition 1.26. An operator $T \in B(H)$ is called *selfadjoint* (or *Hermitian*), if $T = T^*$.

There is a useful formula relating the kernel of T with the image of its adjoint. We denote by $\ker T$ the space of all $x \in H$ such that $Tx = 0$, whereas $\operatorname{ran} T$ denotes the set of all Tx, where $x \in H$.

Lemma 1.27. *For $T \in B(H)$, we have $\ker T = (\operatorname{ran} T^*)^\perp$ and $(\ker T)^\perp = \overline{\operatorname{ran} T^*}$.*

Proof. A vector x is in $(\operatorname{ran} T^*)^\perp$ if and only if $\langle Tx, y\rangle = \langle x, T^*y\rangle = 0$ for all y, i.e. if and only if x is in the kernel of T. Use Lemma 1.6 for the second part. $\square$

Implicitely, we used the following lemma in the proof above.

Lemma 1.28. *Let $T \in B(H)$. If $\langle Tx, y\rangle = 0$ for all $y \in H$, then $Tx = 0$. In particular, $\langle Tx, y\rangle = \langle Sx, y\rangle$ for all $x, y \in H$ implies $S = T$.*

Proof. Put $y = Tx$ for the first part and use the first part for the second. $\square$

1.8. **Algebraic structure of $B(H)$ and C^*-algebras.** Let us now turn to the main structure of these lectures: to C^*-algebras. It turns out that it describes the algebraic structure of $B(H)$ pretty well.

Definition 1.29. We define the following algebraic notions.

(a) An *algebra* A over $\mathbb{C}$ is a complex vector space equipped with a bilinear associative multiplication $\cdot : A \times A \to A$ satisfying $\lambda(xy) = (\lambda x)y = x(\lambda y)$ for $x, y \in A$ and $\lambda \in \mathbb{C}$. The algebra is *unital*, if it contains a unit 1 with respect to the multiplication, i.e. $1x = x1 = x$ for all $x \in A$.

(b) A *normed algebra* A is an algebra which is also a normed vector space and whose norm is *submultiplicative*: It satisfies $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in A$.

(c) A *Banach algebra* is a normed algebra which is complete.

(d) An *involution* on an algebra A is an antilinear map $^* : A \to A$ such that $(x^*)^* = x$ and $(xy)^* = y^*x^*$ for all $x, y \in A$.

(e) A *C^*-algebra* is a Banach algebra A with an involution satisfying the *C^*-identity* $\|x^*x\| = \|x\|^2$ for all $x \in A$.

We conclude that a C^*-algebra combines algebraic structures (algebra with involution) with topological ones (norm and completion). The most important link between these two worlds is the C^*-identity, which turns C^*-algebras into a very special subclass of Banach algebras. We will see later how this identity comes into play. Also, we will discuss basic properties of the above definition in the next lecture. For now, let us be patient and let us only check that $B(H)$ is a C^*-algebra.

Proposition 1.30. *Given a Hilbert space H, the map $T \mapsto T^*$ from Prop. 1.24 give rise to an involution and the composition of maps gives rise to a multiplication. Together with the operator norm, this turns $B(H)$ into a unital C^*-algebra.*

Proof. Using Lemma 1.28, we may directly check that we have an involution on $B(H)$ given by the adjoints. For instance:

$$\langle (T^*)^*x, y\rangle = \langle x, T^*y\rangle = \langle Tx, y\rangle, \qquad \text{for all } x, y \in H$$

By Lemma 1.28 this yields $(T^*)^* = T$. Submultiplicativity of the norm follows from Lemma 1.23 when taking the supremum over $\|S(Tx)\| \leq \|S\|\|Tx\| \leq \|S\|\|T\|\|x\|$.

Let us now check that the involution is isometric (a fact that holds in general in C^*-algebras). Using Cauchy-Schwarz (Prop. 1.3), we have:

$$\|T^*x\|^2 = \langle T^*x, T^*x\rangle = \langle TT^*x, x\rangle \leq \|TT^*x\|\|x\| \leq \|T\|\|T^*x\|\|x\|$$

This implies $\|T^*x\| \leq \|T\|\|x\|$ and taking the supremum, we obtain $\|T^*\| \leq \|T\|$, by Lemma 1.23. On the other hand, $\|T\| = \|(T^*)^*\| \leq \|T^*\|$ which proves that the involution satisfies $\|T^*\| = \|T\|$.

We may now check the C^*-identity. Again, Cauchy-Schwarz yields

$$\|Tx\|^2 \leq \|T^*T\|\|x\|^2 = \|T^*T\|,$$

in case $\|x\| = 1$. Taking the supremum and using that the involution is isometric, we obtain:

$$\|T\|^2 \leq \|T^*T\| \leq \|T^*\|\|T\| = \|T\|^2$$

Hence, we have equality in the above computation.

As for the completeness of $B(H)$ with respect to the operator norm, this is a general fact on Banach spaces, which we omit here.

The unit on $B(H)$ is the identity map $x \mapsto x$, denoted by 1. □

We now have a good example of a C^*-algebra at hand: it is $B(H)$, or $M_N(\mathbb{C})$, if you prefer the finite-dimensional setting. We may easily obtain further examples.

Example 1.31. Any closed *-subalgebra of $B(H)$ is a C^*-algebra. More precisely, let $A \subseteq B(H)$ be a linear subspace, which is closed under taking products and adjoints (i.e. it is a *-subalgebra), and which is also closed in the operator norm topology. Then, A is a C^*-algebra.

Finally, let us remark that $B(H)$ is also closed under taking inverses with respect to the composition, i.e. the inverse as a map is also the inverse with respect to the multiplication.

Proposition 1.32. *Let $T \in B(H)$ be a bijective map. Then also $T^{-1} \in B(H)$ and $(T^{-1})^* = (T^*)^{-1}$.*

Proof (idea): It is easy to see that T^{-1} is linear, but we need the Open Mapping Theorem for boundedness. The second assertion follows from Lemma 1.28. □

1.9. **Algebraic formulations of Hilbert space features.** Being aware of the algebraic structure of $B(H)$ has some advantages: We may express certain properties of operators by purely algebraic means.

Definition 1.33. Let A be a unital C^*-algebra. Let $U, V, P \in A$.

(a) U is called *unitary*, if $U^*U = UU^* = 1$.
(b) V is called *isometry*, if $V^*V = 1$.
(c) P is called *(orthogonal) projection*, if $P = P^* = P^2$.

Let us take a look at the above definition in the special case $A = B(H)$ and see how the naming is motivated. Recall that $1 \in B(H)$ denotes the identity map.

Proposition 1.34. *Let $U, V, P \in B(H)$.*

(a) *U is a unitary if and only if it is a Hilbert space isomorphism of H.*
(b) *V is an isometry if and only if $\langle Vx, Vy\rangle = \langle x, y\rangle$ for all $x, y \in H$.*
(c) *P is a projection if and only if there is a closed subspace $K \subseteq H$ such that $P(x + y) = x$ for $x + y \in K \oplus K^\perp = H$, i.e. $\operatorname{ran} P = K$.*

Proof. Item (b) is an easy consequence of Lemma 1.28. As for (a), assume that U is a unitary. By (b), it is isometric, and from $UU^* = 1$ follows surjectivity. Hence, it is a Hilbert space isomorphism in the sense of Def. 1.17. Conversely, if U is a

Hilbert space isomorphism, we use (b) to deduce $U^*U = 1$. We prove $UU^* = 1$ as follows, making use of Lemma 1.28. Given $x, y \in H$ there is $x_0 \in H$ with $Ux_0 = x$ and hence:

$$\langle UU^*x, y\rangle = \langle UU^*Ux_0, y\rangle = \langle Ux_0, y\rangle = \langle x, y\rangle$$

Showing (c), let us first assume that P is a projection. Put $K := \operatorname{ran} P$, the range of P. Then K is a linear subspace of H. Moreover, any $x \in \operatorname{ran} P$ satisfies $Px = x$, since $P^2 = P$. Thus, for any sequence $x_n \to x$ with $x_n \in \operatorname{ran} P$, we have $x_n = Px_n \to Px$ by continuity of P. As the limit is unique, we have $x = Px \in \operatorname{ran} P$, which means that K is closed. We may hence decompose $H = K \oplus K^\perp$ and we observe that $K^\perp = \ker P$ using Lemma 1.27 and $P = P^*$. Thus, $P(x + y) = x$ for $x \in K$ and $y \in K^\perp$.

Conversely, let $K \subseteq H$ be a closed subspace and $P(x+y) = x$ as in the assertion. Then $P^2 = P$. Moreover, $P^* = P$ holds, since for $x, x' \in K$ and $y, y' \in K^\perp$:

$$\begin{aligned}\langle P^*(x + y), x' + y'\rangle &= \langle x + y, P(x' + y')\rangle = \langle x + y, x'\rangle = \langle x, x'\rangle = \langle x, x' + y'\rangle \\ &= \langle P(x + y), x' + y'\rangle\end{aligned}$$

We then use Lemma 1.28 to finish the proof. □

We conclude, that even in an abstract C^*-algebra A in the sense of Def. 1.29, we may define unitaries, isometries and projections as in Def. 1.33 – and this will allow us to deal abstractly with Hilbert space isomorphisms, the preservation of inner products and closed subspaces even if there is no underlying Hilbert space at hand!

Example 1.35. Let us briefly look at some examples of unitaries and isometries.

(a) In the finite dimensional setting, any isometry is automatically unitary. Indeed, by Prop. 1.34 we know that any isometry $V \in M_N(\mathbb{C})$ is injective: $Vx = 0$ implies $\langle x, x\rangle = \langle Vx, Vx\rangle = 0$. In finite dimensions, injectivity implies surjectivity, thus V is a unitary.

(b) In the infinite dimensional setting, these two notions may differ. Consider the Hilbert space $\ell^2(\mathbb{N})$ with an orthonormal basis e_n, $n \in \mathbb{N}$, see Exm. 1.14 for instance. The *unilateral shift* $S \in B(\ell^2(\mathbb{N}))$ is defined by $Se_n := e_{n+1}$, for all $n \in \mathbb{N}$. It is easy to see that $S^*e_n = e_{n-1}$ for $n \geq 2$ and $S^*e_1 = 0$. So, $S^*S = 1$, but $SS^* \neq 1$. See also Exc. 1.7.

1.10. **Compact operators.** We have seen that $B(H)$ is a unital C^*-algebra. Let us come to another important example of a C^*-algebra, in fact a non-unital one.

Definition 1.36. An operator $T \in B(H)$ is *compact* if one of the following equivalent conditions is satisfied:

(a) For any bounded set $M \subseteq H$, the closed set $\overline{TM}$ is compact.

(b) The closed image $\overline{TB(0,1)}$ of the unit ball $B(0,1) := \{x \in H \mid \|x\| \leq 1\}$ is compact.

(c) For any bounded sequence $(x_n)_{n\in\mathbb{N}}$ in H, the sequence $(Tx_n)_{n\in\mathbb{N}}$ contains a convergent subsequence.

We denote by $\mathcal{K}(H) \subseteq B(H)$ the set of all compact operators.

Example 1.37. (a) In $M_N(\mathbb{C})$, any operator is compact (Heine-Borel).

(b) Let H be infinite dimensional and assume that $T \in B(H)$ has *finite rank*, i.e. its image $\operatorname{ran} T$ is finite dimensional. Then T is compact. This follows again from some Heine-Borel argument, since $\overline{TB(0,1)}$ is contained in $\overline{\{y \in \operatorname{ran} T \mid \|y\| \leq C\}}$ with $C = \|T\|$, by Lemma 1.23.

(c) Let H be infinite dimensional. The operator $1 \in B(H)$ (i.e. the identity map) is *not* compact, since the closed unit ball is not compact. In fact, any normed vector space is finite-dimensional if and only if the closed unit ball is compact. We infer $\mathcal{K}(H) \subsetneq B(H)$ in infinite dimensions.

So, compact operators seem to be close to the finite dimensional setting – that is indeed the case: they may be approximated by finite rank operators as we will see in the next proposition. Thus, compact operators play the role of "small" operators.

Proposition 1.38. *The compact operators have the following properties.*

(a) *$\mathcal{K}(H)$ is a closed two-sided ideal of $B(H)$, i.e. it is a closed linear subspace satisfying $ST, TS \in \mathcal{K}(H)$ for all $S \in \mathcal{K}(H)$ and $T \in B(H)$.*

(b) *Given $T \in \mathcal{K}(H)$, we may find a sequence $T_n \in B(H)$ of finite rank operators approximating T in the operator norm.*

(c) *$\mathcal{K}(H)$ is closed under taking adjoints.*

(d) *$\mathcal{K}(H)$ is a C^*-algebra. It is non-unital, if and only if H is infinite dimensional.*

Proof (idea): The proof of (a) is no fun. That $\mathcal{K}(H)$ is a linear subspace follows easily from the continuity of the addition. Also, the ideal property is doable. However, showing that $\mathcal{K}(H)$ is closed requires some tedious arguments (but no magic).

In order to show (b), let us restrict to the case when H is separable with orthonormal basis e_n, $n \in \mathbb{N}$. We denote by $E(H)$ the set of finite rank operators. By (a) and Exm. 1.37, we know $\overline{E(H)} \subseteq \mathcal{K}(H)$. For the converse inclusion, denote by P_n the projection onto $\operatorname{span}\{e_1, \ldots, e_n\}$. Then $T_n := P_n T$ is of finite rank. One can then directly show that $T_n x \to Tx$, using Lemma 1.12. But this is only pointwise convergence! In order to show convergence in the operator norm, we need to use that T is compact.

Part (c) is known as Schauder's Theorem (which holds for general Banach spaces H). In our case, it follows easily from (b) (but (b) is not true for general Banach spaces H): Let $T \in \mathcal{K}(H)$ and pick a sequence T_n of finite rank operators approximating T. Then T_n^* is also of finite rank, since $P_m T_n = T_n$ for some $m \in \mathbb{N}$ and $T_n^* = T_n^* P_m$, i.e. T_n^* acts only on a finite dimensional subspace. Now, the involution is isometric and hence continuous, i.e. $T_n^* \to T^*$ by (a) and T^* is compact.

(d) We conclude that $\mathcal{K}(H)$ is a closed *-subalgebra and hence it is a C^*-algebra by Exm. 1.31. Why isn't it unital in the infinite dimensional case? Let $(e_i)_{i \in I}$ be an orthonormal basis of H. Assume $P \in \mathcal{K}(H)$ was a unit for $\mathcal{K}(H)$, i.e. $PT = TP = T$

for all $T \in \mathcal{K}(H)$. Then also $PQ_i = Q_iP = Q_i$ where Q_i is the projection onto $\mathbb{C}e_i$, the one-dimensional subspace spanned by the i-th basis vector. This implies $Pe_i = e_i$ for all $i \in I$, and hence $P = 1$. But $1 \notin \mathcal{K}(H)$ by Exm. 1.37. □

1.11. **Exercises.**

Exercise 1.1. (a) Check that property (2) of Def. 1.1 may be derived from (1) and (3).
(b) Check that $\|x + y\|^2 = \|x\|^2 + 2\mathrm{Re}\langle x, y\rangle + \|y\|^2$ holds, where Re is the real part of a complex number.

Exercise 1.2. Prove the Cauchy-Schwarz inequality (Prop. 1.3) and show that equality holds if and only if the vectors are linearly dependent.

Exercise 1.3. Show that the norm induced by an inner product is a norm indeed. Use Cauchy-Schwarz and Exc. 1.1(b). Show that the mapping $x \mapsto \|x\|$ is continuous. This turns a Hilbert space into a topological vector space.

Exercise 1.4. Show that $f_y(x) := \langle x, y\rangle$ is linear and bounded with norm $\|f_y\| = \|y\|$. Thus, f_y is an element in the dual space of a Hilbert space H and the inner product is continuous in the sense that $x \mapsto \langle x, y\rangle$ is continuous.

Exercise 1.5. In Def. 1.1, we defined Hilbert spaces only for complex vector spaces, but the definition of real Hilbert spaces is completely analogous. Let us consider $\mathbb{R}^2$ with the inner product $\langle x, y\rangle = \sum_{i=1}^{2} x_i y_i$.
(a) Describe all unit vectors (i.e. vectors with norm 1) with the help of sine and cosine.
(b) Describe all vectors that are orthogonal to a given vector $x = (x_1, x_2) \in \mathbb{R}^2$.
(c) Show that $\frac{\langle x,y\rangle}{\|x\|\|y\|} = \cos\varphi$, where φ is the angle between x and y.
(d) Convince yourself that Prop. 1.7 is really Pythagoras Theorem for $H = \mathbb{R}^2$.

Exercise 1.6. Prove Lemma 1.18.

Exercise 1.7. Consider the unilateral shift $S \in B(\ell^2(\mathbb{N}))$ from Exm. 1.35.
(a) Verify $S^*e_n = e_{n-1}$ for $n \geq 2$ and $S^*e_1 = 0$. Verify that S is an isometry but no unitary.
(b) Now, consider the bilateral shift $\tilde{S} \in B(\ell^2(\mathbb{Z}))$ given by $\tilde{S}e_n = e_{n+1}$, where e_n, $n \in \mathbb{Z}$ is an orthonormal basis. How about this one, is it an isometry, is it a unitary?
(c) Which matrix is a reasonable analogue of $\tilde{S}$ in $M_N(\mathbb{C})$?

Exercise 1.8. An operator $V \in B(H)$ is called a *partial isometry*, if $VV^*V = V$.
(a) Show that V is a partial isometry if and only if V^*V is a projection (if and only if VV^* is a projection) in the sense of Def. 1.33.
(b) Show that V is a partial isometry if and only if there is a closed subspace $K \subseteq H$ such that $\langle Vx, Vy\rangle = \langle x, y\rangle$ for all $x, y \in K$ and $Vx = 0$ for $x \in K^\perp$. Compare with Prop. 1.34.

2. C^*-algebras, Banach algebras and their spectra

Abstract. We consider Banach algebras and C^*-algebras and study some of their basic properties. We then turn to the spectrum of an element and show that it is compact and non-empty. We define the spectral radius and we prove Beurling's formula. We briefly introduce ideals and quotients for Banach algebras before we turn to unitizations of C^*-algebras.

Throughout Lecture 2, we take a closer look at the special features of C^*-algebras as a particular class of Banach algebras. The book [3] serves as a general reference for Lecture 2; further references are given at the end of the lecture.

2.1. **Banach algebras and C^*-algebras.** Let us recall the definition of Banach and C^*-algebras from Def. 1.29.

Definition 2.1. Building on Def. 1.29, we define the following notions for Banach and C^*-algebras.

(a) An *involution* on a $\mathbb{C}$-algebra A is an antilinear map $^* : A \to A$ such that $(x^*)^* = x$ and $(xy)^* = y^*x^*$. A **-algebra* is an algebra A equipped with an involution; $B \subseteq A$ is a **-subalgebra* of A, if $xy, \lambda x + \mu y, x^* \in B$ for all $x, y \in B$ and $\lambda, \mu \in \mathbb{C}$.
(b) A *Banach algebra* is a normed $\mathbb{C}$-algebra which is complete; its norm satisfies $\|xy\| \leq \|x\|\|y\|$. A *Banach* **-algebra* is a Banach algebra with an involution.
(c) A C^**-algebra* is a Banach *-algebra A satisfying the C^**-identity* $\|x^*x\| = \|x\|^2$. A *-subalgebra $B \subseteq A$ is a C^**-subalgebra*, if B is (topologically) closed.
(d) An algebra is *unital*, if it contains a unit with respect to the multiplication.
(e) An algebra A is *commutative*, if $xy = yx$ for all $x, y \in A$.
(f) An element $x \in A$ in a C^*-algebra is *normal*, if $x^*x = xx^*$. It is *selfadjoint*, if $x^* = x$.

We observe, that C^*-algebras differ from Banach *-algebras only by the C^*-identity. What is so special about it? Some of the immediate consequences are listed in the next remark; others will come up later, for instance when talking about positivity in C^*-algebras. It is hard to believe at this stage, but it is exactly this C^*-identity that turns the class of C^*-algebras into a very well-behaved and very special subclass of Banach algebras.

Remark 2.2. Let A be a C^*-algebra.

(a) Let $x \in A$. If $x^*x = 0$, then $x = 0$. This follows from the C^*-identity.
(b) The involution is bijective, since $(x^*)^* = x$. It is also isometric, since $\|x\|^2 = \|x^*x\| \leq \|x^*\|\|x\|$ implies $\|x\| \leq \|x^*\| \leq \|(x^*)^*\| = \|x\|$.
(c) A Banach *-algebra satisfies the C^*-identity if and only if it satisfies the following inequality:

$$\|x\|^2 \leq \|x^*x\|$$

Indeed, assuming this inequality, we derive as in (b) that the involution is isometric. Then $\|x\|^2 \geq \|x^*x\|$ follows from submultiplicativity.

(d) If A is unital (and $A \neq 0$), then $1^* = 1$ and $\|1\| = 1$. Indeed, $1^*x = (x^*1)^* = (x^*)^* = x$ and $x1^* = x$ for all $x \in A$, hence 1^* is also a unit for A. Thus, $1^* = 1$. Moreover, $\|1\|^2 = \|1^*1\| = \|1\|$, by the C^*-identity. Thus, $\|1\| \in \{0, 1\}$ and we may exclude $\|1\| = 0$ if $A \neq 0$.

In fact, more generally, $\|p\| = 1$ for any non-trivial selfadjoint projection $p \in A$, i.e. $p = p^* = p^2$ and $p \neq 0$, see Def. 1.33.

(e) If $x \in A$ is invertible, then $(x^{-1})^* = (x^*)^{-1}$, since $(x^{-1})^*x^* = (xx^{-1})^* = 1$.

(f) The algebraic operations, i.e. the addition, the multiplication and the involution are continuous, and also the norm is continuous.

(g) Clearly, any selfadjoint element is normal. Normal and selfadjoint elements will play an important role later.

Let us take a look at examples of C^*-algebras.

Example 2.3. (a) From Prop. 1.30 we know that the algebra $B(H)$ of bounded linear operators on a Hilbert space H is a unital C^*-algebra. This is $M_N(\mathbb{C})$ in the finite dimensional case.

(b) Any closed *-subalgebra of $B(H)$ is a C^*-algebra (see Exm. 1.31).

(c) If H is infinite dimensional, then also the compact operators $\mathcal{K}(H)$ form a C^*-algebra, in fact a non-unital one (see Prop. 1.38); if H is finite dimensional, then $\mathcal{K}(H) = B(H)$.

(d) Let X be a compact Hausdorff space. Then

$$C(X) := \{f : X \to \mathbb{C} \mid f \text{ is continuous}\}$$

is a unital C^*-algebra with $(f+g)(x) := f(x) + g(x)$, $(\lambda f) := \lambda f(x)$, $(fg)(x) := f(x)g(x)$, $1(x) := 1$, $f^*(x) := \overline{f(x)}$ for $f, g \in C(X)$, $x \in X$ and $\lambda \in \mathbb{C}$ and the supremum norm

$$\|f\|_\infty := \sup\{|f(x)| \mid x \in X\}.$$

Compactness of X guarantees that $\|f\|_\infty < \infty$ for all $f \in C(X)$, i.e. the supremum norm is a norm indeed. Since $|f(x)g(x)| = |f(x)||g(x)|$ and $|f(x)\overline{f(x)}| = |f(x)|^2$, it is easy to see that the norm is submultiplicative satisfying the C^*-identity. Completeness is a bit more elaborate, but you might recall a proof from your early analysis lectures regarding uniform convergence of sequences of functions.

(e) If X is not compact, then $\|f\|_\infty = \infty$ may happen for some $f \in C(X)$. However, if X is locally compact, we may restrict to a subalgebra of $C(X)$ as follows. We say that a function $f : X \to \mathbb{C}$ vanishes at infinity, if for all $\varepsilon > 0$ there exists a compact set $K \subseteq X$ such that for all $t \notin K$ we have $|f(t)| < \varepsilon$. Put

$$C_0(X) := \{f : X \to \mathbb{C} \mid f \text{ is continuous and vanishes at infinity}\}.$$

We equip this set with the pointwise operations and the supremum norm as above. Then $\|f\|_\infty < \infty$ for all $f \in C_0(X)$ since f vanishes at infinity. One can check that $C_0(X)$ is a C^*-algebra. If X is compact, then $C_0(X) = C(X)$. If X is not compact, then $C_0(X)$ is non-unital.

(f) Let us also give an example of a Banach *-algebra which is not a C^*-algebra. Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$. The *disk algebra*

$$A(\mathbb{D}) := \{f : \overline{\mathbb{D}} \to \mathbb{C} \mid f \text{ is continuous on } \overline{\mathbb{D}} \text{ and holomorphic on } \mathbb{D}\}$$

with the pointwise addition, multiplication and the involution $f^*(z) := \overline{f(\bar{z})}$ is a Banach *-algebra with the supremum norm, but no C^*-algebra.

2.2. **Spectrum of an element.** We know from linear algebra that eigenvalues play an important role when studying matrices. In infinite dimensions, we need to consider spectral values instead of eigenvalues.

Definition 2.4. Let A be a unital Banach algebra and let $x \in A$.

(a) The *spectrum* of x is defined as $\mathrm{sp}(x) := \{\lambda \in \mathbb{C} \mid \lambda 1 - x \text{ is not invertible}\}$. It is also denoted by $\sigma(x)$ sometimes. We also write $\lambda - x$ instead of $\lambda 1 - x$.

(b) The *resolvent set* of x is the complement $\rho(x) = \mathbb{C} \setminus \mathrm{sp}(x)$.

Remark 2.5. (a) Recall the definition of an eigenvalue: Given an operator $T \in B(H)$, a complex number $\lambda \in \mathbb{C}$ is an *eigenvalue*, if $\lambda 1 - T$ is not injective. Indeed, in that case, we find $0 \neq x \in H$ such that $Tx = \lambda x$. Now, in finite dimensions, $\lambda 1 - T$ is not injective if and only if $\lambda 1 - T$ is not invertible. This is not true in the infinite dimensional setting (for instance, the unilateral shift from Exm. 1.35 is injective but not surjective and hence not invertible). This means, that any eigenvalue is a spectral value, but the converse is not true. The set of eigenvalues $\sigma_p(T) := \{\lambda \in \mathbb{C} \mid \lambda 1 - T \text{ is not injective}\}$ is also called the *point spectrum*. The funny thing is, that there are even operators with no eigenvalues at all, i.e. $\sigma_p(T) = \emptyset$ is possible for some $T \in B(H)$ (not in finite dimensions, though).

(b) In linear algebra, there is the well-known spectral theorem. There are analogues in the infinite dimensional setting: a spectral theorem for compact operators as well as a spectral theorem for normal bounded linear operators making use of the above definition of the spectrum.

(c) If A is a unital C^*-algebra and $x \in A$, then $\mathrm{sp}(x^*) = \{\bar{\lambda} \mid \lambda \in \mathrm{sp}(x)\}$. Indeed, from Rem. 2.2(e), we know that $\lambda 1 - x$ is invertible if and only if $(\lambda 1 - x)^*$ is invertible.

In $M_N(\mathbb{C})$, $N \geq 1$, the spectrum coincides with the set of eigenvalues – it is finite and non-empty in that case. What is the situation in $B(H)$ for an infinite dimensional Hilbert space H? Let us prepare the investigation of this question.

Lemma 2.6. *Let A be a unital Banach algebra.*

(a) *If $x \in A$ with $\|1 - x\| < 1$, then x is invertible and $x^{-1} = \sum_{n=0}^{\infty}(1 - x)^n$.*

(b) *If x is invertible and $y \in A$ with $\|x-y\| < (\|x^{-1}\|)^{-1}$, then y is invertible.*
(c) $\mathrm{GL}(A) := \{x \in A \mid x \text{ is invertible}\}$ *is open, and* $\mathrm{GL}(A) \ni x \mapsto x^{-1}$ *is continuous.*

Proof. For the proof of (a), put $z := 1 - x$. Then, $\|z\| < 1$ and hence $\sum_{n=0}^{\infty} z^n$ is absolutely convergent (since $\|z^n\| \leq \|z\|^n$ by submultiplicativity). As a consequence, $\sum_{n=0}^{\infty} z^n$ is convergent. It yields the inverse of x:

$$x\sum_{n=0}^{\infty} z^n \leftarrow (1-z)\sum_{n=0}^{N} z^n = \sum_{n=0}^{N} z^n - \sum_{n=1}^{N+1} z^n = 1 - z^{N+1} \to 1, \qquad \text{as } N \to \infty.$$

As for (b), we have:

$$\|1 - yx^{-1}\| = \|(x-y)x^{-1}\| \leq \|x-y\|\|x^{-1}\| < 1$$

Thus, yx^{-1} is invertible by (a) and hence so is y.

Finally, for showing (c), let $x \in \mathrm{GL}(A)$ and $y \in A$ such that $\|x-y\| < \varepsilon < \|x^{-1}\|^{-1}$. By (b), this shows $y \in \mathrm{GL}(A)$ and hence the ε-ball around x is in $\mathrm{GL}(A)$ proving that $\mathrm{GL}(A)$ is open. Let us now show continuity of taking inverses. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence with $x_n \to x$ and $x_n, x \in \mathrm{GL}(A)$, $n \in \mathbb{N}$. Then $\|x_n - x\| < \|x^{-1}\|^{-1}\frac{\varepsilon}{2}$ for $n \in \mathbb{N}$ large and $0 < \varepsilon < 1$. Hence:

$$\|1 - x_n x^{-1}\| = \|(x - x_n)x^{-1}\| \leq \frac{\varepsilon}{2} < 1$$

By (a), this shows that $x_n x^{-1}$ is invertible with inverse

$$xx_n^{-1} = (x_n x^{-1})^{-1} = \sum_{k=0}^{\infty}(1 - x_n x^{-1})^k = 1 + \sum_{k=1}^{\infty}(1 - x_n x^{-1})^k.$$

Using $\varepsilon^k \leq \varepsilon$, we conclude:

$$\begin{aligned}\|x_n^{-1} - x^{-1}\| = \|x^{-1}(xx_n^{-1} - 1)\| &\leq \|x^{-1}\| \sum_{k=1}^{\infty}\|1 - x_n x^{-1}\|^k \\ &\leq \|x^{-1}\| \sum_{k=1}^{\infty} \varepsilon \frac{1}{2^k} = \varepsilon\|x^{-1}\|\end{aligned}$$

Thus, x_n^{-1} converges to x^{-1}. □

As an immediate consequence, the spectrum is compact.

Proposition 2.7. *Let A be a unital Banach algebra and let $x \in A$. Then* $\mathrm{sp}(x)$ *is compact and* $\mathrm{sp}(x) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| \leq \|x\|\}$.

Proof. Firstly, we notice that the resolvent set $\rho(x)$ is open since it can be written as $\rho(x) = f_x^{-1}(\mathrm{GL}(A))$, where $f_x : \mathbb{C} \to A, \lambda \mapsto \lambda 1 - x$ is continuous and $\mathrm{GL}(A)$ is open by Lemma 2.6. Thus $\mathrm{sp}(x)$ is closed.

Secondly, if $|\lambda| > \|x\| \neq 0$, then $\|\frac{x}{\lambda}\| < 1$. Hence, $\lambda - x = \lambda(1 - \frac{x}{\lambda})$ is invertible by Lemma 2.6, which shows $\lambda \notin \mathrm{sp}(x)$. Thus, $\mathrm{sp}(x) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| \leq \|x\|\}$, which means that $\mathrm{sp}(x)$ is bounded. In summary, $\mathrm{sp}(x)$ is compact by Heine-Borel. $\square$

Let us now prove that the spectrum is non-empty, a fact that some people call the *Fundamental Theorem in Banach Algebras.* It has been proven by Gelfand. Recall that we denote by A' the dual space of A, see Sect. 1.3.

Theorem 2.8. *If A is a unital Banach algebra, $A \neq 0$, then $\mathrm{sp}(x) \neq \emptyset$ for all $x \in A$.*

Proof. Let $x \in A$. For $\lambda \in \rho(x)$, put $R_\lambda(x) := (\lambda - x)^{-1}$.

Claim 1: We have $R_\lambda(x) - R_\mu(x) = (\mu - \lambda)R_\lambda(x)R_\mu(x)$ for all $\lambda, \mu \in \mathbb{C}$.

For proving Claim 1, check that

$$\begin{aligned} R_\lambda(x) - R_\mu(x) &= R_\lambda(x)R_\mu(x)(\mu - x) - (\lambda - x)R_\lambda(x)R_\mu(x) \\ &= (\mu - \lambda)R_\lambda(x)R_\mu(x), \end{aligned}$$

where we used $R_\lambda(x)R_\mu(x)(\mu - x) = (\mu - x)R_\lambda(x)R_\mu(x)$. Note that in principle $ab \neq ba$ for $a, b \in A$, but here this does not cause issues because $(\lambda - x)(\mu - x) = (\mu - x)(\lambda - x)$ implies $(\mu - x)R_\lambda(x) = R_\lambda(x)(\mu - x)$. This proves Claim 1.

Claim 2: Assume that x is invertible and let $f \in A'$ such that $f(x^{-1}) \neq 0$. Then $g\colon \rho(x) \to \mathbb{C}, g(\lambda) := f(R_\lambda(x))$ is holomorphic and $g(0) \neq 0$.

For proving Claim 2, note that $\lambda \mapsto R_\lambda(x)$ is continuous by Lemma 2.6. By Claim 1, we thus have for $\mu \to \lambda$:

$$\frac{g(\lambda) - g(\mu)}{\lambda - \mu} = f\left(\frac{R_\lambda(x) - R_\mu(x)}{\lambda - \mu}\right) = -f(R_\lambda(x)R_\mu(x)) \to -f(R_\lambda^2(x))$$

Thus g is holomorphic with $g(0) = f(R_0(x)) = -f(x^{-1}) \neq 0$. This shows Claim 2.

Finally, assume that $\mathrm{sp}(x) = \emptyset$. Then $0 \notin \mathrm{sp}(x)$, i.e. $0 - x$ is invertible and thus x is invertible. By the Hahn-Banach Theorem, we find a functional $f \in A'$ with $f(x^{-1}) \neq 0$. Thus the function g from Claim 2 is an entire function as $\rho(x) = \mathbb{C}$.

Claim 3: g is bounded, because $g(\lambda) \to 0$ for $\lambda \to \infty$.

In order to show Claim 3, put $z := 1 - \lambda^{-1}x$. Then $\|1 - z\| = |\lambda|^{-1}\|x\| < 1$ for $|\lambda|$ large. Thus, z is invertible by Lemma 2.6 with $z^{-1} = \sum_{n=0}^{\infty}(1 - z)^n$. Hence:

$$\|(1 - \lambda^{-1}x)^{-1}\| = \|z^{-1}\| \leq \sum_{n=0}^{\infty}\|1 - z\|^n = (1 - \|1 - z\|)^{-1} = \frac{1}{1 - \frac{\|x\|}{|\lambda|}}$$

This implies:

$$\begin{aligned} \|R_\lambda(x)\| = \|(\lambda - x)^{-1}\| &= |\lambda|^{-1}\|(1 - \lambda^{-1}x)^{-1}\| \\ &\leq \frac{1}{|\lambda|(1 - \frac{\|x\|}{|\lambda|})} = \frac{1}{|\lambda| - \|x\|} \to 0 \text{ as } |\lambda| \to \infty. \end{aligned}$$

Claim 3 is proven.

As g is a bounded, entire function, it is constant by Liouville's Theorem. From $g(\lambda) \to 0$ as $|\lambda| \to \infty$, we infer $g = 0$, which contradicts $g(0) \neq 0$ from Claim 2. $\square$

An easy consequence: The only Banach algebra which is also a skew field, is $\mathbb{C}$.

Theorem 2.9 (Gelfand-Mazur)**.** *Let A be a unital Banach algebra. If A is also a skew field (i.e. every element $0 \neq a \in A$ is invertible), then $A = \mathbb{C}1$.*

Proof. Let $a \in A$. Then $\mathrm{sp}(a) \neq \emptyset$ by Thm. 2.8. Hence there is some $\lambda \in \mathbb{C}$ such that $\lambda 1 - a$ is not invertible. Since A is a skew field, this implies $\lambda 1 - a = 0$. □

2.3. **Spectral radius.** An important information we can extract from the spectrum is the spectral radius. It is closely linked with the norm providing an attractive alternative way for computing the norm.

Definition 2.10. Let A be a unital Banach algebra and let $x \in A$. The *spectral radius* of x is defined as

$$r(x) := \sup\{|\lambda| \mid \lambda \in \mathrm{sp}(x)\}.$$

Remark 2.11. From Prop. 2.7 we know $r(x) \leq \|x\|$. Also, as the spectrum is compact (by the same proposition), the supremum is in fact a maximum.

Example 2.12. The spectral radius may or may not coincide with the norm:

(a) We have $r(f) = \|f\|_\infty$ for any $f \in C(X)$ whenever X is compact. Indeed, since f is continuous and X is compact, the image $f(X)$ is compact. Thus, there is some $x \in X$ with $|f(x)| = \|f\|_\infty$, i.e. $f(x) = e^{i\alpha}\|f\|_\infty$ for some $\alpha \in [0, 2\pi)$. Then, $e^{i\alpha}\|f\|_\infty - f$ is not invertible and $e^{i\alpha}\|f\|_\infty \in \mathrm{sp}(f)$.

(b) Consider $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{C})$. Then, $\lambda - x = \begin{pmatrix} \lambda & -1 \\ 0 & \lambda \end{pmatrix}$ is invertible for all $\lambda \neq 0$. Thus $\mathrm{sp}(x) = \{0\}$ and $r(x) = 0$ while $\|x\| \neq 0$.

So, when does the spectral radius coincide with the norm? Let us try to find out. The main ingredient is the following amazing formula by Beurling (also known as *Gelfand-Beurling spectral radius formula*). It relates an algebraic quantity (the spectral radius, speaking about invertibility in an algebraic sense) with a topological one (the norm).

Theorem 2.13. *Let A be a unital Banach algebra and $x \in A$. Then the spectral radius formula holds:*

$$r(x) = \lim_{n\to\infty} \sqrt[n]{\|x^n\|}$$

Proof. Let $\lambda \in \mathrm{sp}(x)$. Then $\lambda^n \in \mathrm{sp}(x^n)$, since

$$\lambda^n - x^n = (\lambda - x)(\lambda^{n-1} + \lambda^{n-2}x + \cdots + \lambda x^{n-2} + x^{n-1})$$

cannot have an inverse. Hence, $|\lambda^n| \leq \|x^n\|$ which means $|\lambda| \leq \sqrt[n]{\|x^n\|}$. We infer $r(x) \leq \liminf_{n\to\infty} \sqrt[n]{\|x^n\|}$. Thus, it remains to show $r(x) \geq \limsup_{n\to\infty} \sqrt[n]{\|x^n\|}$. As in the proof of Thm. 2.8, consider

$$R_z(x) = (z - x)^{-1} = \sum_{n=0}^{\infty} \frac{x^n}{z^{n+1}}$$

for $\|x\| < |z|$ (in particular $z \in \rho(x)$ by Prop. 2.7). If this was a power series in $\mathbb{C}$, its radius of convergence would be $\limsup_{n\to\infty} \sqrt[n]{\|x^n\|}$, which is a hint for us to be on the right track. However, it is a series in A unfortunately, so we have to use the same trick as in the proof of Thm. 2.8 in order to "make it a series in $\mathbb{C}$". Let $f \in A'$. Then, the function $g : \rho(x) \to \mathbb{C}, z \mapsto f(R_z(x))$ is holomorphic (see the proof of Thm. 2.8) and $g(z) = \sum_{n=0}^{\infty} \frac{f(x^n)}{z^{n+1}}$ for $|z| > \|x\|$; in fact even for $|z| > r(x)$ (using methods from complex analysis on the holomorphic domain of the Laurent series). Hence

$$\limsup_{n\to\infty} |f(x^n)|^{\frac{1}{n}} \leq r(x)$$

by the formula of Cauchy-Hadamard for convergence radii of power series. This is good, but now we have to get rid of f. For $r > r(x)$, we find some $N \in \mathbb{N}$ such that $|f(x^n)|^{\frac{1}{n}} < r$ for all $n \geq N$. Hence

$$\sup_{n\in\mathbb{N}} \left| \frac{f(x^n)}{r^n} \right| < \infty$$

for all $f \in A'$. By the Principle of Uniform Boundedness, we conclude that the set $\{\frac{x^n}{r^n} \mid n \in \mathbb{N}\}$ is bounded. Hence there is some $C > 0$ such that $\|x^n\| \leq C r^n$. Now, $\|x^n\|^{\frac{1}{n}} \leq C^{\frac{1}{n}} r$, which implies $\limsup_{n\to\infty} \|x^n\|^{\frac{1}{n}} \leq r$ for all $r > r(x)$ and thus also for $r = r(x)$. $\square$

This formula behaves particularly nice with respect to the C^*-identity as we will see in the next corollary. It also answers our question under which conditions the spectral radius and the norm coincide.

Corollary 2.14. *Let A be a unital C^*-algebra and let $x \in A$ be normal (i.e. $x^*x = xx^*$, see Def. 2.1). Then $r(x) = \|x\|$.*

Proof. Using the C^*-identity and the fact that x is normal, we have:

$$\|x^2\|^2 = \|(x^2)^* x^2\| = \|x^* x^* x x\| = \|x^* x x^* x\| = \|(x^* x)^* (x^* x)\| = \|x^* x\|^2 = \|x\|^4$$

Thus $\|x\|^2 = \|x^2\|$. Inductively, we see that $\|x^{2^n}\| = \|x\|^{2^n}$, hence

$$r(x) = \lim_{n\to\infty} \sqrt[2^n]{\|x^{2^n}\|} = \|x\|.$$

$\square$

Another surprising corollary is that a C^*-algebra cannot be equipped with another C^*-norm.

Corollary 2.15. *Let A be a unital $*$-algebra and let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on A such that both $(A, \|\cdot\|_1)$ and $(A, \|\cdot\|_2)$ are C^*-algebras. Then $\|\cdot\|_1 = \|\cdot\|_2$.*

Proof. Let $x \in A$. Then $\|x\|_i^2 = \|x^* x\|_i = r(x^* x)$ by Cor. 2.14, for $i = 1, 2$. $\square$

2.4. **Ideals and quotients.** Let us now briefly discuss some further algebraic structure of Banach algebras and C^*-algebras: Ideals and quotients.

Definition 2.16. Let A be an algebra.

(a) A *(two-sided) ideal* I in A (we write $I \lhd A$) is a linear subspace $I \subseteq A$ such that $xy, yx \in I$, for all $x \in I$ and $y \in A$.
(b) An ideal $I \neq A$ is called *maximal*, if for any other (two-sided) ideal $J \lhd A$ with $I \subseteq J \subseteq A$ either $J = I$ or $J = A$.
(c) If A is a Banach algebra, we say that $I \lhd A$ is a *closed ideal*, if I is a (two-sided) ideal and if it is closed with respect to the topology of A.

Throughout the lectures, all ideals will be two-sided unless specified otherwise. The main use of ideals is that we can take quotients, i.e. we may pass to some "rougher structures" in a way. Let us list some properties of ideals in Banach algebras.

Proposition 2.17. *Let A be a Banach algebra and let $I \lhd A$ be an ideal.*

(a) *If I is a closed ideal, then the quotient A/I is a Banach algebra.*
(b) *The closure $\bar{I}$ of I is an ideal in A.*
(c) *For a unital Banach algebra A, the following are equivalent:*
(i) $I = A$ *(ii)* $I \cap \mathrm{GL}(A) \neq \emptyset$ *(iii)* $1 \in I$
(d) *Let A be unital. Then any maximal ideal is closed.*
(e) *Let A be unital. Then any ideal $I \neq A$ is contained in a maximal ideal.*

Proof. For (a), we first prove a more general (and probably well-known) statement: If A is a Banach space and I is a closed linear subspace, then A/I is a Banach space. For proving it, denote by $\dot{x} = x + I$ the elements in A/I. We define a vector space structure on A/I by $\dot{x} + \dot{y} := (x+y)^{\cdot}$ and $\lambda\dot{x} := (\lambda x)^{\cdot}$. We define a norm on A/I by $\|\dot{x}\| := \inf\{\|x + z\| \mid z \in I\}$. It is not too difficult to prove that this is a norm indeed; note that we need I to be closed in order to deduce $\dot{x} = 0$ from $\|\dot{x}\| = 0$. It is a bit more technical to show that A/I is again a Banach space, i.e. that it is complete with respect to this norm, but no magic is involved. Since $0 \in I$, we have $\|\dot{x}\| \leq \|x\|$.

We then turn back to the situation we are interested in and assume that A is even a Banach algebra while $I \lhd A$ is a closed ideal. In addition to the above structure, we put $\dot{x}\dot{y} := (xy)^{\cdot}$ turning A/I into an algebra. This operation is well-defined since for all $a, b \in I$ and $x, y \in A$, the element $ay + xb + ab$ is in I and hence:

$$((x+a)(y+b))^{\cdot} = (xy + ay + xb + ab)^{\cdot} = (xy)^{\cdot}$$

Moreover, the norm is submultiplicative: Given $\varepsilon > 0$, there are $a, b \in I$ with $\|x + a\| \leq \|\dot{x}\| + \varepsilon$ and $\|y + b\| \leq \|\dot{y}\| + \varepsilon$. Thus:

$$\|\dot{x}\dot{y}\| = \|(x+a)^{\cdot}(y+b)^{\cdot}\| \leq \|(x+a)(y+b)\| \leq \|x+a\|\|y+b\| \leq (\|\dot{x}\| + \varepsilon)(\|\dot{y}\| + \varepsilon)$$

As this holds true for all $\varepsilon > 0$, we just proved $\|\dot{x}\dot{y}\| \leq \|\dot{x}\|\|\dot{y}\|$, and A/I is a Banach algebra.

As for (b), let $x \in \bar{I}$ and $y \in A$. We find a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \in I$ and $x_n \to x$. Then $x_n y \to xy$ with $x_n y \in I$ for all $n \in \mathbb{N}$. Hence $xy \in \bar{I}$ and similarly $yx \in \bar{I}$.

For item (c), it is easy to see that (iii) implies (i), while (i) implies (ii). As for (ii) to (iii), let $x \in I \cap \mathrm{GL}(A)$. Then $1 = xx^{-1} \in I$, since $x \in I$.

For (d), let $I \lhd A$ be a maximal ideal. Then $I \subseteq \bar{I} \subseteq A$. We need to show $\bar{I} \neq A$ in order to deduce $I = \bar{I}$ from (b) and maximality of I. Now, the complement $\mathrm{GL}(A)^c$ of $\mathrm{GL}(A)$ contains I by (c), since $I \neq A$ by the definition of a maximal ideal. Moreover, $\mathrm{GL}(A)^c$ is closed by Lemma 2.6, so it also contains $\bar{I}$. By (c), $\bar{I} \neq A$.

Finally, (e) is a consequence of Zorn's Lemma. ☐

2.5. **Unitization of C^*-algebras.** In Exm. 2.3, we have seen some examples of non-unital C^*-algebras: $\mathcal{K}(H)$, if H is infinite dimensional and $C_0(X)$, if X is locally compact but not compact. In both cases, there are "would-be"-units: If the identity operator $\mathrm{id} : H \to H$ was compact, then it would be a unit for $\mathcal{K}(H)$. Likewise, if the constant function $1 : X \to \mathbb{C}$ vanished at infinity, then it would be a unit for $C_0(X)$. So, it seems that there is some unit in the background, if we enlarge our algebra! Let us do this systematically and study unitizations.

Our idea is to add a copy of a relatively small algebra – ideally: $\mathbb{C}$ – to a non-unital C^*-algebra and to find a unit in this enlarged space. Let us take a look at the direct sum of C^*-algebras first.

Lemma 2.18. *Let A, B be C^*-algebras. Put*

$$A \oplus B := \{(a, b) \mid a \in A, b \in B\}.$$

This is a C^*-algebra with the entrywise operations and $\|(x, y)\| := \max\{\|x\|, \|y\|\}$.

Proof. Straightforward. ☐

When being equipped with the entrywise operations, the only chance for the direct sum of two C^*-algebras to be unital is when A and B are unital, too. In that case, $(1, 1)$ is the unit. So we need to choose different kinds of operations, if we want to obtain some unitization given a non-unital C^*-algebra. As we are looking for a neutral element for the multiplication, we shall modify the multiplication accordingly.

Definition 2.19. Let A be a *-algebra. We define

$$\tilde{A} := \{(a, \lambda) \mid a \in A, \lambda \in \mathbb{C}\}$$

and we equip this set with the following operations, for $a, b \in A$, $\lambda, \mu \in \mathbb{C}$.

(i) $(a, \lambda) + (b, \mu) := (a + b, \lambda + \mu)$
(ii) $\mu(a, \lambda) := (\mu a, \mu\lambda)$
(iii) $(a, \lambda)(b, \mu) := (ab + \lambda b + \mu a, \lambda\mu)$
(iv) $(a, \lambda)^* := (a^*, \bar{\lambda})$

We also write $\lambda + a$ for $(a, \lambda) \in \tilde{A}$.

It is immediately clear that $\tilde{A}$ is a *-algebra again. The crucial point is: It is unital, the unit being $(0, 1)$. Observe that the notation $\lambda + x$ really makes sense, in particular it helps to memorize the multiplication law. It is easy to equip $\tilde{A}$ with a norm turning it into a Banach algebra (provided A is a Banach algebra): We may simply take $\|(a, \lambda)\|_{\mathrm{BA}} := \|a\| + |\lambda|$. Hence, the unitization of Banach algebras is not an issue. However, this norm does not satisfy the C^*-identity, so it does not turn $\tilde{A}$ into a C^*-algebra (even if A is a C^*-algebra). We need to be more creative in finding a C^*-norm on $\tilde{A}$.

Proposition 2.20. *Let A be a C^*-algebra. There exists a unique norm on $\tilde{A}$ turning $\tilde{A}$ into a unital C^*-algebra. We have $A \lhd \tilde{A}$ as an ideal, where A is identified with elements $(a, 0) \in \tilde{A}$ for $a \in A$.*

Proof. If a C^*-norm exists, it must be unique, by Cor. 2.15. So, let us show existence.

In the first case, let us assume that A is already unital. Then $A \oplus \mathbb{C}$ is a unital C^*-algebra by Lemma 2.18. The map $\tilde{A} \to A \oplus \mathbb{C}$, $(a, \lambda) \mapsto (\lambda 1 + a, \lambda)$ is a bijective map preserving the algebra operations such as multiplication, addition and involution and mapping $(0, 1)$ to $(1, 1)$. Thus, $\tilde{A}$ and $A \oplus \mathbb{C}$ are isomorphic as unital *-algebras and we may define a C^*-norm on $\tilde{A}$ simply by using the C^*-norm on $A \oplus \mathbb{C}$.

Now, let us assume that A is not unital. Consider

$$L : \tilde{A} \to B(A) := \{T : A \to A \mid T \text{ is linear and bounded}\}, \qquad x \mapsto L_x,$$

where $L_x(b) := xb$ for $b \in A$ and $x \in \tilde{A}$. This is a left multiplication operator on A: Given $x = \lambda + a$, observe that $xb = \lambda b + ab \in A$. Moreover, L_x is linear and bounded:

$$\|L_x(b)\| = \|xb\| = \|\lambda b + ab\| \leq (|\lambda| + \|a\|)\,\|b\|$$

This implies $\|L_x\| \leq |\lambda| + \|a\|$. We conclude that $L_x \in B(A)$ holds.

Now comes the crucial step: We define the norm on $\tilde{A}$ via the operator norm of L_x by the following.

$$\|x\|_{\tilde{A}} := \|L_x\| = \sup\{\|xb\|_A \mid b \in A, \|b\|_A \leq 1\}$$

Let us now check a couple of properties of $\|\cdot\|_{\tilde{A}}$.

Firstly, we have $\|a\|_{\tilde{A}} = \|a\|_A$ for all $a \in A$. Indeed, we have:

$$\|a\|_A \|a^*\|_A = \|a^*\|_A^2 = \|aa^*\|_A = \|L_a(a^*)\| \leq \|L_a\| \|a^*\|_A$$

This implies $\|a\|_A \leq \|L_a\|$. On the other hand, we have $\|a\|_A \geq \|L_a\|$, since $\|L_a(b)\|_A = \|ab\|_A \leq \|a\|_A \|b\|_A$ for all $b \in A$ with $\|b\|_A \leq 1$. Hence, $\|a\|_A = \|a\|_{\tilde{A}}$.

Secondly, $\|\cdot\|_{\tilde{A}}$ is a norm on $\tilde{A}$. For proving it, the only non-trivial step is that $\|x\|_{\tilde{A}} = 0$ implies $x = 0$. We prove it by contraposition. Let $x = \lambda + a \in \tilde{A}$ with $x \neq 0$. We may assume $\lambda \neq 0$ – otherwise $x \in A$ and then $\|x\|_{\tilde{A}} = \|x\|_A \neq 0$, by the first step above. Let us assume $\|L_x\| = 0$. Then $\lambda b + ab = xb = 0$ for all $b \in A$. But this shows that $e := -\frac{a}{\lambda}$ is a left unit, as $eb = b$ for all $b \in A$. Now, e^* is a right unit, since $be^* = (eb^*)^* = b$ for all $b \in A$. As, $e = ee^* = e^*$, we infer that e is a unit. This contradicts the assumption that A is not unital and we conclude $\|L_x\| \neq 0$.

Thirdly, $\|\cdot\|_{\tilde{A}}$ is submultiplicative. This follows immediately from $L_{xy} = L_x L_y$ and the submultiplicativity of the operator norm:

$$\|xy\|_{\tilde{A}} = \|L_{xy}\| = \|L_x L_y\| \leq \|L_x\| \|L_y\|$$

Fourthly, $\|\cdot\|_{\tilde{A}}$ satisfies the C^*-identity: Let $x \in \tilde{A}$ with $x \neq 0$. Let $\varepsilon > 0$ with $\|L_x\| > \varepsilon$. By the definition of the operator norm, there is some $b \in A$ with $\|b\| \leq 1$ and $\|xb\|_A \geq \|L_x\| - \varepsilon$. Hence:

$$(\|L_x\| - \varepsilon)^2 \leq \|xb\|_A^2 = \|b^* x^* x b\|_A \leq \|b^*\|_A \|L_{x^*x}(b)\|_A \leq \|L_{x^*x}\|$$

As this holds true for all $\varepsilon > 0$, we deduce:

$$\|x\|_{\tilde{A}}^2 = \|L_x\|^2 \leq \|L_{x^*x}\| = \|x^*x\|_{\tilde{A}}$$

From Remark 2.2(c), we deduce that the C^*-identity holds.

Finally, $\tilde{A}$ is complete with respect to $\|\cdot\|_{\tilde{A}}$. For this, first note $B(A)$ is complete by general Banach space arguments (note that A is in particular a Banach space). Furthermore, $L(A) \subseteq B(A)$ is closed, since A is complete and $\|L_a\| = \|a\|_A$ for all $a \in A$. We write $L(\tilde{A}) = L(A) + \mathbb{C}1 \subseteq B(A)$ and we observe that $L(\tilde{A})$ is the sum of a closed subspace and a finite-dimensional one. By general arguments from topology [49, Thm. 1.42], this shows that $L(\tilde{A})$ is closed; hence $\tilde{A}$ is complete. □

Remark 2.21. The unitization $\tilde{A}$ of A is minimal in the following sense: Let B be a unital C^*-algebra and let $A \lhd B$, then there is a unital *-homomorphism $\varphi : \tilde{A} \to B$ with $\varphi(a) = a \in B$ for all $a \in A$, i.e. φ respects $A \lhd B$.

Remark 2.22. The main ingredient of the above unitization was the left multiplication operator $L_b : A \to A$, $a \mapsto ba$. One might wonder whether the right multiplication operator is useful, too, and indeed the unitization may also be performed with the right multiplication operator instead. However, using both of them jointly (or rather an abstraction of them) yields yet another unitization, the so called "maximal" one. As the unitization from Prop. 2.20 will be more important for us, we only want to mention that one may define a multiplier algebra $M(A)$ of A consisting of pairs (L, R) called double centralizers. These double centralizers are an abstraction of the left and right multiplication. One may show that the multiplier algebra $M(A)$ is a unital C^*-algebra and $A \lhd M(A)$. The unitization $M(A)$ of A is maximal in that sense: Let B be a unital C^*-algebra and let $A \lhd B$, then there is a unital *-homomorphism from B to $M(A)$ respecting $A \lhd M(A)$.

Let us come back to the examples of Exm. 2.3.

Example 2.23. Here are the unitizations of $\mathcal{K}(H)$ and $C_0(X)$.

(a) Let H be an infinite dimensional Hilbert space. The minimal unitization of the algebra of compact operators is $\widetilde{\mathcal{K}(H)} = C^*(\mathcal{K}(H), 1) \subsetneq B(H)$, the smallest C^*-subalgebra of $B(H)$ containing $\mathcal{K}(H)$ and $1 \in B(H)$. The maximal unitization (i.e. the multiplier algebra) is $M(\mathcal{K}(H)) = B(H)$.

(b) Let X be locally compact but not compact. The minimal unitization of the algebra of continuous functions on X is $\widetilde{C_0(X)} = C(\hat{X})$, where $\hat{X}$ is the one point compactification of X. The maximal unitization is $M(C_0(X)) = C_b(X) := \{f : X \to \mathbb{C} \mid f \text{ is continuous and bounded}\}$ which is isomorphic to $C(\beta X)$, where βX is the Stone-Čech compactification of X.

2.6. **Exercises.**

Exercise 2.1. Check that the algebraic structures of C^*-algebras are continuous, i.e. the addition, the multiplication, the involution and also the norm are continuous (see Rem. 2.2).

Exercise 2.2. Convince yourself that $C(X)$ is a unital, commutative C^*-algebra (see Exm. 2.3). Moreover, check that a continuous map $h : X \to Y$ between compact Hausdorff spaces induces a *-homomorphism $\alpha_h : C(Y) \to C(X)$ by $f \mapsto f \circ h$. If h is a homeomorphism, then α_h is even an isometric *-isomorphism.

Exercise 2.3. Check that the function $f(z) = z$ in $A(\mathbb{D})$ is selfadjoint and $\mathrm{sp}(f) = \overline{\mathbb{D}}$. Later, we will see that any selfadjoint element in a C^*-algebra has only real spectral values. Thus, $A(\mathbb{D})$ is a Banach *-algebra but no C^*-algebra (see Exm. 2.3).

Exercise 2.4. Consider the unilateral shift $S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$, $Se_n = e_{n+1}$ from Exc. 1.7).

(a) Show that $\lambda - S$ is invertible for $|\lambda| > 1$.

(b) Show that S has no eigenvalues, i.e. the point spectrum of S is empty.

(c) Show that any $\lambda \in \mathbb{C}$ with $|\lambda| < 1$ is an eigenvalue for S^*.

(d) Deduce $\mathrm{sp}(S) = \{\lambda \mid |\lambda| \leq 1\}$.

Exercise 2.5. A C^*-algebra is called *simple*, if it contains no proper closed ideals, i.e. for any closed ideal $I \lhd A$ we have $I = 0$ (shorthand for $I = \{0\}$) or $I = A$. Consider $A = M_N(\mathbb{C})$. By $E_{ij} \in M_N(\mathbb{C})$, $i, j = 1, \ldots, N$ we denote the matrix units, i.e. the i-j-th entry of E_{ij} is one, and zero otherwise.

(a) Let $I \lhd M_N(\mathbb{C})$ be a (two-sided) ideal. Show that if I contains a matrix $T = (t_{ij})$ with $t_{i_0 j_0} \neq 0$ for some $i_0, j_0 \in \{1, \ldots, N\}$, then I contains $E_{i_0 j_0}$. Multiply T with appropriate matrix units in order to see this.

(b) Show that if I contains some matrix unit $E_{i_0 j_0}$, then it contains all matrix units $E_{ij} \in M_N(\mathbb{C})$, $i, j = 1, \ldots, N$. Deduce that I contains $1 = \sum_i E_{ii}$.

(c) Deduce that $M_N(\mathbb{C})$ is simple.

Exercise 2.6. Recall the fact used in the proof of Prop. 2.17: Given a Banach space A and a closed linear subspace $I \subseteq A$, show that A/I is a Banach space.

(a) Show that $\dot{x} + \dot{y}$ and $\lambda \dot{x}$ are well-defined.
(b) Show that $\|\dot{x}\|$ defines a norm and check $\|\dot{x}\| \leq \|x\|$.
(c) Show that A/I is complete with respect to this norm. Hence A/I is a Banach space.

2.7. **Some references on C^*-algebras.** A standard reference for the lectures on C^*-algebras could be the book by Blackadar [3]. Additionally, one could use the books by Murphy [34] or Pedersen [37]. The book by Davidson is a quite friendly approach, but slightly more based on examples [13]. Rather modern but more specialized on nuclearity or group actions is the book by Brown and Ozawa [8].

Historically, the books by Dixmier [16] and Sakai [50] are classic in the literature as well as the encyclopedic series of books by Kadison and Ringrose [28, 29, 26, 27] or by Takesaki [51, 52, 53].

The history of C^*-algebras goes back to the 1943 paper by Gelfand and Naimark [19], see also the 50 years celebration paper by Kadison [25].

As for the context of "quantum/noncommutative mathematics", see also the books by Gracia-Bondía, Várilly and Figueroa [23], the one by Wegge-Olsen [57] or the epic book by Connes [10].

3. Gelfand-Naimark Theorem and functional calculus

Abstract. We begin this lecture with recalling the Stone-Weierstrass (Approximation) Theorem. We then turn to homomorphisms of Banach and C^*-algebras and the spectrum Spec(A) of Banach algebras. We put a focus on the commutative case first: We investigate Spec(A) assuming that A is a commutative, unital Banach algebra. We learn that Spec(A) provides the full description of the maximal ideal space in this case. We define the Gelfand transform and verify that it is a continuous algebra homomorphism which respects the spectrum sp(x) of an element $x \in A$. If A is even a commutative, unital C^*-algebra, we may prove our First Fundamental Theorem of C^*-Algebras (aka Gelfand-Naimark Theorem): The commutative, unital C^*-algebras are exactly the algebras of continuous functions on compact spaces. As a consequence, we obtain a very powerful tool for C^*-algebras: The (continuous) functional calculus. We investigate some properties of this functional calculus.

3.1. **The Stone-Weierstrass Theorem.** We begin this lecture with a little excursus to classical analysis. Initially, Weierstrass asked 1895: Given a continuous function $f \in C([0,1])$ – is there a way to approximate f with respect to the supremum norm by "simpler" functions? The simplest functions we might have in mind are polynomials, and the answer is yes. This is the well-known Weierstrass Approximation Theorem. In 1948, Stone realized that very little of the particular structure of $C([0,1])$ was really needed in the proof and he extracted the main algebraic properties providing a way more general statement, which we now prepare.

Definition 3.1. Let X be a compact Hausdorff space. A *-subalgebra $A \subseteq C(X)$ *separates the points*, if for all $s, t \in X$, $s \neq t$ there is some $f \in A$ with $f(s) \neq f(t)$.

So point separation means that A has sufficiently many functions to "see" that s and t are distinct.

Example 3.2. If $X \subseteq \mathbb{C}$ is compact, then the set of all polynomials in x and its complex conjugate $\bar{x}$ is a unital *-subalgebra of $C(X)$ separating the points.

Recall that A is unital, if the constant function $1 \in C(X)$ is contained in A.

Theorem 3.3 (Stone-Weierstrass Theorem)**.** *Let X be a compact Hausdorff space and $A \subseteq C(X)$ a closed, unital *-subalgebra separating the points. Then $A = C(X)$.*

Proof. We need to show that if $f \in C(X)$, then $f \in A$. In order to do so, let us first show that A is closed under certain operations, the first one being the square root. So, assume that $f \in A$ with $0 \leq f(x) \leq 1$ for all $x \in X$. We put $g := 1 - f$ and we consider the Taylor series expansion of $\sqrt{1-z}$ around $z = 0$. Then

$$\sqrt{f(x)} = \sqrt{1 - g(x)} = 1 - \sum_{n=1}^{\infty} a_n g(x)^n$$

for all $x \in X$ with some coefficients $|a_n| < Cn^{-\frac{3}{2}}$ and some constant $C > 0$. The Taylor series of $\sqrt{1-z}$ converges uniformly on $[-1, 1]$. Thus, $h_m := 1 - \sum_{n=1}^{m} a_n g^n$

converges to $\sqrt{f}$ in norm. As A is closed, $\sqrt{f} \in A$; note that we may drop the assumption $f \leq 1$ by rescaling f. Next, observe that if $f, g \in A$ are real-valued, then $|f-g| = \sqrt{(f-g)^2} \in A$ and hence

$$\max(f,g) = \frac{f+g+|f-g|}{2} \in A \quad \text{and} \quad \min(f,g) = \frac{f+g-|f-g|}{2} \in A.$$

Now, let us begin with the proof of $C(X) \subseteq A$ and let $f \in C(X)$. We may assume that f is real-valued; otherwise we prove the theorem for the real and imaginary parts $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ first. Let $\varepsilon > 0$. We are looking for a function $g \in A$ with

$$f - \varepsilon < g < f + \varepsilon.$$

This will then show $\|f-g\|_\infty \leq \varepsilon$, i.e. f may be approximated by functions in A arbitrarily, and we are done: Since A is closed, $f \in A$. In order to find g, we first convince ourselves that given $s \in X$, there is a function $h_s \in A$ with

$$h_s < f + \varepsilon$$

and $h_s(s) = f(s)$. Indeed, for any $t \in X$ with $s \neq t$, we find a function $\alpha \in A$ with $\alpha(s) \neq \alpha(t)$, as A separates points. We put

$$f_{s,t}(x) := f(t) + (f(s) - f(t))\frac{\alpha(x) - \alpha(t)}{\alpha(s) - \alpha(t)}.$$

Then $f_{s,t} \in A$, $f_{s,t}(s) = f(s)$, $f_{s,t}(t) = f(t)$ and $U_t := \{x \in X \mid f_{s,t}(x) < f(x) + \varepsilon\}$ is open containing t. By compactness of X, we find finitely many $t_1, \ldots, t_m \in X$ such that $X \subseteq \cup_i U_{t_i}$ and we put $h_s := \min(f_{s,t_1}, \ldots, f_{s,t_m}) \in A$.

Finally, having found for all $s \in X$ functions $h_s \in A$ with $h_s < f + \varepsilon$ and $h_s(s) = f(s)$, we observe that the sets $V_s := \{x \in X \mid f(x) - \varepsilon < h_s(x)\}$ are open containing s. So, again by compactness of X, we find finitely many $V_{s_1}, \ldots, V_{s_n}$ covering X and we put $g := \max(h_{s_1}, \ldots, h_{s_n}) \in A$. $\square$

As a corollary, we obtain Weierstrass's Theorem originally formulated for real-valued functions $C_{\mathbb{R}}([0,1]) := \{f : [0,1] \to \mathbb{R} \text{ continuous}\}$.

Corollary 3.4 (Weierstrass Approximation Theorem)**.** *The set of all polynomials (in x) is dense in $C([0,1])$ and the same for $C_{\mathbb{R}}([0,1])$.*

Proof. The set $\mathcal{P}$ of all polynomials is a unital *-subalgebra of $C([0,1])$ separating the points; its closure is all of $C([0,1])$ by Thm. 3.3. Note that we actually also proved a real version of Thm. 3.3, so the result holds for $C_{\mathbb{R}}([0,1])$, too. $\square$

3.2. **Homomorphisms for Banach algebras and C^*-algebras.** We now want to specify the morphisms for our class of objects. Here they are in the case of Banach algebras and C^*-algebras.

Definition 3.5. Let A and B be Banach algebras.

(a) A map $\varphi : A \to B$ is an *(algebra) homomorphism*, if φ is linear and multiplicative (i.e. $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in A$).

(b) Assume that A and B are Banach *-algebras. A *-*homomorphism* is a homomorphism $\varphi : A \to B$ which is also *involutive*, i.e. $\varphi(x^*) = \varphi(x)^*$ holds for all $x \in A$.
(c) A homomorphism is *isometric*, if $\|\varphi(x)\| = \|x\|$ for all $x \in A$.
(d) Assume that A and B are unital. A homomorphism is *unital*, if $\varphi(1) = 1$.

The correct notion of a morphism between Banach algebras is a homomorphism in the sense of Def. 3.5 which is also continuous; for Banach *-algebras, it is a continuous *-homomorphism. Interestingly, we do not have to require continuity for C^*-algebras – it is automatic, as we will show next. Let us first prepare a technical lemma on unitizations. Recall that $\tilde{A}$ denotes the (minimal) unitization of A, see Prop. 2.20.

Lemma 3.6. *Let A, B be C^*-algebras and $\varphi : A \to B$ a *-homomorphism. Then $\tilde{\varphi} : \tilde{A} \to \tilde{B}$ defined as $\lambda + a \mapsto \lambda + \varphi(a)$ is a unital *-homomorphism.*

Proof. Straightforward. □

Definition 3.7 (Compare with Def. 2.4)**.** Let A be a C^*-algebra and let $x \in A$. We define the *spectrum* of x as

$$\mathrm{sp}(x) := \begin{cases} \mathrm{sp}_A(x) & \text{if } A \text{ is unital,} \\ \mathrm{sp}_{\tilde{A}}(x) & \text{if } A \text{ is non-unital.} \end{cases}$$

Lemma 3.8. *Let A and B be C^*-algebras and let $\varphi : A \to B$ be a *-homomorphism. We have:*

(a) *If A, B and φ are unital, we have that $\mathrm{sp}_B(\varphi(x))$ is contained in $\mathrm{sp}_A(x)$.*
(b) *$\|\varphi(x)\| \leq \|x\|$ for all $x \in A$.*
(c) *φ is continuous.*

Proof. For (a), assume that $\lambda - \varphi(x) = \varphi(\lambda - x) \in B$ is not invertible. Hence, $\lambda - x \in A$ cannot be invertible as φ maps invertible elements to invertible elements.

As for (b), assume first that A, B and φ are unital. We then have $r(\varphi(x)) \leq r(x)$ by (a). Thus, using also Cor. 2.14 we have:

$$\|\varphi(x)\|^2 = \|\varphi(x^*x)\| = r(\varphi(x^*x)) \leq r(x^*x) = \|x^*x\| = \|x\|^2$$

Hence, $\|\varphi(x)\| \leq \|x\|$.

Now, if A, B and φ are not necessarily unital, we know by (a) that $\|\tilde{\varphi}(x)\|_{\tilde{B}} \leq \|x\|_{\tilde{A}}$ holds for all $x \in \tilde{A}$. For $x \in A \subseteq \tilde{A}$, we infer $\|\varphi(x)\|_B = \|\tilde{\varphi}(x)\|_{\tilde{B}} \leq \|x\|_{\tilde{A}} = \|x\|_A$.

Item (c) is then a direct consequence of (b). □

Remark 3.9. Note that the above statement holds true even if A is just a Banach *-algebra whose involution is isometric. Indeed, in the proof of (b) we still have $r(\varphi(x^*x)) \leq r(x^*x)$ and we then use Rem. 2.11 and the submultiplicativity of the norm in order to deduce $r(x^*x) \leq \|x^*x\| \leq \|x\|^2$. With these modifications of the above proof, we infer the above statement (b) also in case A is just a Banach *-algebra.

This has an interesting consequence highlighting again the special role of C^*-algebras amongst Banach algebras: Let us consider a unital *-algebra A with two norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Assume that $(A, \|\cdot\|_1)$ is a Banach *-algebra with an isometric involution and assume that $(A, \|\cdot\|_2)$ is a C^*-algebra. Then, the identity map $\varphi : (A, \|\cdot\|_1) \to (A, \|\cdot\|_2)$ is norm decreasing, by the above discussion, i.e. $\|x\|_2 \leq \|x\|_1$. We may interpret this result in the sense that "the C^*-norm is the smallest Banach norm" (turning the involution into an isometric map).

It is good to know that *-homomorphisms between C^*-algebras are automatically continuous. They yield the correct notion of morphisms between C^*-algebras. In order to have a concept of isomorphism, we need isometric *-homomorphisms as they fully preserve the norm. Note that isometric *-homomorphisms are automatically injective – surprisingly, the converse is also true, as we will see later (Lecture 4): Any injective *-homomorphism between C^*-algebras is isometric! Thus, bijective *-homomorphisms are exactly isomorphisms of C^*-algebras.

3.3. **Spectrum of a Banach algebra.** Let us now turn to a special class of homomorphisms: those mapping to the complex numbers.

Definition 3.10. Let A be a Banach algebra. A *character* is a homomorphism $\varphi : A \to \mathbb{C}$ with $\varphi \neq 0$. The set $\mathrm{Spec}(A)$ of all characters is the *spectrum* of A.

We deduce some properties of a character directly from the definition.

Lemma 3.11. *Let A be a unital Banach algebra and let $\varphi \in \mathrm{Spec}(A)$.*

(a) *φ is unital ($\varphi(1) = 1$).*
(b) *If $x \in A$ is invertible, then $\varphi(x) \neq 0$.*
(c) *We have $\varphi(x) \in \mathrm{sp}(x)$ for all $x \in A$.*
(d) *φ is continuous and $\|\varphi\| \leq 1$.*
(e) *If A is a C^*-algebra and $x \in A$ is selfadjoint, then $\varphi(x) \in \mathbb{R}$.*
(f) *If A is a C^*-algebra, then φ is a *-homomorphism with $\|\varphi\| = 1$.*

Proof. For (a), let $x \in A$ with $\varphi(x) \neq 0$; it exists, since $\varphi \neq 0$. Then $\varphi(x) = \varphi(x1) = \varphi(x)\varphi(1)$. Hence $\varphi(1) = 1$.

Item (b) follows from the fact that homomorphisms map invertible elements to invertible elements.

Item (c) is a consequence of (a) and (b): We have $\varphi(\varphi(x)1 - x) = 0$ by (a) and hence $\varphi(x)1 - x$ cannot be invertible by (b).

Also, (d) is immediate: From (c) and Prop. 2.7, we infer $|\varphi(x)| \leq \|x\|$.

For (e), let $x = x^*$ and put $\varphi(x) = \alpha + i\beta \in \mathbb{C}$. Then $\varphi(x) + i\lambda = \varphi(x + i\lambda)$ by (a) and $|\varphi(x) + i\lambda| \leq \|x + i\lambda\|$ by (d), for all $\lambda \in \mathbb{R}$. Hence:

$$\alpha^2 + (\lambda+\beta)^2 = |\varphi(x)+i\lambda|^2 \leq \|x+i\lambda\|^2 = \|(x+i\lambda)^*(x+i\lambda)\| = \|x^2+\lambda^2\| \leq \|x\|^2+\lambda^2$$

Thus, $\alpha^2 + 2\lambda\beta + \beta^2 \leq \|x\|^2$, for all $\lambda \in \mathbb{R}$, which implies $\beta = 0$.

For (f), $\|\varphi\| = 1$ follows easily from $\|1\| = 1$ (see Rem. 2.2) and (a) and (d). For proving that φ is a *-homomorphism, let $x \in A$ and let $\varphi(x) = \alpha + i\beta$ and

$\varphi(x^*) = \gamma + i\delta$. The elements $x_1 := x + x^* \in A$ and $x_2 := i(x - x^*) \in A$ are selfadjoint. We thus have by (e)

$$(\alpha+\gamma) + i(\beta+\delta) = \varphi(x_1) \in \mathbb{R}, \qquad i(\alpha-\gamma) + (\delta-\beta) = \varphi(x_2) \in \mathbb{R},$$

which implies $\beta = -\delta$ and $\alpha = \gamma$ and hence $\varphi(x)^* = \alpha - i\beta = \varphi(x^*)$. □

We now show that $\mathrm{Spec}(A)$ is a nice topological space.

Proposition 3.12. *Let A be a unital Banach algebra. Equipped with the topology of pointwise convergence,* $\mathrm{Spec}(A)$ *becomes a compact Hausdorff space.*

Proof (idea): The proof is not difficult but rather technical. Let us sketch the main ingredients. Consider the pointwise convergence, i.e. a net (φ_λ) converges to φ if and only if $\varphi_\lambda(x) \to \varphi(x)$ for all $x \in A$. It is easy to check that if $\varphi_\lambda \in \mathrm{Spec}(A)$ for all λ, then also $\varphi \in \mathrm{Spec}(A)$. Hence $\mathrm{Spec}(A)$ is a closed subset of the closed unit ball $\{\varphi \in A' \mid \|\varphi\| \leq 1\}$ of the dual space A'. The unit ball in turn is a closed subset of the product $P := \Pi_{x\in A_1}\{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}$, where A_1 denotes the closed unit ball in A. Finally, Tychonoff's Theorem asserts that the cartesian product of compact spaces is compact, hence P is compact – thus, $\mathrm{Spec}(A) \subseteq P$ is compact. □

3.4. **Spectrum of a commutative, unital Banach algebra.** Assuming that A is commutative, $\mathrm{Spec}(A)$ contains the information of the maximal ideal space of A as we will see next. Recall the definition of maximal ideals from Def. 2.16.

Proposition 3.13. [1] *Let A be a commutative, unital Banach algebra. Then, the following assignment is bijective:*

$$\mathrm{Spec}(A) \to \{\text{maximal ideals in } A\}, \qquad \varphi \mapsto \ker\varphi$$

Proof. Before considering the assertion, let us prove that any maximal ideal in A can be written as the kernel of a character. In order to do so, we first show that A/I is a skew field, if $I \lhd A$ is a maximal ideal. So, let $\pi : A \to A/I$ be the quotient map, and $a \in A$ with $\pi(a) \neq 0$. We need to show that $\pi(a)$ is invertible. Put

$$J_a := \{ba + x \mid b \in A, x \in I\} \subseteq A.$$

Then J_a is a two-sided ideal in A, since for $(ba + x), (b'a + x') \in J_a$ and $c \in A$ we have $(ba + x) + (b'a + x') = (b + b')a + (x + x') \in J_a$, $c(ba + x) = (cb)a + cx \in J_a$ and $(ba + x)c = (cb)a + xc \in J_a$ (recall that A is commutative). Furthermore J_a contains I (putting $b = 0$) and $I \neq J_a$ (putting $b = 1$ and $x = 0$; note that $\pi(a) \neq 0$, i.e. $a \notin I$). By the maximality of I, we infer that $J_a = A$, thus there are $b \in A$ and $x \in I$ such that $1 = ba + x$. This shows that $\pi(a)$ is left invertible, since $\pi(b)\pi(a) = \pi(ba + x) = \pi(1) = 1$. By commutativity, it is also right invertible. Thus, $\pi(a)$ is invertible for all $a \in A$ which shows that A/I is a skew field. Morover, it is a Banach algebra by Prop. 2.17, so by the Gelfand-Mazur Theorem, 2.9, A/I is isomorphic to $\mathbb{C}$, i.e. the quotient map π is actually a character and I is its kernel.

[1] This has been Prop. 3.16 in an earlier version.

We now turn to the assingment of the assertion. We first convince ourselves that $\ker\varphi \lhd A$ is a maximal ideal, for all $\varphi \in \mathrm{Spec}(A)$. Given $\varphi \in \mathrm{Spec}(A)$, it is easy to see that $\ker\varphi \lhd A$ is an ideal. Also, $\varphi(1) = 1$ by Lemma 3.11, so $\ker\varphi \neq A$. By Prop. 2.17, we find a maximal Ideal $I \lhd A$ such that $\ker\varphi \subseteq I$. By the above considerations, we know that $I = \ker\psi$ for some character ψ. We thus have $\ker\varphi \subseteq \ker\psi$. But then $\varphi(a)1 - a \in \ker\varphi \subseteq \ker\psi$ for all $a \in A$, i.e. $0 = \psi(\varphi(a)1 - a) = \varphi(a) - \psi(a)$. Thus, $\varphi = \psi$, which shows that $\ker\varphi = I$ is a maximal ideal.

Finally, the assignment is bijective: given a maximal ideal $I \lhd A$, we just showed that it can be written as the kernel of a character; this proves surjectivity. As, for injectivity, let $\varphi, \psi \in \mathrm{Spec}(A)$ with $\ker(\varphi) = \ker(\psi)$. Then as above, $\varphi(a)1 - a \in \ker(\varphi) = \ker(\psi)$ for all $a \in A$, which implies $\varphi = \psi$. □

Let us harvest a little corollary: We relate the spectrum of a Banach algebra (Def. 3.10) with the spectrum of an element (Def. 2.4); this also justifies the usage of the same name for different objects.

Corollary 3.14. [2] *Let A be a commutative, unital Banach algebra and $x \in A$. Then:*

$$\mathrm{sp}(x) = \{\varphi(x) \mid \varphi \in \mathrm{Spec}(A)\}$$

Proof. By Lemma 3.11, $\varphi(x) \in \mathrm{sp}(x)$ for any $\varphi \in \mathrm{Spec}(A)$. Conversely, let $\lambda \in \mathrm{sp}(x)$. Then $I_\lambda := \{b(\lambda - x) \mid b \in A\} \lhd A$ is an ideal in A (two-sided, by commutativity of A) and $1 \notin I_\lambda$ (since $(\lambda - x)$ is not invertible). Hence, it is contained in a maximal ideal by Prop. 2.17, which means $I_\lambda \subseteq \ker\varphi$ for some $\varphi \in \mathrm{Spec}(A)$, by Prop. 3.13. Hence, $\varphi(\lambda - x) = 0$, i.e. $\lambda = \varphi(x)$. □

3.5. **Spectrum of a commutative C^*-algebra.** We investigate the case of commutative C^*-algebras in detail.

Lemma 3.15. *Let A be a commutative C^*-algebra. Then $\mathrm{Spec}(A)$ and $\mathrm{Spec}(\tilde{A})\backslash\{\tilde{0}\}$ are homeomorphic, where $\tilde{0} : \tilde{A} \to \mathbb{C}$ is given by $\lambda + a \mapsto \lambda$.*

Proof. We construct a map $\Psi : \mathrm{Spec}(\tilde{A})\backslash\{\tilde{0}\} \to \mathrm{Spec}(A)$, $\psi \mapsto \psi_{|A}$. Firstly, note that A is a maximal ideal in $\tilde{A}$. This is by construction, but we can also argue that $A = \ker\tilde{0}$ and use Prop. 3.13. Secondly, let $\psi \in \mathrm{Spec}(\tilde{A})$, i.e. $\psi : \tilde{A} \to \mathbb{C}$ is a homomorphism. Then, the restriction $\psi_{|A} : A \to \mathbb{C}$ is a homomorphism, too. If $\psi_{|A} = 0$, then $A \subseteq \ker\psi$. Moreover, $\ker\psi \neq \tilde{A}$ by definition of $\mathrm{Spec}(\tilde{A})$. As A is a maximal ideal, we infer $A = \ker\psi$, and hence $\psi = \tilde{0}$ as the assignment in Prop. 3.13 is bijective. Thus, $\psi \in \mathrm{Spec}(\tilde{A})\backslash\{\tilde{0}\}$ implies $\psi_{|A} \neq 0$ and hence $\psi_{|A} \in \mathrm{Spec}(A)$.

On the other hand, let $\varphi \in \mathrm{Spec}(A)$ and define $\varphi'(a + \lambda) := \varphi(a) + \lambda$ for $a \in A$ and $\lambda \in \mathbb{C}$. This defines a map $\Phi : \mathrm{Spec}(A) \to \mathrm{Spec}(\tilde{A})\backslash\{\tilde{0}\}$ which is inverse to Ψ. Moreover, Φ and Ψ are continuous, so they induce homeomorphisms between $\mathrm{Spec}(A)$ and $\mathrm{Spec}(\tilde{A})\backslash\{\tilde{0}\}$. □

[2] This has been Cor. 3.18 in an earlier version.

Proposition 3.16. *Let A be a commutative C^*-algebra. Then, its spectrum* $\mathrm{Spec}(A)$ *is locally compact. If A is unital, then* $\mathrm{Spec}(A)$ *is compact.*

Proof. By Prop. 3.12, $\mathrm{Spec}(\tilde{A})$ is compact. Hence, $\mathrm{Spec}(A)$ is locally compact by Lemma 3.15. The statement on the unital case is a special case of Prop. 3.12. □

Remark 3.17. One can define a unitization of a general Banach algebra A and show that it contains A as a maximal ideal. One can thus prove an analogue of Lemma 3.15 and Prop. 3.16 for commutative Banach algebras in general.

So, $\mathrm{Spec}(A)$ seems to be all we need, given A is a commutative C^*-algebra. Wait, let us do a quick check with a commutative C^*-algebra we know: Let X be a compact Hausdorff space and consider $A = C(X)$, the algebra of continuous functions on X. By Exm. 2.3 and Exc. 2.2, we know that $C(X)$ is a commutative, unital C^*-algebra.

Proposition 3.18. [3] *Let X be a compact Hausdorff space. Then* $\mathrm{Spec}(C(X))$ *is homeomorphic to X.*

Proof. The homeomorphism is given by $\Psi : X \to \mathrm{Spec}(C(X))$, $t \mapsto \mathrm{ev}_t$, with $\mathrm{ev}_t : C(X) \to \mathbb{C}$ defined as $\mathrm{ev}_t(f) := f(t)$ for $f \in C(X)$. The fact that it is a homeomorphism is left as Exc. 3.2 for the case of the metric space $X = ([0,1], d)$. For a general compact Hausdorff space X, the proof is analogous, but we need Urysohn's Lemma for showing that Ψ is injective. □

So, the spectrum contains all information about $C(X)$ and we may fully recover the C^*-algebra from its spectrum in that case, nice! As a caveat, note that the spectrum is less helpful, if A is *not* commutative: For instance, $\mathrm{Spec}(M_N(\mathbb{C})) = \emptyset$, as may be deduced easily from Exc. 2.5! So, the commutativity assumption is really crucial. The moral reason is: As $\varphi(xy) = \varphi(x)\varphi(y) = \varphi(y)\varphi(x) = \varphi(yx)$ for all $\varphi \in \mathrm{Spec}(A)$ of an *arbitrary* C^*-algebra A, the spectrum does not "see" any noncommutativity in A. Thus, if A is highly noncommutative, this particular information is not contained in $\mathrm{Spec}(A)$.

3.6. **Gelfand transform for commutative, unital Banach algebras.** We now define one of the most important tools for commutative, unital Banach algebras. Note that the spectrum of such algebras is always non-empty, by Cor. 3.14.

Definition 3.19. Let A be a commutative, unital Banach algebra. The *Gelfand transform* $\chi : A \to C(\mathrm{Spec}(A))$ is defined by $\chi(x) := \hat{x}$ and $\hat{x}(\varphi) := \varphi(x)$ for $\varphi \in \mathrm{Spec}(A)$.

Lemma 3.20. *The Gelfand transform is a continuous, unital algebra homomorphism with $\|\hat{x}\|_\infty \leq \|x\|$.*

Proof. First note that $\hat{x}$ is continuous with respect to the topology of pointwise convergence on $\mathrm{Spec}(A)$: If $\varphi_\lambda \to \varphi$, then in particular $\varphi_\lambda(x) \to \varphi(x)$. So, $\hat{x} \in$

[3] This has been Prop. 3.17 in an earlier version.

$C(\mathrm{Spec}(A))$. Checking $\widehat{\lambda x + \mu y} = \lambda\hat{x} + \mu\hat{y}$ and $\widehat{xy} = \hat{x}\hat{y}$ is straightforward, since characters are additive and multiplicative; here $x, y \in A$ and $\lambda, \mu \in \mathbb{C}$. By Lemma 3.11, we have $|\hat{x}(\varphi)| = |\varphi(x)| \leq \|x\|$, which implies $\|\hat{x}\|_\infty \leq \|x\|$. The Gelfand transform is unital by Lemma 3.11. □

Lemma 3.21. *Let A be a commutative, unital Banach algebra. Then $\|\hat{x}\|_\infty = r(x)$ and $\hat{x}(\mathrm{Spec}(A)) = \mathrm{sp}(x)$.*

Proof. By Cor. 3.14, $\hat{x}(\mathrm{Spec}(A)) = \mathrm{sp}(x)$. Thus, $r(x)$ is the supremum over all $|\hat{x}(\varphi)|$ for $\varphi \in \mathrm{Spec}(A)$. This is exactly the definition of the supremum norm $\|\hat{x}\|_\infty$. □

3.7. **Gelfand transform for commutative, unital C^*-algebras.** If A is in fact a commutative, unital C^**-algebra*, the Gelfand transform is even nicer – it provides us a very powerful isomorphism. Let us quickly prove a lemma first.

Lemma 3.22. *Let A and B be C^*-algebras and let $\varphi : A \to B$ be an isometric *-homomorphism. Then $\varphi(A) \subseteq B$ is a closed *-subalgebra; in particular $\varphi(A)$ is a C^*-algebra.*

Proof. Since φ is a *-homomorphism, it is clear that $\varphi(A) \subseteq B$ is a *-subalgebra. Moreover, $\varphi(A)$ is closed by a general fact for isometric maps: Let $(\varphi(x_n))_{n\in\mathbb{N}}$ be a Cauchy sequence in $\varphi(A)$. Then $\|x_n - x_m\| = \|\varphi(x_n) - \varphi(x_m)\|$, since φ is isometric. Hence, (x_n) is a Cauchy sequence in A and we find a limit $x_n \to x$ since A is complete. Then $\varphi(x_n) \to \varphi(x) \in \varphi(A)$, i.e. $\varphi(A)$ is complete and hence closed. □

We are now ready to prove the First Fundamental Theorem for C^*-Algebras. Note that this is not a very common name (we allow ourselves to use this in order to emphasize the role of this theorem) – it is more common to call it *(commutative) Gelfand-Naimark Theorem*. As both names of the mathematicians are Russian, you will also find Gel'fand or Neumark as alternative spellings.

Theorem 3.23 (1st Fundamental Theorem of C^*-Algebras, Gelfand-Naimark 1943)**.** *The Gelfand transform is an isometric *-isomorphism for commutative, unital C^*-algebras. Hence, we have the following equivalence given a unital C^*-algebra A:*

$$A \text{ is commutative} \quad \Longleftrightarrow \quad \exists X \text{ compact}: A \cong C(X)$$

The space X is then given by $\mathrm{Spec}(A)$*. In the non-unital case, we have $A \cong C_0(X)$ for some locally compact space X.*

Proof. Let A be a commutative, unital C^*-algebra. By Lemma 3.20, the Gelfand transform $\chi : A \to C(\mathrm{Spec}(A))$ is a unital algebra homomorphism. By Lemma 3.11, any character $\varphi \in \mathrm{Spec}(A)$ is a *-homomorphism. Hence,

$$\widehat{x^*}(\varphi) = \varphi(x^*) = \varphi(x)^* = (\hat{x}(\varphi))^*.$$

Thus, the Gelfand transform is a unital *-homomorphism. It is isometric (and hence also injective), since any element in A is normal, thanks to commutativity. Thus, $\|\hat{x}\|_\infty = r(x)$ by Lemma 3.21 and $r(x) = \|x\|$ by Cor. 2.14.

As for surjectivity, we employ the Stone-Weierstrass (Approximation) Theorem (Thm. 3.3). Note that $X := \mathrm{Spec}(A)$ is a compact Hausdorff space by Prop. 3.12. By Lemma 3.22, $B := \chi(A)$ is a closed *-subalgebra of $C(\mathrm{Spec}(A))$. It remains to show that B separates the points. Let $\varphi, \psi \in \mathrm{Spec}(A)$ be two points in our compact space with $\varphi \neq \psi$. We then find some $\hat{x} \in B$ distinguishing these two points, i.e. $\hat{x}(\varphi) = \varphi(x) \neq \psi(x) = \hat{x}(\psi)$. Hence, we verified all conditions of the Stone-Weierstrass Theorem, and we conclude $\chi(A) = B = C(\mathrm{Spec}(A))$, i.e. χ is surjective. This settles the unital case.

If A is non-unital, we restrict the isomorphism $\chi : \tilde{A} \to C(\mathrm{Spec}(\tilde{A}))$ to $A \subseteq \tilde{A}$ and we observe that for $x \in A \subseteq \tilde{A}$ we have $\hat{x}(\tilde{0}) = \tilde{0}(x) = 0$. We then conclude that under the Gelfand isomorphism, A is isomorphic to

$$\{f : \mathrm{Spec}(\tilde{A}) \to \mathbb{C} \mid f \text{ is continuous and } f(\tilde{0}) = 0\} \subseteq C(\mathrm{Spec}(\tilde{A})).$$

This in turn is isomorphic to $C_0(\mathrm{Spec}(A))$, see also Prop. 3.16. □

There is no way to overestimate the importance of the Gelfand-Naimark Theorem; we give a brief laudatio on this theorem at the end of this lecture, see Sect. 3.12.

Remark 3.24. The Gelfand-Naimark Theorem is not true for arbitrary Banach algebras. One can check that $\ell^1(\mathbb{Z})$ is a commutative, unital Banach algebra, the multiplication being the convolution and the unit being $(a_n)_{n\in\mathbb{Z}}$ with $a_n := \delta_{n0}$. One can show that $\mathrm{Spec}(\ell^1(\mathbb{Z})) = \mathbb{T}$ holds, where $\mathbb{T}$ is the closed unit circle in $\mathbb{C}$. Eventually, one can prove that the Gelfand transform maps $\ell^1(\mathbb{Z})$ to $C(\mathbb{T})$, and it is just the Fourier transform; and χ is no isomorphism here.

By the way, one can prove one of Wiener's theorems using the Gelfand transform: If $f \in C(\mathbb{T})$ has an absolute convergent Fourier series and $f(z) \neq 0$ for all $z \in \mathbb{T}$, then also $\frac{1}{f}$ has an absolute convergent Fourier series. For the proof, one needs to investigate the image of $\ell^1(\mathbb{Z})$ under the Gelfand transform. See [49, Lemma 11.6].

3.8. **C^*-subalgebras generated by subsets.** Let us prepare some applications of the Gelfand-Naimark Theorem. For doing so, we need to study C^*-subalgebras generated by subsets.

Definition 3.25. Let A be a C^*-algebra and let $M \subseteq A$ be a subset. We denote by $C^*(M)$ the smallest C^*-subalgebra of A containing M. If A is unital and $x \in A$, we denote by $C^*(x, 1)$ the smallest C^*-subalgebra of A containing x and 1.

Note that $C^*(M)$ is the intersection of all C^*-subalgebras $B \subseteq A$ with $M \subseteq B$. If $M = \{x_1, \ldots, x_n\}$ is finite, then $C^*(M)$ is given by the closure of all a priori noncommutative polynomials in x_i and x_i^*, $i = 1, \ldots, n$. More precisely, a *noncommutative *-monomial* in $x_1, \ldots, x_n$ is an expression

$$x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_m}^{k_m},$$

where $i_1, \ldots, i_m \in \{1, \ldots, n\}$ and $k_1, \ldots, k_m \in \{1, *\}$. In particular, note that x_1x_2 might differ from x_2x_1. A *noncommutative *-polynomial* is a linear combination of noncommutative *-monomials.

Now, $C^*(x,1)$ consists in limits of linear combinations of expressions of the form $x^{s_1}(x^*)^{s_2}x^{s_3}(x^*)^{s_4}\cdots x^{s_m}$ with $s_i \in \mathbb{N}_0$. So, we have:

$$C^*(x,1) = \overline{\{\text{noncommutative polynomials in } x \text{ and } x^*\}}$$

We are going to prove that $C^*(x,1)$ is particularly nice, if x is normal. We first prove a preparatory lemma.

Lemma 3.26. *Let A, B be C^*-algebras and let $M \subseteq A$ be a subset. Let $\varphi, \psi : A \to B$ be two *-homomorphisms. If $\varphi(x) = \psi(x)$ for all $x \in M$, then $\varphi(x) = \psi(x)$ for all $x \in C^*(M)$.*

Proof. Since φ and ψ are *-homomorphisms, $D := \{x \in A \mid \varphi(x) = \psi(x)\} \subseteq A$ is a C^*-subalgebra of A containing M. Thus $C^*(M) \subseteq D$. □

Thus, *-homomorphisms are uniquely determined on the generators of C^*-algebras. Let us now study $C^*(x,1)$ when x is normal.

Lemma 3.27. *Let A be a unital C^*-algebra and let $x \in A$ be normal.*
(a) *$C^*(x,1) \subseteq A$ is commutative.*
(b) *Let $y \in C^*(x,1)$ and $\lambda \in \mathbb{C}$. If $\lambda - y$ is invertible (in A), then its inverse belongs to $C^*(x,1)$. In particular,* $\mathrm{sp}_A(y) = \mathrm{sp}_{C^*(x,1)}(y)$.
(c) *The map $\hat{x} : \mathrm{Spec}(C^*(x,1)) \to \mathrm{sp}(x)$ mapping $\varphi \mapsto \varphi(x)$ is a homeomorphism, i.e.* $\mathrm{Spec}(C^*(x,1)) \cong \mathrm{sp}(x)$ *as topological spaces. If A is not necessarily unital,* $\mathrm{Spec}(C^*(x))$ *is homeomorphic to* $\mathrm{sp}(x)\backslash\{0\}$.

Proof. For (a), if x is normal, then the noncommutative monomials in x and x^* are of the form $x^k(x^*)^l$ for $k, l \in \mathbb{N}_0$ – and they actually commute. Since arbitrary elements in $C^*(x,1)$ are limits of linear combinations of such monomials, all elements in $C^*(x,1)$ commute.

As for (b), let $y \in C^*(x,1)$ and $\lambda \in \mathbb{C}$ and assume that $\lambda - y$ is invertible, i.e. we have $(\lambda - y)^{-1} \in A$. We need to show $(\lambda - y)^{-1} \in C^*(x,1)$. We do so by proving that $C^*(x,1)$ coincides with $B := C^*(\{x, (\lambda - y)^{-1}, 1\}) \subseteq A$.

Observe that $x(\lambda - y) = (\lambda - y)x$, since $C^*(x,1)$ is commutative by (a). We infer $x(\lambda - y)^{-1} = (\lambda - y)^{-1}x$. Thus, B is commutative and unital. By Thm. 3.23, the Gelfand transform $\chi : B \to C(\mathrm{Spec}(B))$ is an isomorphism, in particular, it is isometric. Thus, also the restriction to $C^*(x,1) \subseteq B$ is isometric and we infer that $\chi(C^*(x,1)) \subseteq C(\mathrm{Spec}(B))$ is a closed *-subalgebra by Lemma 3.22. Moreover, it separates the points: Let φ and ψ be in $\mathrm{Spec}(B)$ with $\varphi \neq \psi$. Assuming that they coincide on $C^*(x,1)$, we have $\varphi(\lambda - y) = \psi(\lambda - y)$ in particular. Hence, $\varphi((\lambda - y)^{-1}) = \psi((\lambda - y)^{-1})$, which implies that φ and ψ coincide on B, by Lemma 3.26, in contradiction to $\varphi \neq \psi$. We infer that φ and ψ differ on $C^*(x,1)$, which means that $\chi(C^*(x,1))$ separates the points.

By Stone-Weierstrass (Thm. 3.3), $\chi(C^*(x,1)) = C(\mathrm{Spec}(B))$. From surjectivity of χ we also have $\chi(B) = C(\mathrm{Spec}(B))$, which yields $C^*(x,1) = B$ by injectivity of χ. Hence, $(\lambda - y)^{-1} \in B = C^*(x,1)$.

In particular, if $\lambda \in \mathrm{sp}_{C^*(x,1)}(y)$, then $\lambda - y$ has no inverse in $C^*(x,1)$ nor in A. Thus, $\mathrm{sp}_{C^*(x,1)}(y) \subseteq \mathrm{sp}_A(y)$, the other inclusion being trivial.

As for (c), note that $\hat{x}$ is continuous by definition. It is surjective by Lemma 3.21 and injective by Lemma 3.26. The inverse map $\hat{x}^{-1}$ is continuous by general facts from topology: Given a closed subset $M \subseteq \mathrm{Spec}(C^*(x,1))$, the set M is compact and so is $\hat{x}(M)$. Hence, $(\hat{x}^{-1})^{-1}(M) = \hat{x}(M) \subseteq \mathrm{sp}(x)$ is closed and $\hat{x}^{-1}$ is continuous.

The non-unital case may be treated accordingly. □

3.9. **Functional calculus for continuous functions.** The preceding lemma looks pretty technical and kind of boring. But it is the key to one of the most powerful applications of the Gelfand-Naimark Theorem: the functional calculus. How comes? Well, if A is a noncommutative C^*-algebra, there is no chance to apply the Gelfand-Naimark Theorem. Really? From Lemma 3.27(a) we learn that $C^*(x,1) \subseteq A$ is commutative, if x is normal – so, we may apply Gelfand-Naimark at least *locally*! And we even learned that $\mathrm{Spec}\, C^*(x,1)$ is of a pretty nice form: It is $\mathrm{sp}(x) \subseteq \mathbb{C}$.

It seems to pay off to study such technicalities: we obtain a very useful tool.

Theorem 3.28 (Continuous functional calculus)**.** *Let A be a unital C^*-algebra and $x \in A$ be normal. There is an isometric *-isomorphism $\Phi : C(\mathrm{sp}(x)) \to C^*(x,1) \subseteq A$ mapping $\Phi(\mathrm{id}) = x$ and $\Phi(1) = 1$. We also write $f(x) := \Phi(f)$.*

If A is not unital, we have $\Phi : C_0(\mathrm{sp}(x)\backslash\{0\}) \to C^*(x) \subseteq A$ mapping $\Phi(\mathrm{id}) = x$.

Proof. By Lemma 3.27(c) and Exc. 2.2, $C(\mathrm{sp}(x))$ is isomorphic to $C(\mathrm{Spec}(C^*(x,1)))$ mapping $f \mapsto f \circ \hat{x}$. By the Gelfand-Naimark Theorem, Thm. 3.23, and Lemma 3.27, $C(\mathrm{Spec}(C^*(x,1)))$ is isomorphic to $C^*(x,1)$ mapping $\hat{x}$ to x. The *-isomorphism Φ (or equivalently Φ^{-1}) is unique by Lemma 3.26. The non-unital case is similar. □

Some quick comments: Firstly, note that this is the *continuous* functional calculus, since it allows us to apply continuous functions to normal elements in C^*-algebras. There are many other functional calculi such as the measurable functional calculus (related to von Neumann algebras), the holomorphic functional calculus (for general Banach algebras) and many others – there has even been an Internetseminar on this subject, ISem21![4]

How about a polynomial functional calculus? Well, that is trivial. Of course, we are allowed to apply polynomials to elements in A: it is an algebra! This also explains the notation $f(x) = \Phi(f)$: For polynomials f, the element $\Phi(f) \in A$ is obtained exactly by plugging x into the polynomial f. So, the continuous functional calculus is basically the evaluation homomorphism extended from polynomials to continuous functions. And if you trace back the proof of Thm. 3.28 and Thm. 3.23, you will see where this extension comes from: From the Stone-Weierstrass Theorem, Thm. 3.3, our core theorem of such extensions. It all adds up – nice, isn't it?

Back to concrete math. Here are some properties of the functional calculus.

[4]ISem21, Functional Calculus, virtual lectures by Markus Haase (Kiel), `https://www.math.uni-kiel.de/isem21/en/course`

Proposition 3.29. *Let A be a C^*-algebra, $x \in A$ be normal, $f, g \in C(\mathrm{sp}(x))$. Then:*

(a) $(f+g)(x) = f(x) + g(x)$, $(fg)(x) = f(x)g(x)$, $\bar{f}(x) = f(x)^*$. *In particular, $f(x)$ is selfadjoint, if f is real-valued.*
(b) $\mathrm{sp}(f(x)) = f(\mathrm{sp}(x))$.
(c) *For $h \in C(f(\mathrm{sp}(x))$, we have $(h \circ f)(x) = h(f(x))$.*
(d) *If $x \in A$ is selfadjoint, then $\mathrm{sp}(x) \subseteq \mathbb{R}$ and we may decompose $x = x_+ - x_-$ with $x_+x_- = x_-x_+ = 0$ and $\mathrm{sp}(x_+), \mathrm{sp}(x_-) \subseteq [0,\infty)$ and $\|x_+\|, \|x_-\| \leq \|x\|$.*
(e) *Let A, B be unital and let $\varphi : A \to B$ be a unital *-homomorphism. Then $\varphi(f(x)) = f(\varphi(x))$.*

Proof. Item (a) follows directly from the fact that Φ from Thm. 3.28 is a *-homomorphism. Items (b) and (c) are left as Exc. 3.3. As for (d), note that $\Phi(\bar{\mathrm{id}}) = \Phi(\mathrm{id})^* = x^* = x = \Phi(\mathrm{id})$. So $\bar{\mathrm{id}} = \mathrm{id}$, i.e. $\mathrm{sp}(x) \subseteq \mathbb{R}$. Put:

$$h_+(t) := \begin{cases} t & t \geq 0 \\ 0 & \text{otherwise} \end{cases}, \qquad h_-(t) := \begin{cases} -t & t \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

Then h_+ and h_- are continuous functions on $\mathrm{sp}(x)$ and we may put $x_+ := h_+(x)$, $x_- := h_-(x)$ by functional calculus. Then $\mathrm{id} = h_+ - h_-$ and everything is transferred via Φ making use of (a) and (b).

Item (e) is clear for polynomials, as φ is a *-homomorphism. Let p_n be polynomials approximating f by Stone-Weierstrass. Then $p_n(x) = \Phi(p_n) \to \Phi(f) = f(x)$ and we apply φ. Note that $\mathrm{sp}(\varphi(x)) \subseteq \mathrm{sp}(x)$, thus $f(\varphi(x))$ exists. $\square$

3.10. **An application of the functional calculus.** Let us end this lecture with an application, further ones will be treated for instance in the next lecture. Recall the definition of unitary elements from Def. 1.33.

Proposition 3.30. *Let A be a unital C^*-algebra and let $u \in A$ be unitary.*

(a) *We have* $\mathrm{sp}(u) \subseteq S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$.
(b) *If $\mathrm{sp}(u) \neq S^1$, then there is a selfadjoint element $x \in A$ such that $u = e^{ix}$, i.e. u can be written in polar coordinates.*

Proof. Item (a) is part of Exc. 3.4. As for (b), let $\lambda_0 \in S^1$ with $\lambda_0 \notin \mathrm{sp}(u)$. Let f be a branch of the logarithm mapping $z \in S^1$ to $\vartheta \in \mathbb{R}$ with $e^{i\vartheta} = z$ such that f is continuous on $\mathrm{sp}(u)$. Recall that the logarithm on all of S^1 is *not* continuous, since it needs to jump at some point. But as $\mathrm{sp}(u)$ is an honest subset of S^1, we may choose a branch of the logarithm avoiding this jump and we obtain a continuous function f on $\mathrm{sp}(u)$. This means, we are allowed to apply the functional calculus and we put $x := f(u) \in A$. The element x is selfadjoint, by Prop. 3.29(a). Since $\mathrm{id} = e^{if}$ as functions on $\mathrm{sp}(u)$, we have $u = e^{ix}$. $\square$

3.11. **Exercises.**

Exercise 3.1. Denote the circle by $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$. Show that we may approximate functions in $C(S^1)$ by polynomials $p(z) = \sum_{n=-N}^{N} a_n z^n$, where $a_n \in \mathbb{C}$ and $z^{-n} = (\bar{z})^n$ for $n > 0$.

One can use this result to show that the functions $e_n(t) := \frac{1}{\sqrt{2\pi}} e^{int}$, $t \in [0, 2\pi]$, $n \in \mathbb{Z}$ form an orthonormal basis for $L^2([0, 2\pi])$. This also shows that $L^2([0, 2\pi])$ is isomorphic to $\ell^2(\mathbb{N})$ as a Hilbert space.

Exercise 3.2. Consider the metric space $([0, 1], d)$ with the usual metric $d(s, t) = |s - t|$. Then $C([0, 1])$ is a commutative, unital Banach algebra (in fact, even a C^*-algebra) by Exm. 2.3 and Exc. 2.2. We want to compute its spectrum. For $t \in [0, 1]$, define $\mathrm{ev}_t : C([0, 1]) \to \mathbb{C}$ via $\mathrm{ev}_t(f) := f(t)$ for $f \in C([0, 1])$. Consider $\Psi : [0, 1] \to \mathrm{Spec}(C([0, 1]))$, $t \mapsto \mathrm{ev}_t$.

(a) Show that ev_t is in $\mathrm{Spec}(C([0, 1]))$ for all $t \in [0, 1]$.
(b) Show that Ψ is injective. Use $f_s(y) := d(s, y) = |s - y|$.
(c) Let $I \lhd C([0, 1])$ be an ideal such that for any $t \in [0, 1]$ there is some $f \in I$ with $f(t) \neq 0$. Show that $I = C([0, 1])$. In order to do so, use compactness of $[0, 1]$ to find an invertible element in I; then use Prop. 2.17(c).
(d) Show that Ψ is surjective: Given $\varphi \in \mathrm{Spec}\, C([0, 1])$, use (c) and Prop. 3.13 in order to show that $\ker \varphi = \ker \mathrm{ev}_t$ for some $t \in [0, 1]$. Observe that $\ker \mathrm{ev}_t = \{f \in C([0, 1]) \mid f(t) = 0\}$, for $t \in [0, 1]$.
(e) Show that Ψ is continuous. And deduce that Ψ^{-1} is also continuous, by some general topological argument. Deduce that $[0, 1]$ and $\mathrm{Spec}(C([0, 1]))$ are homeomorphic (i.e. "the same" as topological spaces).

Exercise 3.3. Prove items (b) and (c) of Prop. 3.29. For (c), study

$$A := \{h \in C(\mathrm{sp}(f(x))) \mid \Phi_1(h \circ f) = \Phi_2(h)\} \subseteq C(\mathrm{sp}(f(x))),$$

where Φ_1 and Φ_2 are suitable functional calculi.

Exercise 3.4. Let A be a unital C^*-algebra.

(a) Show that if $x \in A$ is invertible, then $\mathrm{sp}(x^{-1}) = \{\lambda^{-1} \mid \lambda \in \mathrm{sp}(x)\}$.
(b) Show that if $u \in A$ is unitary in the sense of Def. 1.33, then $\|u\| = \|u^*\| = 1$. Use (a) to deduce that $\mathrm{sp}(u) \subseteq S^1$.

Exercise 3.5. Let A be a unital C^*-algebra and let $x \in A$ be selfadjoint. The following are easy statements needed in the next lecture.

(a) Recall that x is invertible if and only if $0 \notin \mathrm{sp}(x)$.
(b) Let x be invertible. Show that x^{-1} is selfadjoint.
(c) Let $\mathrm{sp}(x) \subseteq (0, \infty)$. Use the functional calculus to show $\mathrm{sp}(x^{-1}) \subseteq (0, \infty)$.
(d) Show that if $f, g \in C(\mathrm{sp}(x))$, then $f(x)$ and $g(x)$ commute; in particular, $f(x)$ and x commute.
(e) Show that $\mathrm{sp}(x - 1) \subseteq [0, \infty)$ if and only if $\mathrm{sp}(x) \subseteq [1, \infty)$.
(f) Show that if $\mathrm{sp}(x) \subseteq [1, \infty)$, then x is invertible and $\mathrm{sp}(1 - x^{-1}) \subseteq [0, \infty)$.

3.12. **Some comments on Gelfand duality.** Let us elaborate more on the philosophical impact of the Gelfand-Naimark Theorem, see also [55] for some easy account. What does Gelfand-Naimark say exactly? A unital C^*-algebra is commutative if and only if it is an algebra of continuous functions on a compact space. So, what does it say philosophically? It says that compact topological spaces are in "duality" with commutative, unital C^*-algebras. What is this famous "Gelfand duality" about?

Let us speak in the language of category theory first. Recall from Exc. 2.2 that given a continuous map $h : X \to Y$ between compact Hausdorff spaces, we obtain a *-homomorphism $\alpha_h : C(Y) \to C(X)$ by composition with h: We map $f \mapsto f \circ h$. So, a morphism on the level of topological space induces a morphism on the level of C^*-algebras. One can push this further and show that there is an equivalence of categories between the category of commutative, unital C^*-algebras and the category of compact topological spaces, induced by the functors $A \mapsto \mathrm{Spec}(A)$ on the one hand and $X \mapsto C(X)$ on the other. See [1] for more on this.

Now, in a more philosophical language, we may say that commutative C^*-algebras correspond to (classical) topology – while noncommutative C^*-algebras correspond to a kind of "noncommutative (or quantum) topology". Indeed, Gelfand-Naimark tells us, that within the class of all (possibly noncommutative) C^*-algebras, the commutative ones are exactly those coming from classical topology. Hence, the others, noncommutative ones must correspond to some noncommutative topology. Therefore, we sometimes view noncommutative C^*-algebras as algebras of functions on some "noncommutative spaces". And it makes sense to do so! As mathematicians, we need precise objects, precise definitions and precise theorems – but we also need some intuition! So, intuitively, we may want to think of noncommutative C^*-algebras as function algebras on some underlying noncommutative spaces – which do not "exist" in a precise way, but indirectly, via their function algebras.

Why is this way of thinking useful? Because it allows us to transfer concepts, ideas and possibly even techniques from the "classical" world to the "noncommutative" one. If you want to learn more about such a noncommutative topology, you may take a look at the nice noncommutative topology dictionaries in [23, Introduction to Ch. 1 + end of Sect. 1.3], [57, Sect. 1.11] revealing the dual concepts to connected components, closed subsets, compactifications etc .within noncommutative topology aka the theory of C^*-algebras.

Moreover, this Gelfand-Naimark philosophy (commutativity corresponds to the classical world, noncommutativity to the quantum world) is the basis also for other quantum theories: Murray-von Neumann's von Neumann algebras, Voiculescu's Free Probability Theory, Connes's Noncommutative Geometry and Woronowicz's Quantum Groups – they all share the same philosophy about the role of commutativity as a classical counterpart. Nowadays, the following areas may be counted to such a "non-commutative analysis" or "quantum mathematics":

Classical theory	Quantum/noncomm. version	Founders and pioneers
Topology	C^*-Algebras	Gelfand-Naimark 1940s
Measure Theory	von Neumann Algebras	Murray-vonNeumann 1930s
Probability Theory	Free Probability Theory,	Voiculescu 1980s
	Quantum Probability Theory	Accardi,
		Hudson-Parthasarathy 1970s
Differential Geometry	Noncommutative Geometry	Connes 1980s
(Compact) Groups	(Compact) Quantum Groups	Woronowicz 1980s
Information Theory	Quantum Information Theory	Feynman, Deutsch 1980s
Complex Analysis	Free Analysis	J. L. Taylor 1970s

In many of the above theories, there is an analog of the Gelfand-Naimark Theorem. We conclude that quantum/noncommutative mathematics follows the philosophy:

$$\begin{array}{rcl} \text{commutative algebras} & \longleftrightarrow & \text{classical situation} \\ \text{noncommutative algebras} & \longleftrightarrow & \text{quantum/noncommutative situation} \end{array}$$

By the way, the name "C^*-algebra" has been coined by Segal in 1947, the letter "C" referring to "closed" *-subalgebras of $B(H)$ as a major example of C^*-algebras, see Exm. 1.31. Gelfand and Naimark themselves used the term "normed *-ring" in their 1943 article. For more on the history of C^*-algebras and the Gelfand-Naimark Theorem, see [25].

4. Positive elements, approximate units

Abstract. In this lecture, we turn to one of the key features of C^*-algebras: positivity. We define the notion of positive elements in a C^*-algebra and we prove the very important algebraic characterization that positive elements are exactly elements of the form x^*x. We show that the set of positive elements of a C^*-algebra forms a cone and we derive a number of useful observations on the induced order structure. From positivity, the amazing fact may be deduced that any injective *-homomorphism is already isometric. Besides, we observe that every positive element has a unique positive square root.

We then introduce the concept of approximate units and we show that any C^*-algebra and any ideal in a C^*-algebra possesses an approximate unit. This allows us to conclude that the quotient of a C^*-algebra by a closed two-sided ideal is a C^*-algebra again. Thus, the theory of C^*-algebras is admissible for homological tools such as short exact sequences.

From now on, we leave the general framework of Banach algebras and in the remaining lectures we will deal with the particular subclass of C^*-algebras only.

4.1. **Definition of positive elements and sums of positive elements.** In the first lecture, we briefly mentioned the importance of the C^*-identity

$$\|x\|^2 = \|x^*x\|$$

for C^*-algebras. Let us now explore in detail some powerful consequences of this harmless identity. The main aspect is that it introduces positivity to C^*-algebras.

Definition 4.1. Let A be a C^*-algebra. An element $x \in A$ is *positive* (we write $x \geq 0$), if $x = x^*$ and $\mathrm{sp}(x) \subseteq [0, \infty)$.

Recall from Prop. 3.29, that $\mathrm{sp}(x) \subseteq \mathbb{R}$ whenenver $x = x^*$. So, positive elements form a natural subclass of selfadjoint elements – their spectrum is positive! Note that we already encountered positivity in Prop. 3.29: We saw that any selfadjoint element may be decomposed as $x = x_+ - x_-$ with x_+ and x_- being positive. So, if x is positive itself, then $x = x_+$ in the sense of Prop. 3.29. Moreover, *-homomorphisms respect positivity, by Lemma 3.8.

Let us now characterize positivity via some norm estimate. This is basically the functional calculus point of view on positivity.

Lemma 4.2. *Let A be a unital C^*-algebra and let $x \in A$ be selfadjoint. Let $\lambda \geq \|x\|$. We have:*

$$x \geq 0 \qquad \Longleftrightarrow \qquad \|\lambda 1 - x\| \leq \lambda$$

In particular, if $x \geq 0$, then $\|x\| = \inf\{\lambda \geq 0 \mid \lambda 1 - x \geq 0\}$.

Proof. Let x be selfadjoint and $\lambda \geq \|x\|$. Let us first work in $C(\mathrm{sp}(x))$. Let id be the identity function and 1 the constant function both on $\mathrm{sp}(x) \subseteq \mathbb{R}$. From Prop. 2.7, we infer that $\mathrm{sp}(x) \subseteq [-\lambda, \lambda]$. We observe that $\mathrm{sp}(x) \subseteq [0, \infty)$ holds if and only if $\|\lambda 1 - \mathrm{id}\|_\infty = \sup\{|\lambda - \mu| \mid \mu \in \mathrm{sp}(x)\} \leq \lambda$, as a statement on functions in $C(\mathrm{sp}(x))$.

Coming back to $x \in A$, we employ the functional calculus. As it is isometric, we conclude that x is positive if and only if $\|\lambda 1 - x\| = \|\lambda 1 - \mathrm{id}\|_\infty \leq \lambda$.

If now $x \geq 0$ – i.e. $\mathrm{sp}(x) \subseteq [0, \infty)$ – then $\lambda 1 - \mathrm{id} \geq 0$ as a function on $\mathrm{sp}(x)$ if and only if $\lambda \geq \|\mathrm{id}\|_\infty$. Thus, $\|\mathrm{id}\|_\infty = \inf\{\lambda \geq 0 \mid \lambda 1 - \mathrm{id} \geq 0\}$. Now, by Prop. 3.29, $\lambda 1 - \mathrm{id} \geq 0$ if and only if $\lambda 1 - x \geq 0$ and we infer $\|x\| = \inf\{\lambda \geq 0 \mid \lambda 1 - x \geq 0\}$. $\square$

As an easy consequence, we see that sums of positive elements are positive.

Lemma 4.3. *Let A be a C^*-algebra and $x, y \in A$ be positive. Then $x + y \geq 0$.*

Proof. Assume first that A is unital. Put $\lambda := \|x\| + \|y\|$. By the triangle inequality, $\lambda \geq \|x + y\|$. By Lemma 4.2, $\|\|x\| - x\| \leq \|x\|$ and $\|\|y\| - y\| \leq \|y\|$. Then

$$\|\lambda - (x + y)\| \leq \|\|x\| - x\| + \|\|y\| - y\| \leq \|x\| + \|y\| = \lambda$$

and we apply Lemma 4.2 again.

If A is not unital, we view x and y as elements in $\tilde{A}$. Then, they are also positive in $\tilde{A}$, see also Def. 3.7, and we conclude that $x + y \in A \subseteq \tilde{A}$ is positive. $\square$

4.2. **Positive square root.** While the functional calculus was the key to the above characterization of positivity, it also allows us to write expressions such as

$$\sqrt{T} = \sqrt{\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}} \qquad \text{for} \qquad T = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} \in M_2(\mathbb{C}).$$

What exactly do we mean by it? Let us clarify it in the next proposition.

Proposition 4.4. *Let A be a C^*-algebra and $x \in A$ be positive. Then there is a unique positive element $y \in A$ such that $y^2 = x$, i.e. any positive element x possesses a unique positive square root $\sqrt{x}$.*

Proof. The existence is by functional calculus: Since $\sqrt{\cdot}$ is continuous on $[0, \infty)$, we are allowed to put $y := \sqrt{x}$ by functional calculus and we check $y = y^*$ and $\mathrm{sp}(y) = \sqrt{\mathrm{sp}(x)} \subseteq [0, \infty)$ and $y^2 = x$ using Prop. 3.29.

As for uniqueness, assume first that A is unital. Let $\tilde{y} \in A$ be positive with $\tilde{y}^2 = x$. If $\tilde{y} \in C^*(x, 1)$, then we may find a positive function $\tilde{f} \in C(\mathrm{sp}(x))$ with $\tilde{f}(x) = \tilde{y}$ by the Gelfand-Naimark Thm. But then $\tilde{f}^2 = \mathrm{id}$, i.e. $\tilde{f} = \sqrt{\cdot}$ and $\tilde{y} = y$. We are left with proving that for general $\tilde{y} \in A$, we always have $\tilde{y} \in C^*(x, 1)$.

It is clear that $C^*(x, 1) \subseteq C^*(\tilde{y}, 1)$ as $x = \tilde{y}^2$. Now, $\tilde{y}$ is normal and we have a functional calculus $\tilde{\Phi} : C(\mathrm{sp}(\tilde{y})) \to C^*(\tilde{y}, 1)$. Then, $\tilde{\Phi}^{-1}(C^*(x, 1)) \subseteq C(\mathrm{sp}(\tilde{y}))$ is a unital closed *-subalgebra by Lemma 3.22. Let us show that $\tilde{\Phi}^{-1}(C^*(x, 1))$ separates the points. Recall $\mathrm{sp}(\tilde{y}) \cong \mathrm{Spec}(C^*(\tilde{y}, 1))$ from Lemma 3.27. So, let $\varphi_1, \varphi_2 \in \mathrm{Spec}(C^*(\tilde{y}, 1))$ with $\varphi_1 \neq \varphi_2$. Then $\varphi_1(\tilde{y}) \neq \varphi_2(\tilde{y})$ by Lemma 3.26. This implies $\varphi_1(x) \neq \varphi_2(x)$ since $\tilde{y}^2 = x$. We conclude that $\tilde{\Phi}^{-1}(C^*(x, 1))$ separates the points. Thus, $\tilde{\Phi}^{-1}(C^*(x, 1)) = C(\mathrm{sp}(\tilde{y})) = \tilde{\Phi}^{-1}(C^*(\tilde{y}, 1))$ by Stone-Weierstrass (Thm. 3.3) and hence $C^*(x, 1) = C^*(\tilde{y}, 1)$ which shows $\tilde{y} \in C^*(x, 1)$.

Now, assume that A is not unital. We then consider $x \in A \subseteq \tilde{A}$ and we just showed, that x has a unique positive square root in $\tilde{A}$ – so, there cannot be another positive square root in $A \subseteq \tilde{A}$. $\square$

Coming back to our above example, our knowledge from linear algebra allows us to compute the eigenvalues (i.e. the spectrum) of the given 2×2 matrix T and we deduce that it is positive. We then have by Prop. 4.4

$$\sqrt{T} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \in M_2(\mathbb{C}).$$

4.3. **Algebraic characterization of positivity.** We now turn to the most important subsection of Lecture 4: The algebraic characterization of positivity. We want to show that positive elements are exactly those of the form x^*x.

As a motivation, we interpret Exc. 4.1 as a hint. There, we show that an operator $T \in B(H)$ is positive if and only if $\langle Tx, x\rangle \geq 0$ for all $x \in H$. As a consequence, any operator $T = S^*S \in B(H)$ is positive. The same holds true in arbitrary C^*-algebras, as we are going to show soon. Let us prepare some technical tools.

Lemma 4.5. *Let A be a C^*-algebra and let $x, y \in A$. Then*

$$\mathrm{sp}(xy) \cup \{0\} = \mathrm{sp}(yx) \cup \{0\}.$$

Proof. The proof is left as Exc. 4.3. $\square$

Definition 4.6. Let A be a C^*-algebra and let $x \in A$. The *real part* $\mathrm{Re}(x)$ and the *imaginary part* $\mathrm{Im}(x)$ of x are defined as follows:

$$\mathrm{Re}(x) := \frac{x + x^*}{2}, \qquad \mathrm{Im}(x) := \frac{x - x^*}{2i}$$

We may then decompose $x = \mathrm{Re}(x) + i\mathrm{Im}(x)$.

Note that the real and imaginary parts are defined in clear analogy to the situation in $\mathbb{C}$. The whole point of the above decomposition of x is that $\mathrm{Re}(x)$ and $\mathrm{Im}(x)$ are selfadjoint. We learned already that selfadjoint (or weaker: normal) elements behave particularly nice – for instance, they are admissible for functional calculus!

Lemma 4.7. *Let A be a C^*-algebra and $x \in A$. If $-x^*x \geq 0$, then $x = 0$.*

Proof. We may assume that A is unital; otherwise, we consider $x \in \tilde{A}$. We decompose $x = x_1 + ix_2$ in its real and imaginary part as in Def. 4.6. We then have $x^* = x_1 - ix_2$ and thus:

$$x^*x + xx^* = (x_1^2 + ix_1x_2 - ix_2x_1 + x_2^2) + (x_1^2 + ix_2x_1 - ix_1x_2 + x_2^2) = 2x_1^2 + 2x_2^2$$

By functional calculus and Prop. 3.29(b), we know that $\mathrm{sp}(2x_j^2) \subseteq [0, \infty)$ for $j = 1, 2$. Using Lemma 4.3, we then infer that $xx^* = 2x_1^2 + 2x_2^2 + (-x^*x)$ is positive as a sum of three positive elements. By Lemma 4.5, we thus infer

$$\mathrm{sp}(x^*x) \subseteq \mathrm{sp}(x^*x) \cup \{0\} = \mathrm{sp}(xx^*) \cup \{0\} \subseteq [0, \infty).$$

On the other hand, $-x^*x \geq 0$, i.e. $\mathrm{sp}(x^*x) \subseteq (-\infty, 0]$ using Prop. 3.29(b) again. Thus, $\mathrm{sp}(x^*x) = \{0\}$ which implies

$$\|x\|^2 = \|x^*x\| = r(x^*x) = 0$$

by Cor. 2.14. This shows $x = 0$. □

We are now ready for the main result of this lecture: The algebraic characterization of positivity.

Theorem 4.8. *Let A be a C^*-algebra and $x \in A$. The following are equivalent.*

(a) $x \geq 0$.
(b) *There is a selfadjoint element $y \in A$ with $y^2 = x$.*
(c) *There is an element $z \in A$ with $x = z^*z$.*

Proof. The implication from (a) to (b) is by Prop. 4.4, putting $y := \sqrt{x}$. The step from (b) to (c) is trivial, putting $z := y$.

Now assume that x is of the form $x = z^*z$. We want to show that x is posititive. We decompose x as $x = x_+ - x_-$ with $x_+, x_- \geq 0$ and $x_+x_- = x_-x_+ = 0$, see Prop. 3.29. By functional calculus, we infer $x_-^3 \geq 0$. Put $y := zx_-$. Then:

$$-y^*y = -x_-z^*zx_- = -x_-(x_+ - x_-)x_- = x_-^3 \geq 0$$

By Lemma 4.7, $y = 0$, which implies $x_-^3 = 0$. Thus also $x_- = 0$ by functional calculus (or as $\|x_-\|^4 = \|x_-^4\|$, see for instance the proof of Cor. 2.14). We conclude $x = x_+ \geq 0$. □

4.4. **Induced partial order structure.** As a corollary of the above theorem, we obtain a partial order structure on C^*-algebras.

Corollary 4.9. *Let A be a C^*-algebra. We put*

$$A_{\mathrm{sa}} := \{x \in A \mid x = x^*\} \subseteq A, \qquad A_+ := \{x \in A \mid x \geq 0\} \subseteq A_{\mathrm{sa}} \subseteq A.$$

Then A_+ is a convex cone, i.e. we have:

(i) *If $x \in A_+$ and $\lambda \geq 0$, then $\lambda x \in A_+$.*
(ii) *If $x, y \in A_+$, then $x + y \in A_+$.*

Moreover, $A_+ \cap (-A_+) = \{0\}$, $A_{\mathrm{sa}} = A_+ - A_+$ and A_+ is (topologically) closed.

Proof. Item (i) is by functional calculus. Item (ii) is by Lemma 4.3. The assertion $A_+ \cap (-A_+) = \{0\}$ is by Lemma 4.7 and Thm. 4.8. The decomposition $A_{\mathrm{sa}} = A_+ - A_-$ is by Prop. 3.29. The cone A_+ is closed as an easy consequence of Lemma 4.2. □

Definition 4.10. For elements $x, y \in A$ in a C^*-algebra, we write $x \leq y$, if $y - x \geq 0$.

The order structure defined above is a partial order (reflexive, antisymmetric, transitive), by Cor. 4.9.

Note that *-homomorphisms respect the order structure. Indeed, let us recap that *-homomorphisms preserve positivity, as we mentioned before. We can employ

Lemma 3.8 for a proof, but with the characterization in Thm. 4.8, it is even easier: Since any positive element is of the form x^*x, it is clear that also $\varphi(x^*x) = \varphi(x)^*\varphi(x)$ is positive, provided that $\varphi : A \to B$ is a *-homomorphism and $x \in A$. Then, of course, preservation of positivity implies preservation of the order structure, so $x \leq y$ implies $\varphi(x) \leq \varphi(y)$.

Remark 4.11. For some instances, the preservation of the positivity structure (aka the order structure) on C^*-algebras is so important that also weakenings of *-homomorphisms are considered: A linear map $\varphi : A \to B$ between C^*-algebras is called positive, if it maps positive elements to positive elements, so $x \geq 0$ implies $\varphi(x) \geq 0$. If such a positive map respects some matrix structure over A and B, it is called completely positive. Completely positive maps are generalizations of *-homomorphisms and a key ingredient in the theory of nuclear C^*-algebras [8].

Let us now prove some properties of the order structure.

Proposition 4.12. *Let A be a C^*-algebra and let $x, y \in A$.*

(a) *If $x \leq y$, then $z^*xz \leq z^*yz$ for all $z \in A$.*
(b) *If $x \geq 0$, then $\|x\| = \inf\{\lambda \geq 0 \mid \lambda 1 \geq x\}$. In particular $x \leq \|x\|1$. Note that we view $A \subseteq \tilde{A}$ here, in case A is not unital.*
(c) *If $0 \leq x \leq y$, then $\|x\| \leq \|y\|$.*
(d) *If A is unital, $0 \leq x \leq y$ and x, y are invertible, then $0 \leq y^{-1} \leq x^{-1}$.*
(e) *If $0 \leq x \leq y$ and $\beta \in [0,1]$, then $0 \leq x^\beta \leq y^\beta$, in particular $0 \leq \sqrt{x} \leq \sqrt{y}$.*

Proof. Item (a) is easy: We use Thm. 4.8 in order to write $y - x = w^*w$ and we infer $z^*yz - z^*xz = z^*(y-x)z = (wz)^*(wz) \geq 0$, again by Thm. 4.8.

Item (b) follows directly from Lemma 4.2.

Item (c) follows from (b).

As for (d), we use Exc. 3.5(c) in order to see $x^{-1} \geq 0$. Thus, the expression $\sqrt{x^{-1}}$ makes sense and this element commutes with x, by functional calculus (see Exc. 3.5(d)). Thus,

$$1 = \sqrt{x^{-1}}x\sqrt{x^{-1}} \leq \sqrt{x^{-1}}y\sqrt{x^{-1}}$$

by (a). Put $z := \sqrt{x}y^{-1}\sqrt{x}$ and observe that

$$z = \sqrt{x}y^{-1}\sqrt{x} = (\sqrt{x^{-1}}y\sqrt{x^{-1}})^{-1} \leq 1.$$

For the latter inequality, we used that $w \geq 1$ implies $w^{-1} \leq 1$ for any self-adjoint element w, again by functional calculus, see Exc. 3.5(f). We conclude $y^{-1} = \sqrt{x^{-1}}z\sqrt{x^{-1}} \leq \sqrt{x^{-1}}\sqrt{x^{-1}} = x^{-1}$, where we used (a).

Finally, (e) is more complicated and we omit a complete proof; see [37, Prop. 1.3.8], [3, Prop. II.3.1.10]. The idea is to write the real-valued function $[0,\infty) \ni t \mapsto g(t) := t^\beta$ as an integral $g(t) = \frac{1}{\gamma}\int_0^\infty f_\alpha(t)\alpha^{-\beta}d\alpha$, where $f_\alpha(t) = \frac{t}{1+\alpha t}$, $\alpha > 0$ and $\gamma > 0$. One may then check $f_\alpha(y) - f_\alpha(x) \geq 0$ and derive $y^\beta - x^\beta \geq 0$ from the integral presentation. $\square$

Remark 4.13. Item (e) of Prop. 4.12 is *not* true in general for $\beta > 1$. In particular, an implication from $0 \leq x \leq y$ to $0 \leq x^2 \leq y^2$ is wrong in general. We even have: If there is some $\beta > 1$ such that for all $x, y \in A$ with $0 \leq x \leq y$ the inequality $x^\beta \leq y^\beta$ holds, then A must be commutative. So, we must be careful in our intuition for order relations as in item (e) and we shall not be driven by our understanding from the commutative case.

4.5. **Injective *-homomorphisms are isometric.** Let us now come to an amazing fact: For *-homomorphisms between C^*-algebras, injectivity implies the preservation of the norm! Before we prove this nice result, let us mention a little lemma.

Lemma 4.14. *Let A be a unital C^*-algebra and let $x \in A$ with $x \geq 0$. Then $\|x\| \in \mathrm{sp}(x)$.*

Proof. Since x is selfadjoint, we have $r(x) = \|x\|$ by Cor. 2.14. Hence, $\|x\| \in \mathrm{sp}(x)$ or $-\|x\| \in \mathrm{sp}(x)$ by the definition of the spectral radius. Now, $\mathrm{sp}(x) \subseteq [0, \infty)$ since x is positive, so $\|x\| \in \mathrm{sp}(x)$ must hold. □

Proposition 4.15. *Let A, B be C^*-algebras and $\varphi : A \to B$ a *-homomorphism. Then, φ is injective if and only if φ is isometric (i.e. $\|\varphi(x)\| = \|x\|$ for all $x \in A$).*

Proof. If φ is isometric, then it is clearly also injective. So, let us prove the converse. In the first case, assume that A is unital. We may then also assume that B and φ are unital; otherwise we consider instead of B the closure of $\varphi(A)$ – which is a C^*-subalgebra of B with unit $\varphi(1)$.

Assume that φ is not isometric. Hence, there is some $x \in A$ such that $\|\varphi(x^*x)\| = \|\varphi(x)\|^2 < \|x\|^2 = \|x^*x\|$. Here, we used that $\|\varphi(x)\| \leq \|x\|$ holds by Lemma 3.8.

Let $f : \mathrm{sp}(x^*x) \to \mathbb{R}$ be a continuous function with $0 \leq f \leq 1$ which is zero on $[0, \|\varphi(x^*x)\|]$ and $f(\|x^*x\|) = 1$. Then $\|f\|_\infty = 1$, since the supremum $\|f\|_\infty$ is taken over the set $\mathrm{sp}(x^*x) \subseteq [0, \|x^*x\|]$ and we have $\|x^*x\| \in \mathrm{sp}(x^*x)$ by Lemma 4.14. Then $\|f(x^*x)\| = \|f\|_\infty = 1$ by the functional calculus for x^*x. In particular $f(x^*x) \neq 0$ holds.

It remains to show that $\varphi(f(x^*x)) = 0$, which then implies that φ is not injective. From, $\mathrm{sp}(\varphi(x^*x)) \subseteq [0, \|\varphi(x^*x)\|]$ we infer that $\mathrm{sp}(f(\varphi(x^*x))) = f(\mathrm{sp}(\varphi(x^*x))) = \{0\}$ holds by Prop. 3.29(b). Then $\|f(\varphi(x^*x))\| = r(f(\varphi(x^*x))) = 0$ using Cor. 2.14. Hence, $\varphi(f(x^*x)) = f(\varphi(x^*x)) = 0$ by Prop. 3.29(e).

In the second case, assume that A is not unital and that φ is injective. Then $\tilde{\varphi} : \tilde{A} \to \tilde{B}$ given as in Lemma 3.6 is injective: given $x = a + \lambda \in \tilde{A}$ with $\lambda \neq 0$, assume $\tilde{\varphi}(x) = 0$. Thus $\varphi(a) = -\lambda$, i.e. we find some $e \in A$ with $\varphi(e) = 1 \in \tilde{B}$. But injectivity of φ shows that e is a unit of A, which is a contradiction. Thus, $\tilde{\varphi}(x) \neq 0$, i.e. $\tilde{\varphi}$ is injective. Hence, $\tilde{\varphi}$ is isometric by the first case and so is its restriction φ. □

4.6. **Definition of approximate units.** We now come to a different topic: to approximate units. We learned already that some C^*-algebras do not possess a unit, see Exm. 2.3. In the sequel, we will show that they always possess at least an approximate unit. We first recall the notion of a net.

Definition 4.16. Let X be a topological space. A family $(x_\lambda)_{\lambda\in\Lambda} \subseteq X$ is called a *net*, if Λ is a partially ordered, directed set, i.e. there is a relation $\leq$ on Λ such that for all $\lambda, \mu, \nu \in \Lambda$:

(i) $\lambda \leq \lambda$
(ii) If $\lambda \leq \mu$ and $\mu \leq \lambda$, then $\lambda = \mu$.
(iii) If $\lambda \leq \mu$ and $\mu \leq \nu$, then $\lambda \leq \nu$.
(iv) For all $\lambda, \mu \in \Lambda$ there is some $\nu \in \Lambda$ with $\lambda \leq \nu$ and $\mu \leq \nu$.

A net (x_λ) converges to $x \in X$ (we write $x_\lambda \to x$), if for any neighborhood U of x there is some $\lambda_0 \in \Lambda$ with $x_\lambda \in U$ for all $\lambda \geq \lambda_0$.

Our favourite choice is $\Lambda = \mathbb{N}$ with its natural order. Then nets are simply sequences.

Definition 4.17. Let A be a C^*-algebra and $I \subseteq A$ be a subset. An *approximate unit* for I is a net $(u_\lambda)_{\lambda\in\Lambda} \subseteq I$ such that:

(i) $0 \leq u_\lambda$ and $\|u_\lambda\| \leq 1$ for all $\lambda \in \Lambda$.
(ii) If $\lambda \leq \mu$, then $u_\lambda \leq u_\mu$.
(iii) We have $u_\lambda x \to x$ and $x u_\lambda \to x$ for all $x \in I$.

It is clear, that the unit 1 in a unital C^*-algebra is an approximate unit.

Example 4.18. Again, we take a look at our favourite non-unital C^*-algebras.

(a) For $C_0(\mathbb{R})$, we choose functions $0 \leq u_n \leq 1$ being 1 on $[-n, n]$ and zero outside of $[-n-1, n+1]$. They form an approximate unit with $\Lambda = \mathbb{N}$ and its natural order.
(b) For the compact operators $\mathcal{K}(H)$ on an infinite dimensional separable Hilbert space with orthonormal basis $(e_n)_{n\in\mathbb{N}}$, we let u_n be the projection onto the finite dimensional subspace of H spanned by $e_1, \ldots, e_n$. By Prop. 1.38, (u_n) is an approximate unit.

4.7. **Existence of approximate units.** Let us now show that approximate units always exist. We prepare some technical lemma first. Recall that $I \lhd A$ denotes a two-sided ideal in a C^*-algebra A. If in addition I is closed (in topology) and closed under taking adjoints, it is a C^*-algebra itself. Later in this lecture, we will see that a two-sided closed ideal is automatically closed under taking adjoints.

Lemma 4.19. *Let A be a C^*-algebra and let $I \lhd A$ be a closed ideal. We put*

$$\Lambda := \{h \in I \mid h \geq 0, \|h\| < 1\}.$$

(a) *The set Λ is a partially ordered, directed set.*
(b) *Let $h \in I$ with $h \geq 0$ and $n \in \mathbb{N}$. Then $h(\frac{1}{n} + h)^{-1} \in \Lambda$ and $\|h(1 - h(\frac{1}{n} + h)^{-1})h\| \leq \frac{1}{n}\|h\|$.*

(c) *Let $h \in I$ with $h \geq 0$ and $n \in \mathbb{N}$. Let $g \in \Lambda$ with $h(\frac{1}{n}+h)^{-1} \leq g$, then $\|h-gh\|^2 \leq \frac{1}{n}\|h\|$ and $\|h-hg\|^2 \leq \frac{1}{n}\|h\|$.*

Proof. We begin with (a). Items (i), (ii) and (iii) of Def. 4.16 follow from the fact that we have a partial order on C^*-algebras, see Sect. 4.4. We are left with proving directedness, i.e. item (iv) of Def. 4.16. Let $a, b \in \Lambda$. We need to find some $c \in \Lambda$ with $a \leq c$ and $b \leq c$.

Note that $\|a\| < 1$ implies $\mathrm{sp}(a) \subseteq [0,1)$. The function $z \mapsto z(1-z)^{-1}$ is continuous and positive on $[0,1)$. We may therefore define

$$a' := a(1-a)^{-1}, \qquad b' := b(1-b)^{-1}$$

and derive $a' \geq 0$ and $b' \geq 0$ by functional calculus, see also Exc. 4.5. Hence also $c' := a' + b'$ is positive, by Lemma 4.3. We may thus define

$$c := c'(1+c')^{-1}$$

again by functional calculus. We need to show that $c \in \Lambda$, $a \leq c$ and $b \leq c$ hold.

Firstly, the function $z \mapsto z(1+z)^{-1}$ is positive on $[0,\infty)$ and strictly smaller than 1. Thus, by functional calculus, $c \in \Lambda$. Secondly, we have $0 \leq a' \leq c'$ and hence also $0 \leq 1 + a' \leq 1 + c'$. By Prop. 4.12, $(1+c')^{-1} \leq (1+a')^{-1}$. Thus:

$$a = a'(1+a')^{-1} = 1-(1+a')^{-1} \leq 1-(1+c')^{-1} = c'(1+c')^{-1} = c$$

Similarly, $b \leq c$ and we conclude that Λ is directed.

Let us comment on some subtlety of the proof. We did *not* assume that A is unital, nor that I is. Nevertheless, we were using the symbol 1 all the time – weren't we mistaken to do so? No, we were not. We may view $A \subseteq \tilde{A}$ and we therefore have a unit at hand, the one of $\tilde{A}$. However, we may check in each and every case where 1 was involved, that the resulting element was in A or in I resp., simply, because A and I are ideals in $\tilde{A}$. We address these issues in Exc. 4.5.

For (b), we can immediately use the functional calculus.

For (c), note that $(1-g)-(1-g)^2 = g(1-g) \geq 0$ by functional calculus, since $\mathrm{sp}(g) \subseteq [0,1)$. Thus $0 \leq (1-g)^2 \leq 1-g$, as a comparison of elements in $\tilde{A}$. By Prop. 4.12, we then have $0 \leq h(1-g)^2h \leq h(1-g)h$. Moreover, $0 \leq 1-g \leq 1-h(\frac{1}{n}+h)^{-1}$ by assumption, which yields $0 \leq h(1-g)h \leq h(1-h(\frac{1}{n}+h)^{-1}h$. Applying Prop. 4.12(c) twice and using (b), this implies:

$$\|h-gh\|^2 = \|h(1-g)^2h\| \leq \|h(1-g)h\| \leq \|h(1-h(\frac{1}{n}+h)^{-1})h\| \leq \frac{1}{n}\|h\|$$

Similarly, $\|h-hg\|^2 \leq \frac{1}{n}\|h\|$. □

Theorem 4.20. *Let A be a C^*-algebra and let $I \lhd A$ be a two-sided closed ideal (possibly $I = A$). Then, I possesses an approximate unit.*

Proof. The idea is to take all positive elements in I (which have a small norm) as an approximate unit and to index this set by itself. The technical part of the proof has been shifted to the previous lemma, so we may now simply put everything together.

We define Λ as in Lemma 4.19 and we put $u_\lambda := \lambda$ for $\lambda \in \Lambda$. Then, (i) and (ii) of Def. 4.17 are satisfied.

As for Def. 4.17(iii), let $x \in I$ and let $\varepsilon > 0$. Put $h := xx^* \geq 0$. Let $n \in \mathbb{N}$ such that $\frac{1}{n}\|h\| < \varepsilon^2$. Put $\lambda_0 := h(\frac{1}{n} + h)^{-1}$. Then $\lambda_0 \in \Lambda$, by Lemma 4.19. Let $g \in \Lambda$ with $g \geq \lambda_0$. Then $\|h - gh\| \leq \left(\frac{1}{n}\|h\|\right)^{\frac{1}{2}} < \varepsilon$ by Lemma 4.19. Using the C^*-identity, we deduce:

$$\|x - gx\|^2 = \|(1-g)h(1-g)\| \leq \|h - gh\|(1 + \|g\|) < 2\|h - gh\| < 2\varepsilon$$

This shows that $u_g x = gx \in U$ for an ε-neighborhood U of x and hence $u_g x - x \to 0$. Similarly $xu_g - x \to 0$. $\square$

Just to make sure, note that an approximate unit of $I \lhd A$ approximates only elements x from I in the sense of $u_\lambda x \to x$ and $xu_\lambda \to x$ – we cannot say anything about the approximation of elements x from A.

Remark 4.21. If A is a separable C^*-algebra (i.e. there is a countable, dense subset in A), then A admits a countable approximate unit (i.e. an approximate unit $(u_n)_{n\in\mathbb{N}}$ with $u_1 \leq u_2 \leq u_3 \leq \ldots$). Indeed, by Thm. 4.20, there is an approximate unit $(u_\lambda)_{\lambda\in\Lambda}$ in A. Given a dense subset $\{x_n \mid n \in \mathbb{N}\} \subseteq A$, we choose a sequence $(\lambda_n)_{n\in\mathbb{N}} \subseteq \Lambda$ with $\lambda_{n+1} \geq \lambda_n$ and $\|u_\lambda x_i - x_i\|, \|x_i u_\lambda - x_i\| < \frac{1}{n}$ for all $\lambda \geq \lambda_n$ and all $i = 1, \ldots, n$. Put $u_n := u_{\lambda_n}$.

4.8. **Consequence for ideals of C^*-algebras.** The existence of approximate units has some consequences for the ideal structure of C^*-algebras.

Lemma 4.22. *Let A be a C^*-algebra.*

(a) *Any closed ideal $I \lhd A$ is closed under taking adjoints. Hence, I is a C^*-algebra.*
(b) *If $I \lhd J \lhd A$ are closed ideals, then $I \lhd A$.*

Proof. For (a), let (u_λ) be an approximate unit for I (which exists by Thm. 4.20) and let $x \in I$. We use the convergence $u_\lambda x \to x$ and the fact that the involution is continuous in order to derive $x^* u_\lambda \to x^*$. As all elements $x^* u_\lambda$ are in I and since I is closed, we infer $x^* \in I$. Thus, $I \subseteq A$ is a closed *-subalgebra and hence it is a C^*-algebra.

For (b), let again (u_λ) be an approximate unit for I, $x \in I$ and $a \in A$. Now, $a \in A$ and $u_\lambda \in I \subseteq J$ implies $u_\lambda a \in J$, since J is an ideal in A. But as $x \in I$ and I is an ideal in J, we infer $x(u_\lambda a) \in I$. Then, $xu_\lambda \to x$ implies $I \ni xu_\lambda a \to xa$ and we infer $xa \in I$, since I is closed. Likewise $ax \in I$. $\square$

4.9. **Quotients of C^*-algebras.** Did you ever wonder whether a C^*-algebraic version of Prop. 2.17(a) holds? Why shall we investigate ideals in C^*-algebras if we are not allowed to take quotients? Well, we are! Let's prove it. Note that approximate units are a crucial ingredient in the proof.

Theorem 4.23. *Let A be a C^*-algebra and let $I \lhd A$ be a closed ideal. Then A/I is a C^*-algebra.*

Proof. By Prop. 2.17, A/I is a Banach algebra via $\dot{x} + \dot{y} := (x+y)\dot{}$, $\lambda\dot{x} := (\lambda x)\dot{}$, $\dot{x}\dot{y} := (xy)\dot{}$ and $\|\dot{x}\| := \inf\{\|x+z\| \mid z \in I\}$.

We equip it with an involution $\dot{x}^* := (x^*)\dot{}$. By Lemma 4.22, this is well-defined: Assume $\dot{x} = \dot{y}$. Then $x - y \in I$, but also $x^* - y^* \in I$, since I is closed under taking adjoints. Hence $(x^*)\dot{} = (y^*)\dot{}$.

It remains to show that the norm satisfies the C^*-identity. Let (u_λ) be an approximate unit for I and let $x \in A$. We then have

$$\|\dot{x}\| = \lim_{\lambda\in\Lambda} \|x - xu_\lambda\|.$$

Let's prove this description of the norm. Let $\varepsilon > 0$. By the definition of the norm, we find some $z \in I$ with $\|x+z\| \le \|\dot{x}\| + \varepsilon$. On the other hand, there is some $\lambda_0 \in \Lambda$ with $\|z - zu_\lambda\| < \varepsilon$ for all $\lambda \ge \lambda_0$, due to the convergence $zu_\lambda \to z$. Note that $xu_\lambda \in I$, so $\|\dot{x}\| \le \|x - xu_\lambda\|$. Moreover, $\|1 - u_\lambda\| \le 1$ since $\mathrm{sp}(u_\lambda) \subseteq [0,1]$ implies $\mathrm{sp}(1-u_\lambda) \subseteq [0,1]$. Thus, for all $\lambda \ge \lambda_0$:

$$\|\dot{x}\| \le \|x - xu_\lambda\| \le \|(x+z)(1-u_\lambda)\| + \|z(1-u_\lambda)\| \le \|1-u_\lambda\|\|x+z\| + \varepsilon \le \|\dot{x}\| + 2\varepsilon$$

We infer $\|\dot{x}\| = \lim_{\lambda\in\Lambda} \|x - xu_\lambda\|$. This implies, for any $z \in I$:

$$\begin{aligned}
\|\dot{x}\|^2 &= \lim_{\lambda\in\Lambda} \|x - xu_\lambda\|^2 \\
&= \lim_{\lambda\in\Lambda} \|(1-u_\lambda)x^*x(1-u_\lambda)\| \\
&= \lim_{\lambda\in\Lambda} \|(1-u_\lambda)(x^*x+z)(1-u_\lambda)\| \\
&\le \|x^*x + z\|
\end{aligned}$$

Here, we used $\lim_{\lambda\in\Lambda} \|(1-u_\lambda)z(1-u_\lambda)\| = 0$; thus $\|(1-u_\lambda)x^*x(1-u_\lambda)\|$ and $\|(1-u_\lambda)x^*x(1-u_\lambda) + (1-u_\lambda)z(1-u_\lambda)\|$ must have the same limit. Finally, taking the infimum over all $z \in I$, we infer $\|\dot{x}\|^2 \le \|\dot{x}^*\dot{x}\|$. By Remark 2.2(c), we thus obtain the C^*-identity. □

Theorem 4.23 showed that we found the right concept of ideals, when investigating closed two-sided ideals. Together with the fact that injective *-homomorphisms are already isometric, we may prove that images of C^*-algebras under *-homomorphisms are C^*-algebras again.

Proposition 4.24. *Let A, B be C^*-algebras and let $\varphi : A \to B$ be a *-homomorphism. Then, $\varphi(A)$ is a C^*-algebra which is isomorphic to $A/\ker\varphi$.*

Proof. It is easy to check that $\ker\varphi \lhd A$ is an ideal. By Thm. 4.23, $A/\ker\varphi$ is a C^*-algebra. Let us denote the quotient map by $\pi : A \to A/\ker\varphi$, $x \mapsto \dot{x}$. We define a map $\dot{\varphi} : A/\ker\varphi \to B$ by $\dot{\varphi}(\dot{x}) := \varphi(x)$. This is an injective *-homomorphism with range $\varphi(A)$. By Prop. 4.15, $\dot{\varphi}$ is even isometric. Hence, any Cauchy sequence in $\varphi(A)$ is then also a Cauchy sequence in $A/\ker\varphi$. This implies that $\varphi(A)$ is complete.

Hence $\varphi(A) \subseteq B$ is a closed *-subalgebra and hence it is a C^*-algebra. We conclude that $\dot{\varphi} : A/\ker\varphi \to \varphi(A)$ is a *-isomorphism. $\square$

Remark 4.25. The above proposition has some nice homological consequences for C^*-algebras: We may work with short exact sequences. Recall the concept of exact sequences from homological algebra. In our context, it means the following. Let $(A_n)_{n\in J}$ be C^*-algebras for $J = \{1, \dots, N\}$, and let $\varphi_n : A_n \to A_{n+1}$ be *-homomorphisms for $n = 1, \dots, N-1$. This chain of C^*-algebras and *-homomorphisms is called exact, if $\ker\varphi_{n+1} = \operatorname{ran}\varphi_n$ for all $n = 1, \dots, N-1$. A short exact sequence is an exact sequence

$$0 \to I \to A \to B \to 0$$

of C^*-algebras I, A and B. This encodes exactly the situation of ideals and quotients: If $I \lhd A$, then

$$0 \to I \to A \to A/I \to 0$$

is a short exact sequence. See also Exc. 4.6.

4.10. **Exercises.**

Exercise 4.1. Let H be a Hilbert space and let $T \in B(H)$. Show that T is positive if and only if $\langle Tx, x\rangle \geq 0$ for all $x \in H$. In order to do so, show that $T - \lambda$ is bounded from below and surjective for $\lambda \notin [0, \infty)$.

Exercise 4.2. Let H be a Hilbert space and let $T \in B(H)$. We want to show that T admits a polar decomposition $T = V|T|$.

(a) Convince yourself that $|T| := \sqrt{T^*T}$ exists by functional calculus.
(b) Show that $\ker|T| = \ker T$ and that the map $V_0 : \operatorname{ran}|T| \to \operatorname{ran} T$ given by $|T|x \mapsto Tx$ is well-defined and isometric. It thus has an isometric extension $\overline{V_0} : \overline{\operatorname{ran}|T|} \to \overline{\operatorname{ran} T}$ and we may define $V(x_1 + x_2) := \overline{V_0}x_1$ for $x_1 + x_2 \in \overline{\operatorname{ran}|T|} \oplus \overline{\operatorname{ran}|T|}^{\perp}$.
(c) Show that V is a partial isometry in the sense of Exc. 1.8. Show that V^*V is the projection onto $(\ker T)^{\perp}$ while VV^* is the projection onto $\overline{\operatorname{ran} T}$.
(d) Show that $T = V|T|$. Show that V is the unique partial isometry with $T = V|T|$ and $\ker V = \ker T$.
(e) Show that V is unitary in the sense of Def. 1.33, if T is invertible.
(f) What is the polar decomposition of T in the one-dimensional case $H = \mathbb{C}$?

Exercise 4.3. Let A be a C^*-algebra and let $x, y \in A$. Show that

$$\operatorname{sp}(xy) \cup \{0\} = \operatorname{sp}(yx) \cup \{0\}.$$

Find an example for $\operatorname{sp}(xy) \neq \operatorname{sp}(yx)$.

Exercise 4.4. Let A be a C^*-algebra and $x, h \in A$ be selfadjoint with $h \geq 0$ and $h \geq x$. The positive part $x_+ \geq 0$ of x is defined as in Prop. 3.29.

(a) Show that $h \geq x_+$, if A is commutative.

(b) Show that $h \not\geq x_+$ in general by giving a counterexample in $A = M_2(\mathbb{C})$.

Exercise 4.5. Let A be a C^*-algebra and let $I \lhd A$ be a closed ideal in A. Let $a, b \in I$ with $a, b \geq 0$ and $\|a\|, \|b\| < 1$.

(a) Show that we may define $a' := a(1-a)^{-1}$. Be careful: We do not assume that A is unital! Why can we still write down the expression a' and why do we even have $a' \in I$?

(b) Let $b' := b(1-b)^{-1}$, $c' := a' + b'$ and $c := c'(1+c')^{-1}$. Show that we may define b', c' and c and that they all lie in I (again, taking care of the issue with the unit).

(c) Show that a', b', c' and c are positive. Show $\|c\| < 1$.

(d) Show $1 - (1+a')^{-1} \leq 1 - (1+c')^{-1}$.

(e) Show $a = a'(1+a')^{-1}$.

(f) Show that $0 \leq b(1-a)^2 b \leq b(1-a)b$ holds. Again, why are all these elements in I?

Exercise 4.6. Let I, A and B be C^*-algebras and let $\iota : I \to A$ and $\pi : A \to B$ be *-homomorphisms.

(a) Show that if the sequence

$$0 \to I \to A \to B \to 0$$

is exact, then ι is injective, π is surjective, $\iota(I) \lhd A$ is a closed ideal in A and $B \cong A/\iota(I)$.

(b) Conversely, if ι is injective and $\iota(I) \lhd A$ is a closed ideal in A, show that

$$0 \to I \to A \to A/\iota(I) \to 0$$

is exact, where $A \to A/\iota(I)$ is the canonical quotient map.

5. States, representations and the GNS construction

Abstract. We introduce and study positive linear functionals and states. We briefly recall the Hahn-Banach Theorem. We then turn to representations of C^*-algebras and we discuss the famous GNS construction. This yields our Second Fundamental Theorem of C^*-Algebras: Any abstractly defined C^*-algebra may be represented concretely on a Hilbert space – as a subalgebra of all bounded linear operators.

5.1. **Positive linear functionals.** In the last lecture, we discussed positivity for elements in a C^*-algebra and the induced order structure. We now turn to a class of linear functionals preserving these structures.

Definition 5.1. Let A be a C^*-algebra. A linear functional $\varphi : A \to \mathbb{C}$ is *positive* (we write $\varphi \geq 0$), if $\varphi(x) \geq 0$ for all $x \in A$ with $x \geq 0$.

We infer that positive linear functionals preserve the order structure: Given $x, y \in A$, the relation $x \leq y$ implies $\varphi(x) \leq \varphi(y)$. Indeed, recall that $x \leq y$ holds if and only if $z := y - x \geq 0$. Thus, $z \geq 0$ implies $\varphi(z) \geq 0$ from which we deduce $\varphi(x) \leq \varphi(y)$, see also the discussion around Rem. 4.11.

Example 5.2. Let us take a look at some examples.

(a) Let $A = C([0,1])$ as in Exm. 2.3 and let $t \in [0,1]$. Then $\mathrm{ev}_t : C([0,1]) \to \mathbb{C}$ given by $\mathrm{ev}_t(f) := f(t)$ is a positive linear functional. In fact, it is even an algebra homomorphism, see Prop. 3.18 and Exc. 3.2, but forgetting the multiplicative structure, we obtain a positive linear functional. More generally, let μ be a Radon measure on $[0,1]$ (i.e. for all $x \in [0,1]$ there is an open neighborhood U_x such that $\mu(U_x) < \infty$; and for any Borel set B, the value $\mu(B)$ is the supremum of all $\mu(K)$, where $K \subseteq B$ is compact). Then

$$\varphi(f) := \int_0^1 f(t) d\mu(t)$$

is a positive linear functional, $f \in C([0,1])$. In fact, *any* positive linear functional is of exactly this form! You might have met the Representation Theorem by Riesz-Markov in some of your analysis lectures. It states that the positive linear functionals on $[0,1]$ are in bijection with Radon measures on $[0,1]$ via the above correspondence. If μ is the Dirac measure on $t \in [0,1]$, we obtain ev_t under this correspondence.

(b) In Exc. 5.3, we will see the following. The trace $\mathrm{Tr} : M_N(\mathbb{C}) \to \mathbb{C}$ with

$$\mathrm{Tr}(T) = \mathrm{Tr}((t_{ij})) := \sum_{i=1}^{N} t_{ii}, \quad T = (t_{ij}) = (t_{ij})_{i,j=1,\dots,N} \in M_N(\mathbb{C})$$

is a positive linear functional on $M_N(\mathbb{C})$. Likewise, the normalized trace

$$\operatorname{tr}(T) = \operatorname{tr}((t_{ij})) := \frac{1}{N}\sum_{i=1}^{N} t_{ii}$$

is a positive linear functional. More generally, if $B \in M_N(\mathbb{C})$ is a positive matrix, then

$$\tau_B(T) := \operatorname{tr}(BT)$$

is a positive linear functional. In fact, *any* positive linear functional on $M_N(\mathbb{C})$ is of exactly this form, see Exc. 5.3.

(c) Let H be a Hilbert space and let $x \in H$. Consider $A = B(H)$. Then

$$\varphi_x : B(H) \to \mathbb{C}, \quad \varphi_x(T) := \langle Tx, x\rangle, \quad T \in B(H)$$

is a positive linear functional. Indeed, recall from Thm. 4.8 that positive elements T are of the form $T = S^*S$. Thus, $\varphi_x(S^*S) = \langle Sx, Sx\rangle \geq 0$.

If $H = \mathbb{C}^N$ is finite dimensional with the standard basis $e_1, \ldots, e_N$, and $x = \sum_j x_j e_j \in H$, then the above positive linear functional φ_x is of the form $\varphi_x(T^t) = \tau_B(T)$ with $B = X^*X$. Here, $X = (x_{ij}) \in M_N(\mathbb{C})$ is the matrix given by $x_{ij} = \sqrt{N}\delta_{i1}x_j$. The matrix T^t is the transpose of $T \in M_N(\mathbb{C})$, so $T^t = (t_{ji})$, if $T = (t_{ij})$.

We learn from Exm. 5.2(a), that positive linear functionals may be seen as generalized evaluation maps.

5.2. **Induced positive sesquilinear form.** Before we proceed investigating some properties of positive linear functionals, let us make an important observation: Positive linear functionals induce positive sesquilinear forms.

Lemma 5.3. *Let A be a C^*-algebra, $\varphi : A \to \mathbb{C}$ a positive linear functional. Then*

$$\langle x, y\rangle := \varphi(y^*x)$$

is a positive sesquilinear form on A, i.e. we have:

(i) $\langle \lambda x_1 + \mu x_2, y\rangle = \lambda\langle x_1, y\rangle + \mu\langle x_2, y\rangle$ *for all* $x_1, x_2, y \in A, \lambda, \mu \in \mathbb{C}$.
(ii) $\langle x, \lambda y_1 + \mu y_2\rangle = \bar{\lambda}\langle x, y_1\rangle + \bar{\mu}\langle x, y_2\rangle$ *for all* $x, y_1, y_2 \in A, \lambda, \mu \in \mathbb{C}$.
(iii) $\langle x, x\rangle \geq 0$ *for all* $x \in A$.

As a consequence, we have $\overline{\langle x, y\rangle} = \langle y, x\rangle$ *for all* $x, y \in A$.

Proof. It is immediately clear that $\langle\cdot,\cdot\rangle$ is a positive sesquilinear form. A direct computation shows that any sesquilinear form satisfies the polarisation identity, see also Prop. 1.4:

$$\langle x, y\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k\langle x + i^k y, x + i^k y\rangle$$

Here, $i \in \mathbb{C}$ denotes the imaginary unit and $x, y \in A$. We infer $\overline{\langle x, y\rangle} = \langle y, x\rangle$. □

These sesquilinear forms will be important later in the present lecture.

5.3. **Properties of positive linear functionals.** Positive linear functionals are automatically continuous and involutive (i.e. they preserve the involution).

Lemma 5.4. *Let A be a C^*-algebra and let $\varphi : A \to \mathbb{C}$ be a positive linear functional.*

(a) *φ is bounded (and hence continuous), i.e. $|\varphi(x)| \leq \|\varphi\|\|x\|$ for all $x \in A$.*
(b) *φ is involutive, i.e. $\varphi(x^*) = \overline{\varphi(x)}$ for all $x \in A$.*
(c) *We have $|\varphi(x)|^2 \leq \|\varphi\|\varphi(x^*x)$ for all $x \in A$.*

Proof. For (a), let $S := \{x \in A \mid x \geq 0, \|x\| \leq 1\}$. We first show, that φ is bounded on S. Assume the converse. Hence, we find a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \in S$ and $\varphi(x_n) \geq 2^n$, for all $n \in \mathbb{N}$. For $N \in \mathbb{N}$ put $s_N := \sum_{n=1}^{N} \frac{1}{2^n} x_n \in A$ and $x := \lim_{N\to\infty} s_N$. All these elements are positive by Cor. 4.9 and we have $x \geq s_N$ for all $N \in \mathbb{N}$. Thus, $\varphi(x) \geq \varphi(s_N) \geq N$ for all $N \in \mathbb{N}$ which is a contradiction.

Now, as φ is bounded on S, we find some constant $K \geq 0$ such that $\varphi(x) \leq K$ for all $x \in S$. This implies for $x \in A$ with $x \geq 0$ and $x \neq 0$ that $x' := \|x\|^{-1}x \in S$ and thus $\varphi(x) = \|x\|\varphi(x') \leq \|x\|K$. Finally, let us consider an arbitrary element $x \in A$. By Prop. 3.29 and Def. 4.6, we may decompose it into a linear combination of four positive elements each with norm less or equal to $\|x\|$:

$$x = (\mathrm{Re}(x))_+ - (\mathrm{Re}(x))_- + i(\mathrm{Im}(x))_+ - i(\mathrm{Im}(x))_-$$

Then $|\varphi(x)| \leq 4K\|x\|$.

Let us prove (b). Let (u_λ) be an approximate unit for A, which exists by Thm. 4.20. Using (a) and the induced sesquilinear form from Lemma 5.3, we deduce:

$$\varphi(x^*) \leftarrow \varphi(x^*u_\lambda) = \langle u_\lambda, x\rangle = \overline{\langle x, u_\lambda\rangle} = \overline{\varphi(u_\lambda^* x)} = \overline{\varphi(u_\lambda x)} \to \overline{\varphi(x)}$$

As for (c), this is basically the Cauchy-Schwarz inequality (Prop. 1.3) for positive sesquilinear forms. Let again (u_λ) be an approximate unit for A. We deduce $\varphi(u_\lambda^2) \leq \|\varphi\|$ from (a), since $\|u_\lambda^2\| = \|u_\lambda\|^2 \leq 1$ by the C^*-identity and Def. 4.17. Then:

$$|\varphi(x)|^2 \leftarrow |\varphi(u_\lambda x)|^2 = |\langle x, u_\lambda\rangle|^2 \leq \langle x, x\rangle\langle u_\lambda, u_\lambda\rangle = \varphi(x^*x)\varphi(u_\lambda^2) \leq \|\varphi\|\varphi(x^*x)$$

□

Approximate units were used in the previous lemma and they also help us to give a characterization of positive linear functionals in the next proposition. Let us insert two preparatory lemmas first.

Lemma 5.5. *Let A be a C^*-algebra and let $x, y \in A$ be positive with $\|x\|, \|y\| \leq 1$. Then $\|x - y\| \leq 1$.*

Proof. We have $\mathrm{sp}(x) \subseteq [0, 1]$. Thus, $1 - x \geq 0$ by functional calculus, when viewing $1 - x$ as an element in the unitization $\tilde{A}$, in case A does not have a unit. Hence, $(1 - x) + y \geq 0$, if $y \geq 0$, by Lemma 4.3. This shows $x - y \leq 1$. The argument is symmetric and we have $-1 \leq x - y \leq 1$. Thus $\|x - y\| \leq 1$ by functional calculus. □

The next lemma is a modification of Lemma 3.11(e).

Lemma 5.6. *Let A be a C^*-algebra and let $\varphi : A \to \mathbb{C}$ be linear and continuous with $\|\varphi\| = 1$. Assume there is some approximate unit $(u_\lambda)_{\lambda\in\Lambda}$ of A with $\|\varphi\| = \lim_{\lambda\to\infty} \varphi(u_\lambda)$. Then, $\varphi(x) \in \mathbb{R}$, for all selfadjoint elements $x \in A$.*

Proof. We mimic the proof of Lemma 3.11, but we may not assume that φ is unital. So, let $\alpha, \beta \in \mathbb{R}$ with $\varphi(x) = \alpha + i\beta$, for a given selfadjoint element $x \in A$. Let $\mu \in \mathbb{R}$. Then, for all $\lambda \in \Lambda$:

$$\begin{aligned}|\varphi(x + i\mu u_\lambda)|^2 &\leq \|x + i\mu u_\lambda\|^2\\ &= \|(x + i\mu u_\lambda)^*(x + i\mu u_\lambda)\|\\ &\leq \|x\|^2 + |\mu|\|xu_\lambda - u_\lambda x\| + \mu^2\|u_\lambda\|^2\end{aligned}$$

As $\|xu_\lambda - u_\lambda x\| \to 0$ and $\|u_\lambda\| \leq 1$, we infer:

$$\begin{aligned}\alpha^2 &+ \mu^2 + 2\mu\beta + \beta^2\\ &= |\varphi(x) + i\mu|^2 \leftarrow |\varphi(x + i\mu u_\lambda)|^2 \leq \|x\|^2 + |\mu|\|xu_\lambda - u_\lambda x\| + \mu^2 \to \|x\|^2 + \mu^2\end{aligned}$$

Since this holds true for all $\mu \in \mathbb{R}$, we conclude $\beta = 0$, i.e. $\varphi(x) \in \mathbb{R}$. □

Proposition 5.7. *Let A be a C^*-algebra and let $\varphi : A \to \mathbb{C}$ be linear and continuous. The following are equivalent:*

(i) *φ is positive.*
(ii) *For all approximate units (u_λ) of A, we have $\|\varphi\| = \lim_{\lambda\to\infty} \varphi(u_\lambda)$.*
(iii) *For some approximate unit (u_λ) of A, we have $\|\varphi\| = \lim_{\lambda\to\infty} \varphi(u_\lambda)$.*

Proof. The case $\varphi = 0$ is trivial, so let us assume $\|\varphi\| = 1$; otherwise we work with the positive linear functional $\varphi' := \|\varphi\|^{-1}\varphi$.

We begin with proving the implication from (i) to (ii). Assume that φ is positive. Let (u_λ) be an approximate unit for A. Then, $\lambda \leq \mu$ implies $u_\lambda \leq u_\mu$ by Def. 4.17, and hence $\varphi(u_\lambda) \leq \varphi(u_\mu)$ as positive maps preserve the order structure. Moreover, $\varphi(u_\lambda) \leq 1$ for all $\lambda \in \Lambda$. Hence $(\varphi(u_\lambda))$ is an increasing, bounded net in $\mathbb{C}$. We may thus find some $\alpha \leq 1$ such that $\varphi(u_\lambda) \to \alpha$ converges from below. Note that we have $u_\lambda^2 \leq u_\lambda$ by functional calculus and Def. 4.17; thus, $\varphi(u_\lambda^2) \leq \varphi(u_\lambda)$. Employing the Cauchy-Schwarz inequality as in the proof of Lemma 5.4, we deduce for all $x \in A$ with $\|x\| \leq 1$:

$$|\varphi(x)|^2 \leftarrow |\varphi(u_\lambda x)|^2 = |\langle x, u_\lambda\rangle|^2 \leq \langle x, x\rangle\langle u_\lambda, u_\lambda\rangle = \varphi(x^*x)\varphi(u_\lambda^2) \leq \varphi(x^*x)\varphi(u_\lambda) \leq \alpha$$

Hence, $1 = \|\varphi\| \leq \sqrt{\alpha} \leq 1$, which shows $\|\varphi\| = 1 = \alpha = \lim_{\lambda\to\infty} \varphi(u_\lambda)$.

The implication from (ii) to (iii) is trivial (provided the existence of approximate units is ensured, which it is, by Thm. 4.20).

As for (iii) to (i), assume that there is an approximate unit (u_λ) with $\varphi(u_\lambda) \to 1$. Let $x \in A$ with $x \geq 0$ and $x \neq 0$. Assume $\|x\| \leq 1$; otherwise we work with $x' := \|x\|^{-1}x$. By Lemma 5.5, $\|u_\lambda - x\| \leq 1$ for all $\lambda \in \Lambda$; hence $|\varphi(u_\lambda - x)| \leq 1$. Moreover, $\varphi(u_\lambda - x) \to 1 - \varphi(x)$, as $\lambda \to \infty$, by assumption. This shows $|1 - \varphi(x)| \leq 1$. On the other hand, $1 - \varphi(x) \in \mathbb{R}$ by Lemma 5.6. Thus, $1 - \varphi(x) \leq |1 - \varphi(x)| \leq 1$. This shows $\varphi(x) \geq 0$. □

In the unital case, this gives a very easy – and amazing! – characterization of positive linear functionals: Positivity is encoded in $\varphi(1)$.

Corollary 5.8. *Let A be a unital C^*-algebra and let $\varphi : A \to \mathbb{C}$ be linear and continuous. Then φ is positive if and only if $\varphi(1) = \|\varphi\|$.*

Surprisingly, the norm behaves additive on positive linear functionals turning the space of positive linear functionals on C^*-algebras into an abstract L-space (or AL-space).

Corollary 5.9. *Let A be a C^*-algebra and let $\varphi, \psi : A \to \mathbb{C}$ be two positive linear functionals. Then*

$$\|\varphi + \psi\| = \|\varphi\| + \|\psi\|.$$

Proof. Let (u_λ) be an approximate unit of A. By Prop. 5.7, we have:

$$\|\varphi + \psi\| \leftarrow (\varphi + \psi)(u_\lambda) = \varphi(u_\lambda) + \psi(u_\lambda) \to \|\varphi\| + \|\psi\| \qquad \square$$

5.4. **Hahn-Banach Theorem.** Let us quickly recall a classic theorem from functional analysis. In fact, it exists in hundreds of variations, so let us state a suitable one for our purposes.

Theorem 5.10 (Hahn-Banach)**.** *Let E be a normed complex vector space and let $F \subseteq E$ be a linear subspace. Let $f : F \to \mathbb{C}$ be linear and continuous. Then, there is a linear and continuous map $\tilde{f} : E \to \mathbb{C}$ extending f (i.e. $\tilde{f}(x) = f(x)$ for all $x \in F$) such that $\|\tilde{f}\| = \|f\|$.*

The crucial point is, that we may extend linear, continuous maps from subspaces to the whole space – in a norm preserving way!

5.5. **States.** We now turn to a special subclass of positive linear functionals: states.

Definition 5.11. Let A be a C^*-algebra. A *state* on A is a positive linear functional $\varphi : A \to \mathbb{C}$ with $\|\varphi\| = 1$.

So, states are simply positive linear functionals with some normalization. In view of Cor. 5.8, this normalization is reasonable.

Proposition 5.12. *Let A be a unital C^*-algebra and let $\varphi : A \to \mathbb{C}$ be a linear functional. Then, φ is a state if and only if φ is positive and unital (i.e. $\varphi(1) = 1$).*

Proof. Follows immediately from Cor. 5.8 and Def. 5.11. $\square$

Next, we formulate some Hahn-Banach Theorem for C^*-algebras.

Theorem 5.13. *Let A be a C^*-algebra and let $x \in A$ be normal. There is a state $\varphi : A \to \mathbb{C}$ with $|\varphi(x)| = \|x\|$.*

Proof. The proof is beautiful. We consider $C^*(x,1) \subseteq \tilde{A}$. This is a unital Banach algebra and we consider the Gelfand transform $\chi : C^*(x,1) \to C(\mathrm{Spec}(C^*(x,1)))$. Now, for $\hat{x} = \chi(x) \in C(\mathrm{Spec}(C^*(x,1))$ we find some character $\varphi_0 \in \mathrm{Spec}(C^*(x,1))$ such that $|\hat{x}(\varphi_0)| = \|\hat{x}\|_\infty$. As φ_0 is a character, we also have $\varphi_0(1) = 1$. Moreover, φ_0 is in particular a positive linear functional, so $\|\varphi_0\| = \varphi_0(1) = 1$, by Cor. 5.8.

Since x is normal, $C^*(x,1) \subseteq \tilde{A}$ is in fact a commutative, unital C^*-algebra. Hence, χ is an isometric *-isomorphism by the Gelfand-Naimark Theorem (Thm. 3.23), which means that $\|\hat{x}\|_\infty = \|x\|$. Now comes the first funny aspect: Forgetting some of the structure of φ_0, we conclude that we found a linear and continuous map $\varphi_0 : C^*(x,1) \to \mathbb{C}$ with $\varphi_0(1) = 1$ and

$$|\varphi_0(x)| = |\hat{x}(\varphi_0)| = \|\hat{x}\|_\infty = \|x\|.$$

We may hence apply the Hahn-Banach Theorem (Thm. 5.10), and find a linear and continuous extension $\tilde{\varphi} : \tilde{A} \to \mathbb{C}$ with $\|\tilde{\varphi}\| = \|\varphi_0\|$. Is this extension still positive? It is, surprisingly: since $\tilde{\varphi}$ coincides with φ_0 on $C^*(x,1)$, we have

$$\tilde{\varphi}(1) = \varphi_0(1) = \|\varphi_0\| = \|\tilde{\varphi}\|.$$

By Cor. 5.8, we obtain that $\tilde{\varphi} : \tilde{A} \to \mathbb{C}$ is a positive linear functional. Thus, the restriction of $\tilde{\varphi}$ to $A \subseteq \tilde{A}$ yields a positive linear functional $\varphi : A \to \mathbb{C}$ with

$$|\varphi(x)| = |\tilde{\varphi}(x)| = |\varphi_0(x)| = \|x\|.$$

Moreover, $\|\varphi\| \le \|\tilde{\varphi}\| = 1$ and $|\varphi(x)| = \|x\|$ imply $\|\varphi\| = 1$, i.e. φ is a state. $\square$

5.6. **Representations.** Having discussed positive linear functionals and states in detail, we now come to a different subject: Representations. In the next definition, we define representations and some related notions.

Recall the definition of a unitary $U : H \to H$ on a Hilbert space from Def. 1.33. From Prop. 1.34 we know that unitaries are exactly isomorphisms of the Hilbert space H. In accordance with Def. 1.17 we adapt the notion and call a map $U : H_1 \to H_2$ between two Hilbert spaces H_1 and H_2 unitary, if U is surjective and $\langle Ux, Uy\rangle_{H_2} = \langle x, y\rangle_{H_1}$ for all $x, y \in H_1$.

Recall the definition of the direct sum of Hilbert spaces: Given Hilbert spaces H_i, $i \in I$, we define $\oplus_{i\in I} H_i$ as the set of all families $(x_i)_{i\in I}$ with $x_i \in H_i$ for $i \in I$, and $\sum_{i\in I} \|x_i\|_{H_i}^2 < \infty$. The inner product is given by $\langle (x_i), (y_i)\rangle := \sum_{i\in I} \langle x_i, y_i\rangle_{H_i}$.

Definition 5.14. Let A be a C^*-algebra.

(a) Let H be a Hilbert space. A *representation* of A on H is a *-homomorphism $\pi : A \to B(H)$.
(b) Two representations $\pi_i : A \to B(H_i)$, $i = 1, 2$ are *equivalent*, if there is a unitary $U : H_1 \to H_2$ such that $\pi_2(x) = U\pi_1(x)U^*$ for all $x \in A$.
(c) Given representations $\pi_i : A \to B(H_i)$, $i \in I$, let $\oplus_{i\in I}\pi_i : A \to B(\oplus_{i\in I} H_i)$ be the representation given by $((\oplus_{i\in I}\pi_i)(a))\,((x_j)_{j\in I}) := ((\pi_j(a))(x_j))_{j\in I}$ for $(x_i)_{i\in I} \in \oplus_{i\in I} H_i$ and $a \in A$.

(d) A representation $\pi : A \to B(H)$ is *non-degenerate*, if we have $\overline{\pi(A)H} = H$ for the closure of the range of π.
(e) A representation $\pi : A \to B(H)$ is *cyclic* if there is some $x \in H$ (a *cyclic vector*) such that $\overline{\pi(A)x} = H$.
(f) Given a closed linear subspace $K \subseteq H$, we say that K is *invariant* under a representation $\pi : A \to B(H)$, if $\pi(A)K \subseteq K$, i.e. $\pi(a)x \in K$ for all $a \in A$ and all $x \in K$.
(g) A representation is *faithful*, if it is injective.

Before we comment a bit on the above definition, let us prove a little lemma.

Lemma 5.15. *Let A be a C^*-algebra, $\pi : A \to B(H)$ a representation and $K \subseteq H$ a closed linear subspace. If K is invariant under π, then also its orthogonal complement $K^\perp$ is invariant under π. We may then write $\pi = \pi_1 \oplus \pi_2$, where $\pi_1 : A \to B(K)$ and $\pi_2 : A \to B(K^\perp)$ are defined as restrictions of π.*

Proof. Let $x \in K$ and $a \in A$. Then $\pi(a^*)x \in K$ since K is invariant under π. Hence, $\langle x, \pi(a)y\rangle = \langle \pi(a^*)x, y\rangle = 0$ for $y \in K^\perp$. As this is true for all $x \in K$, this shows $\pi(a)y \in K^\perp$ for all $y \in K^\perp$. Thus, we have $\pi(a) : K \to K$ and $\pi(a) : K^\perp \to K^\perp$ for the restrictions and thus $\pi = \pi_1 \oplus \pi_2$ on the Hilbert space $H = K \oplus K^\perp$. □

Remark 5.16. (a) The notion in Def. 5.14(b) is an equivalence relation indeed, as can be checked easily.
(b) Non-degeneracy means that π transports vectors to "all of H". Also, any representation $\pi : A \to B(H)$ may be written as a direct sum of a non-degenerate representation and a zero representation.
Indeed, $K := \overline{\pi(A)H} \subseteq H$ is a closed linear subspace which is invariant under π, since $\pi(A)K \subseteq \pi(A)H \subseteq K$. This also shows that the restriction of π to K is non-degenerate. By Lemma 5.15, also $K^\perp$ is invariant and we have $\pi = \pi_{|K} \oplus \pi_{|K^\perp}$. Now, $\pi_{|K^\perp} = 0$, since for $a \in A$ and $x \in K^\perp$, the element $\pi(a)x$ is in $K^\perp$ (by invariance), but also in K (by definition of K). Hence, $\pi(a)x = 0$.
(c) A cyclic representation transports the cyclic vector to any place in H. In particular, cyclic representations are non-degenerate. One may show that any non-degenerate representation $\pi : A \to B(H)$ is the direct sum of cyclic representations. This follows from Zorn's Lemma (note that any vector $x \in H$ is cyclic for the restriction of π to $K := \overline{\pi(A)x} \subseteq H$).
(d) Let $\pi : A \to B(H)$ be a representation. Then $\pi(A) \subseteq B(H)$ is a C^*-subalgebra by Prop. 4.24. If π is faithful, then A is isomorphic to $\pi(A)$.
(e) Let (u_λ) be an approximate unit for A and let $\pi : A \to B(H)$ be a non-degenerate representation. Then $\pi(u_\lambda)x \to x$ for all $x \in H$. Indeed, let $a \in A$ and $y \in H$. Let $x := \pi(a)y$. Then $\pi(u_\lambda)x = \pi(u_\lambda a)y \to \pi(a)y = x$. Since such vectors x are dense in H, thanks to the non-degeneracy, this shows $\pi(u_\lambda)x \to x$ for all $x \in H$. We conclude that $(\pi(u_\lambda))$ approximates the unit on $B(H)$.

5.7. **GNS construction.** Given a representation $\pi : A \to B(H)$ of a C^*-algebra, and a cyclic vector $x \in H$, it is easy to see that

$$\varphi(a) := \langle \pi(a)x, x\rangle, \qquad a \in A,$$

defines a positive linear functional $\varphi : A \to \mathbb{C}$. Moreover, $\|\varphi\| = \|x\|^2$, as a combination of Prop. 5.7 and Rem. 5.16(e). Interestingly, given a positive linear functional, we can also go the way back: We will find a representation such that φ can be written as above. This is the famous GNS construction, which we now prepare. Let us prove a lemma on uniqueness first.

Lemma 5.17. *Let A be a C^*-algebra and let $\pi_i : A \to B(H_i)$, $i = 1, 2$, be two cyclic representations with cyclic vectors $x_i \in H_i$, $i = 1, 2$. Let $\varphi_i : A \to \mathbb{C}$ be the positive linear functionals given by $\varphi_i(a) = \langle \pi_i(a)x_i, x_i\rangle$, $a \in A$. If $\varphi_1 = \varphi_2$, then there exists a unitary $U : H_1 \to H_2$ such that $\pi_2(a) = U\pi_1(a)U^*$ for all $a \in A$, and $Ux_1 = x_2$.*

Proof. Put $H_i^0 := \pi_i(A)x_i \subseteq H_i$, $i = 1, 2$, and let $a \in A$. Then:

$$\|\pi_2(a)x_2\|^2 = \langle \pi_2(a)x_2, \pi_2(a)x_2\rangle = \langle \pi_2(a^*a)x_2, x_2\rangle = \varphi_2(a^*a) = \varphi_1(a^*a) = \|\pi_1(a)x_1\|^2$$

We define $U_0 : H_1^0 \to H_2^0$ by $U_0\pi_1(a)x_1 := \pi_2(a)x_2$, for $a \in A$. This is well-defined, since given some $a, b \in A$ with $\pi_1(a - b)x_1 = 0$, we have $\pi_2(a - b)x_2 = 0$, by the above equation. Moreover, $U_0 : H_1^0 \to H_2^0$ is surjective. Furthermore, U_0 preserves the inner product, as can be seen directly from the polarization identity (Prop. 1.4). Finally, $U_0\pi_1(a)U_0^* = \pi_2(a)$ holds for all $a \in A$, since

$$(U_0\pi_1(a)U_0^*)(\pi_2(b)x_2) = U_0\pi_1(a)\pi_1(b)x_1 = U_0\pi_1(ab)x_1 = \pi_2(ab)x_2 = (\pi_2(a))(\pi_2(b)x_2)$$

for any $b \in A$. Now, H_i^0 is dense in H_i, as x_i is cyclic, for $i = 1, 2$. As U_0 preserves the norm, it can be extended to $U : H_1 \to H_2$. This operator is a unitary with $\pi_2(a) = U\pi_1(a)U^*$ for all $a \in A$, as all these properties hold on a dense subset.

Finally, let (u_λ) be an approximate unit for A. By Rem. 5.16:

$$Ux_1 \leftarrow U\pi_1(u_\lambda)x_1 = U_0\pi_1(u_\lambda)x_1 = \pi_2(u_\lambda)x_2 \to x_2$$ □

We are ready for the GNS construction. It can be found in the 1943 Gelfand-Naimark article [19], but it has been refined by Segal a few years later; the letters GNS stand for Gelfand-Naimark-Segal.

Theorem 5.18. *Let A be a C^*-algebra and let $\varphi : A \to \mathbb{C}$ be a state. There are a Hilbert space H_φ, a representation $\pi_\varphi : A \to B(H_\varphi)$, and a cyclic vector $x_\varphi \in H_\varphi$ such that $\varphi(a) = \langle \pi_\varphi(a)x_\varphi, x_\varphi\rangle$ for all $a \in A$. With these properties, the triple $(H_\varphi, \pi_\varphi, x_\varphi)$ is unique up to equivalence (by Lemma 5.17).*

Proof. We construct the triple $(H_\varphi, \pi_\varphi, x_\varphi)$ step by step.

(1) Construction of the Hilbert space H_φ. We consider the C^*-algebra A as a vector space. The state φ induces a positive sesquilinear form $\langle x, y\rangle := \varphi(y^*x)$ on A, by Lemma 5.3. It might fail to satisfy the implication from $\langle x, x\rangle = 0$ to $x = 0$, so let us mod out the bad elements: Consider the closed linear subspace

$N_\varphi := \{x \in A \mid \langle x, x\rangle = 0\} \subseteq A$ and let $K_\varphi := A/N_\varphi$ be the quotient space, $\gamma : A \to K_\varphi$ be the quotient map.

Then K_φ is a pre Hilbert space with $\langle \gamma(x), \gamma(y)\rangle := \langle x, y\rangle$. Using Cauchy-Schwarz, we infer that this is a well-defined inner product, see Exc. 5.1. Besides, note that the quotient map is continuous, since $\|\gamma(x)\|^2 = \varphi(x^*x) \leq \|x\|^2$, by Lemma 5.4. Finally, the completion of K_φ is our Hilbert space H_φ. Summarizing:

$$H_\varphi := \overline{K_\varphi}^{\|\cdot\|} = \overline{A/\{x \in A \mid \varphi(x^*x) = 0\}}^{\|\cdot\|}$$

(2) Construction of the representation $\pi_\varphi : A \to B(H_\varphi)$. The most natural representation of A on itself is left multiplication. As H_φ is basically A itself (up to the defect N_φ), the idea is to define π_φ as a left multiplication operator, which goes as follows. For $a \in A$, we first consider

$$\pi^0_\varphi(a) : K_\varphi \to K_\varphi, \qquad \pi^0_\varphi(a)(\gamma(y)) := \gamma(ay), y \in A.$$

Note that $a^*a \leq \|a^*a\|1$ (in $\tilde{A}$) by Prop. 4.12(b), and hence $y^*a^*ay \leq \|a^*a\|y^*y$ (in A) by Prop. 4.12(a). Thus, the map $\pi^0_\varphi(a)$ is continuous, as

$$\|\pi^0_\varphi(a)(\gamma(y))\|^2 = \|\gamma(ay)\|^2 = \varphi(y^*a^*ay) \leq \|a^*a\|\varphi(y^*y) = \|a\|^2\|\gamma(y)\|^2,$$

for all $y \in A$; this shows $\|\pi^0_\varphi(a)\| \leq \|a\|$ for all $a \in A$. In particular, $\pi^0_\varphi(a)$ is well-defined: If $\gamma(y) = \gamma(y')$, then

$$\|\gamma(ay) - \gamma(ay')\|^2 = \|\gamma(a(y - y'))\|^2 \leq \|a\|^2\|\gamma(y - y')\|^2 = 0.$$

Moreover, $a \mapsto \pi^0_\varphi(a)$ is obviously linear and multiplicative. It is involutive, as

$$\langle\pi^0_\varphi(a)\gamma(y), \gamma(z)\rangle = \langle\gamma(ay), \gamma(z)\rangle = \varphi(z^*ay) = \varphi((a^*z)^*y) = \langle\gamma(y), \pi^0_\varphi(a^*)\gamma(z)\rangle,$$

for all $y, z \in A$; this shows $\pi^0_\varphi(a)^* = \pi^0_\varphi(a^*)$ for all $a \in A$. Finally, for $a \in A$, we extend $\pi^0_\varphi(a) : K_\varphi \to K_\varphi$ to $\pi_\varphi(a) : H_\varphi \to H_\varphi$. Then $\|\pi_\varphi(a)\| \leq \|a\|$. Summarizing,

$$\pi_\varphi : A \to B(H_\varphi), \qquad a \mapsto \pi_\varphi(a), \qquad \text{where } \pi_\varphi(a)(\gamma(y)) = \gamma(ay), \qquad y \in A,$$

is a *-homomorphism, i.e. it is a representation of A on H_φ.

(3) Construction of the cyclic vector $x_\varphi \in H_\varphi$. Let $(u_\lambda)_{\lambda\in\Lambda}$ be an approximate unit for A (which exists by Thm. 4.20). Then $(\gamma(u_\lambda))_{\lambda\in\Lambda}$ is a Cauchy net in K_φ. Indeed, let $\lambda \geq \mu$. Then $u_\lambda \geq u_\mu$ by Def. 4.17 and we have $1 \geq u_\lambda \geq u_\lambda - u_\mu \geq 0$. We infer $(u_\lambda - u_\mu)^2 \leq u_\lambda - u_\mu$ by functional calculus. As φ is order preserving, this implies

$$\|\gamma(u_\lambda) - \gamma(u_\mu)\|^2 = \|\gamma(u_\lambda - u_\mu)\|^2 = \varphi((u_\lambda - u_\mu)^2) \leq \varphi(u_\lambda - u_\mu).$$

Since $(\varphi(u_\lambda))_{\lambda\in\Lambda}$ is an increasing bounded net in $\mathbb{C}$, see the proof of Prop. 5.7, we conclude that $(\gamma(u_\lambda))_{\lambda\in\Lambda}$ is a Cauchy net in K_φ. Hence, we may define:

$$x_\varphi := \lim_\lambda \gamma(u_\lambda) \in H_\varphi$$

Now, for any $a \in A$, the net $\pi_\varphi(a)\gamma(u_\lambda) = \gamma(au_\lambda)$ converges to $\gamma(a)$, from which we deduce $\pi_\varphi(a)x_\varphi = \gamma(a)$ for all $a \in A$. Thus, $K_\varphi \subseteq \pi_\varphi(A)x_\varphi$, which shows that x_φ is a cyclic vector.

(4) We have $\varphi(a) = \langle \pi_\varphi(a)x_\varphi, x_\varphi\rangle$ for all $a \in A$. Indeed, we check

$$\langle \pi_\varphi(a)x_\varphi, x_\varphi\rangle = \lim_\lambda \langle \pi_\varphi(a)\gamma(u_\lambda), \gamma(u_\lambda)\rangle = \lim_\lambda \varphi(u_\lambda a u_\lambda) = \varphi(a)$$

for all $a \in A$. $\square$

We thus have a one-to-one correspondence between states and cyclic representations (up to equivalence classes of representations).

5.8. **Existence of faithful representations.** We derive the Second Fundamental Theorem of C^*-algebras as a consequence of the GNS construction. As in Lecture 3, note that the naming as "Fundamental Theorem" is not common – we only use it here in these lectures. The theorem below is sometimes called *(noncommutative) Gelfand-Naimark Theorem* or *2nd Gelfand-Naimark Theorem.*

Theorem 5.19. *Any C^*-algebra A possesses a faithful representation $\pi : A \to B(H)$ on some Hilbert space H. Thus, A is isomorphic to a C^*-subalgebra of $B(H)$.*

Proof. Let $a \in A$ with $a \neq 0$. Then $a^*a \in A$ is normal and by Thm. 5.13, we find a state $\varphi : A \to \mathbb{C}$ with $\varphi(a^*a) = \|a^*a\| = \|a\|^2 \neq 0$. By the GNS construction (Thm. 5.18), we obtain a Hilbert space H_φ and a representation $\pi_\varphi : A \to B(H_\varphi)$ with a cyclic vector $x_\varphi \in H_\varphi$ such that

$$\|\pi_\varphi(a)x_\varphi\|^2 = \langle \pi_\varphi(a^*a)x_\varphi, x_\varphi\rangle = \varphi(a^*a) \neq 0.$$

Hence, $\pi_\varphi(a) \neq 0$.

We then put $H := \oplus_{\varphi \in S(A)} H_\varphi$ and $\pi := \oplus_{\varphi\in S(A)}\pi_\varphi$, where $S(A)$ is the set of all states on A. By the previous consideration, we have $\pi(a) \neq 0$ for all $a \in A$ with $a \neq 0$, i.e. π is faithful. By Prop. 4.24, A is then isomorphic to $\pi(A) \subseteq B(H)$. $\square$

We conclude that any C^*-algebra may be represented concretely on a Hilbert space. This is a quite deep insight, for various reasons. On a practical level, it allows us to add a Hilbert space structure to our given C^*-algebra, if we need it. Thus, we may also use Hilbert space techniques when working with C^*-algebras. On a more philosophical level, it means that we could also define C^*-algebras as norm closed *-subalgebras of $B(H)$ – and this definition would be equivalent to the more abstract, axiomatic one given in Def. 2.1. Some remarks on these possible definitions and the history behind the GNS construction may be found in Sect. 5.10.

Remark 5.20. If A is a separable C^*-algebra, then there is a faithful representation $\pi : A \to B(H)$ on a *separable* Hilbert space H, see Exc. 5.5.

Remark 5.21. In Def. 5.14, we learned that a closed subspace $K \subseteq H$ of a Hilbert space is invariant under a representation $\pi : A \to B(H)$, if $\pi(A)K \subseteq K$. In that case, we say that K reduces π. If π has no reducing subspaces (i.e. the only invariant subspaces are 0 and H), we say that π is irreducible.

The theory of irreducible representations is a subject of its own and we could easily devote a whole lecture to it. However, we decided to skip this part of the theory as it won't be really necessary for the remainder of the lectures. Just to mention a few facts on irreducible representations [3]: We can write representations as direct sums of irreducible representations; there is a definition of a pure state and the corresponding GNS construction yields an irreducible representation; on $C(X)$, the pure states correspond exactly to Dirac measures, see also Exm. 5.2(a); the pure states (together with 0) form the extremal points of all positive linear functionals with norm less or equal to one, on a given C^*-algebra; by the Krein-Milman Theorem, we may show that for any non-zero element $x \in A$ in a C^*-algebra A, there is an irreducible representation π such that $\|\pi(x)\| = \|x\|$, in analogy to Thm. 5.13 and Thm. 5.19.

5.9. **Exercises.**

Exercise 5.1. Verify the details of the proof of Lemma 5.3: Let A be a C^*-algebra and let $\varphi : A \to \mathbb{C}$ be a positive linear functional. Put $\langle x, y\rangle := \varphi(y^*x)$, for $x, y \in A$.

(a) Check that $\langle \cdot, \cdot \rangle$ is a positive sesquilinear form.
(b) Check that $\langle \cdot, \cdot \rangle$ satisfies the polarisation identity and that we have $\overline{\langle x, y\rangle} = \langle y, x\rangle$ for all $x, y \in A$. (This holds true for any positive sesquilinear form.)
(c) Convince yourself of a Cauchy-Schwarz inequality in the following form: $|\langle x, y\rangle|^2 \leq \varphi(x^*x)\varphi(y^*y)$, for all $x, y \in A$.
(d) Show that $N_\varphi := \{x \in A \mid \langle x, x\rangle = 0\} \subseteq A$ is a closed linear subspace of A.
(e) Let $\gamma : A \to A/N_\varphi$ be the quotient map. Let $x, x' \in A$ with $x - x' \in N_\varphi$. Show that $\langle x, y\rangle = \langle x', y\rangle$ for all $y \in A$ using (c). Deduce that $\langle \gamma(x), \gamma(y)\rangle := \langle x, y\rangle$ is a well-defined inner product on A/N_φ.

Exercise 5.2. Let A be a C^*-algebra and let $\varphi : A \to \mathbb{C}$ be a state. Let $\pi_\varphi : A \to B(H_\varphi)$ be the associated GNS representation. A state φ is called faithful, if $\varphi(x^*x) = 0$ implies $x = 0$, for $x \in A$.

(a) Let $I \lhd A$ be a closed ideal. Show that $I \subseteq \ker \pi_\varphi$ holds if and only if $I \subseteq \ker \varphi$.
(b) Show that π_φ is faithful, if φ is faithful.

Exercise 5.3. Let $\mathrm{tr} : M_N(\mathbb{C}) \to \mathbb{C}$, $(t_{ij}) \mapsto \frac{1}{N}\sum_i t_{ii}$ be the normalized trace as in Exm. 5.2. Let $B \in M_N(\mathbb{C})$. We define $\tau_B(T) := \mathrm{tr}(BT)$ for $T \in M_N(\mathbb{C})$.

(a) Let B be positive. Show that τ_B is a positive linear functional with $\|\tau_B\| = \mathrm{tr}(B)$.
(b) Compute the values $\tau_B(E_{ij})$, $i, j = 1, \ldots, N$, where $E_{ij} \in M_N(\mathbb{C})$ are the matrix units, i.e. the i-j-th entry of E_{ij} is one, and zero otherwise.
(c) Let τ be a positive linear functional on $M_N(\mathbb{C})$. Show that $\tau = \tau_B$ for some matrix $B \in M_N(\mathbb{C})$ (in fact, it can be shown that B must be positive). Use (b) in order to find B.
(d) Characterize all states on $M_N(\mathbb{C})$.

(e) Show that the trace tr on $M_N(\mathbb{C})$ is a faithful state.

Exercise 5.4. Consider $A = M_N(\mathbb{C})$ and its trace $\mathrm{tr} : M_N(\mathbb{C}) \to \mathbb{C}$.

(a) What does the GNS construction yield for tr? Determine all components: $(H_{\mathrm{tr}}, \pi_{\mathrm{tr}}, x_{\mathrm{tr}})$. Is this GNS representation faithful?
(b) Let $B = E_{11} \in M_N(\mathbb{C})$, see Exc. 5.3 for a definition. Consider τ_B as in Exc. 5.3. Determine all components of its GNS construction. Is this GNS representation faithful? Is τ_B faithful? Compare with Exc. 5.2.

Exercise 5.5. Let A be a separable C^*-algebra, i.e. there is a countable dense subset $\{a_n \mid n \in \mathbb{N}\} \subseteq A$. For $n \in \mathbb{N}$, let φ_n be a state with $\varphi_n(a_n^* a_n) = \|a_n\|^2$, by Thm. 5.13. Let (H_n, π_n, x_n) be the corresponding GNS construction and put $H := \oplus_{n\in\mathbb{N}} H_n$, $\pi := \oplus_{n\in\mathbb{N}} \pi_n$.

(a) Show that H_n is separable, for $n \in \mathbb{N}$ and deduce that H is separable.
(b) Given $a \in A$ and $\varepsilon < \frac{1}{2}\|a\|$, let $n \in \mathbb{N}$ such that $\|a - a_n\| < \varepsilon$. Show that $\|\pi_n(a)\| > 0$.
(c) Deduce that $\pi : A \to B(H)$ is faithful.

5.10. **Comments on the Second Fundamental Theorem of C^*-algebras.** Let us comment a bit on the history of the Second Fundamental Theorem of C^*-algebras. Representing certain algebras on Hilbert spaces (i.e. as subalgebras of some $B(H)$) is an old business. For instance, let G be a locally compact group. If G is abelian, its dual, consisting in all continuous group homomorphisms $\varphi : G \to S^1 \subseteq \mathbb{C}$ (characters) forms a locally compact group again, the so called dual group $\hat{G}$. Here, S^1 is seen as a multiplicative group. Now, we can take the dual group of the dual group – and we obtain $\hat{\hat{G}} \cong G$, i.e. we may reconstruct G from its dual group. This is the famous Pontryagin Duality.

Now, how about a non-abelian locally compact group G? Unfortunately, the duality principle breaks down. So, we need to come up with some more sophisticated notion of a dual, if we want to have some object from which we may reconstruct G. One idea is to replace group homomorphisms $\varphi : G \to \mathbb{C}$ by group homomorphisms $\varphi : G \to M_N(\mathbb{C})$. Or $\varphi : G \to B(H)$, if you want. The philosophy is then to "understand" G by "understanding" all of its representations.

One may associate a group algebra $\mathbb{C}G$ to G. Interestingly, representations of G on $B(H)$ correspond to representations of $\mathbb{C}G$ on $B(H)$. In other words: The representation theory of groups (on Hilbert spaces) boils down to the representation theory of group *algebras* on Hilbert spaces.

This was the starting point for Gelfand's work on C^*-algebras: Being interested in the representation theory of groups on Hilbert spaces, he wanted to understand the theory of subalgebras of $B(H)$. As we are dealing with unitary representations, these subalgebras should be closed under taking adjoints (i.e. they are *-subalgebras) and since we are interested in topological groups, we also want to require some topological closure. Choosing the operator norm topology, we end up with: C^*-algebras! These are the right subalgebras of $B(H)$ to consider. By the way, choosing the closure in the weak or the strong operator topology, we obtain von Neumann algebras.

Summarizing: From the perspective of representation theory of groups, we might be interested in norm closed *-subalgebras of $B(H)$. Shouldn't this be our definition of a C^*-algebra then? Why do we define C^*-algebras in an abstract way instead, as certain *-Banach algebras obeying some strange C^*-identity? Because it is more conceptual in its axiomatic nature – and it is equivalent!

The latter is the content of Thm. 5.19: Given any abstractly defined C^*-algebra (in the sense of Def. 2.1), it is isomorphic to some concrete C^*-algebra, i.e. to some C^*-subalgebra of some $B(H)$. Conversely, any such concrete C^*-algebra is also an abstract C^*-algebra in the sense of Def. 2.1, see Prop. 1.30. We conclude: The axiomatic definition of C^*-algebras (Def. 2.1) and the concrete one (as norm closed *-subalgebra of some $B(H)$) are equivalent.

In that respect, one could say in retrospective: the merit of the seminal Gelfand-Naimark article [19] is to provide an alternative, axiomatic definition of C^*-algebras – which is equivalent to the concrete definition (2nd Fundamental Theorem). And it includes classical topology in form of commutative algebras (1st Fundamental Theorem).

6. Universal C^*-algebras

Abstract. We introduce the concept of universal C^*-algebras. We show that the following C^*-algebras may be viewed as universal C^*-algebras: The algebra of functions on the circle S^1, the matrix algebras $M_N(\mathbb{C})$, the algebra of compact operators $\mathcal{K}(H)$ on a separable Hilbert space, as well as the so-called Toeplitz algebra. We explain how the latter one may be seen as an extension of the function algebra $C(S^1)$ by the compact operators $\mathcal{K}(H)$.

6.1. **Definition of universal C^*-algebras.** In this lecture, we turn to a quite modern way of dealing with C^*-algebras: universal C^*-algebras. First, recall the definition of a C^*-algebra (Def. 2.1): a C^*-algebra is a complex algebra equipped with an involution; moreover, it posseses a norm which is submultiplicative and which satisfies the C^*-identity; finally, the algebra is complete with respect to this norm. This provides us with a recipe how to cook up C^*-algebras abstractly: we begin with a *-algebra; we find a good norm on it; and we complete. Let's do this systematically in terms of generators and relations.

Definition 6.1. Let elements $E = \{x_i \mid i \in I\}$ be given, where I is some index set.

(a) A *noncommutative monomial* in E is a word $x_{i_1} \cdots x_{i_m}$ with $i_1, \ldots, i_m \in I$ and $m \in \mathbb{N}\backslash\{0\}$.
(b) A *noncommutative polynomial* in E is a formal complex linear combination of noncommutative monomials: $\sum_{k=1}^{N} \alpha_k y_k$ with $N \in \mathbb{N}$, $\alpha_k \in \mathbb{C}$ and $y_1, \ldots, y_N$ being noncommutative monomials in E.
(c) On noncommutative monomials, we consider the *concatenation of words*, i.e.
$$(x_{i_1} \cdots x_{i_m}) \cdot (x_{j_1} \cdots x_{j_n}) := x_{i_1} \cdots x_{i_m} x_{j_1} \cdots x_{j_n},$$
where $x_{i_1} \cdots x_{i_m}$ and $x_{j_1} \cdots x_{j_n}$ are two monomials.
(d) The *free (complex) algebra on the generator set* E is given as the set of noncommutative polynomials in E together with the canonical addition and scalar multiplication, and the multiplication of elements given by the concatenation. The elements in E are understood as being distinct.

Note that the order of the elements plays a role for such noncommutative monomials, i.e. $x_1 x_2 \neq x_2 x_1$ in the free algebra. Moreover, the algebra is "free" in the sense that the elements x_i satisfy no relations, i.e. the only polynomial in the generators which is zero, is the zero polynomial itself. Hence, the free algebra has the following universal property: Whenever B is some algebra containing elements $\{y_i \mid i \in I\}$ (where we even allow $y_i = y_j$ for some $i, j \in I$), there is a replacement homomorphism from the free algebra to B sending x_i to y_i, for all $i \in I$.

Given $E = \{x_i \mid i \in I\}$, we add another set (disjoint with E) of generators $E^* := \{x_i^* \mid i \in I\}$ and we define an involution on the free algebra on $E \cup E^*$ by extending
$$(\alpha x_{i_1}^{\varepsilon_1} \cdots x_{i_m}^{\varepsilon_m})^* := \bar{\alpha} x_{i_m}^{\bar{\varepsilon}_m} \cdots x_{i_1}^{\bar{\varepsilon}_1}$$

to linear combinations; here $\alpha \in \mathbb{C}$, $\varepsilon_k \in \{1, *\}$ and $\bar{\varepsilon}_k := \begin{cases} 1 & \text{if } \varepsilon_k = * \\ * & \text{if } \varepsilon_k = 1 \end{cases}$. In this way, we obtain the *free* **-algebra* $P(E)$ on the generator set E. Note that any polynomial $p \in P(E)$ can be viewed as an algebraic relation when considering the equation $p = 0$, see also Exm. 6.3.

Definition 6.2. We consider the following data:

(i) Let $E = \{x_i \mid i \in I\}$ be a set of elements, I some index set.
(ii) Let $R \subseteq P(E)$ be a set of polynomials.

Let $J(R) \subseteq P(E)$ be the two-sided *-ideal generated by R. The *universal* **-algebra* with *generators* E and *relations* R is defined as the quotient $A(E \mid R) := P(E)/J(R)$.

The image of an element $x_i \in E$ in $A(E \mid R)$ is denoted by x_i again, by some slight (but very common) abuse of notation.

Example 6.3. Let $E = \{x\}$ and $R = \{x^2, xx^*x - x\} \subseteq P(E)$. Then $A(E \mid R)$ is the universal *-algebra with generator x such that the relations $x^2 = 0$ and $xx^*x = x$ hold. Using these relations, we see that the only monomials in $A(E \mid R)$ are x, x^*, xx^* and x^*x, hence $A(E \mid R)$ is at most four-dimensional.

We now want to find a C^*-norm on $A(E \mid R)$. Let's consider C^*-seminorms first.

Definition 6.4. Let A be a *-algebra (i.e. a complex algebra with an involution). A C^**-seminorm* on A is a map $p : A \to [0, \infty)$ such that

(i) $p(\lambda x) = |\lambda| p(x)$ and $p(x + y) \le p(x) + p(y)$ for all $x, y \in A$, $\lambda \in \mathbb{C}$,
(ii) $p(xy) \le p(x)p(y)$ for all $x, y \in A$,
(iii) and $p(x^*x) = p(x)^2$ for all $x \in A$.

We are now ready for the main definition of today's lecture.

Definition 6.5. Let E be a set of generators and $R \subseteq P(E)$ be relations. Put

$$\|x\| := \sup\{p(x) \mid p \text{ is a } C^*\text{-seminorm on } A(E \mid R)\}.$$

If $\|x\| < \infty$ for all $x \in A(E \mid R)$, it is not difficult to show that $\|\cdot\|$ is a C^*-seminorm (see Lemma 6.6) and $\{x \in A(E \mid R) \mid \|x\| = 0\}$ is a two-sided *-ideal. In that case (i.e. if $\|x\| < \infty$ for all $x \in A(E \mid R)$), we define the *universal* C^**-algebra* $C^*(E \mid R)$ as the completion with respect to $\|\cdot\|$:

$$C^*(E \mid R) := \overline{A(E \mid R)/\{x \in A(E \mid R) \mid \|x\| = 0\}}^{\|\cdot\|}$$

In the same way, we may define enveloping C^*-algebras for any given *-algebra A. Let us quickly check that our definition makes sense.

Lemma 6.6. *Let $E = \{x_i \mid i \in I\}$ be a set of generators and $R \subseteq P(E)$ be relations.*

(a) *If $\|x\| < \infty$ for all $x \in A(E \mid R)$, then $C^*(E \mid R)$ is a C^*-algebra and we say that the universal C^*-algebra of E and R exists.*

(b) *If there is a constant $C > 0$ such that $p(x_i) < C$ for all $i \in I$ and all C^*-seminorms p on $A(E \mid R)$, then $\|x\| < \infty$ for all $x \in A(E \mid R)$.*

Proof. By definition, it is clear that $\|\cdot\|$ is a C^*-seminorm; hence we obtain a norm on the quotient $A(E \mid R)/\{x \in A(E \mid R) \mid \|x\| = 0\}$ and the completion yields a C^*-algebra. This shows (a). As for (b), the norm of any monomial in E of length N is bounded by C^N and hence any polynomial in $A(E \mid R)$ has bounded norm. □

6.2. **The universal property.** So, we have some criterion for the existence of $C^*(E \mid R)$, see Lemma 6.6. However, it could still be the case, that the construction yields the trivial C^*-algebra: we could have $C^*(E \mid R) = 0$. In order to exclude triviality, we need to find a non-trivial *-homomorphism from our universal C^*-algebra to another (non-trivial) C^*-algebra. For this the following property is very useful, ensuring the existence of many *-homomorphisms.

Let $E = \{x_i \mid i \in I\}$ be a set of generators and $R \subseteq P(E)$ be relations. We say that elements $\{y_i \mid i \in I\}$ in some *-algebra B satisfy the relations R, if all polynomials $p \in R$ are zero, when we replace each x_i by y_i, for all $i \in I$.

Proposition 6.7. *Let $E = \{x_i \mid i \in I\}$ be generators and $R \subseteq P(E)$ be relations such that the universal C^*-algebra $C^*(E \mid R)$ exists. Let B be a C^*-algebra containing a subset $E' = \{y_i \mid i \in I\}$. If the elements E' satisfy the relations R, then there is a unique *-homomorphism $\varphi : C^*(E \mid R) \to B$ sending x_i to y_i, for all $i \in I$.*

Proof. The two-sided *-ideal generated by R vanishes in B by assumption. Hence, the replacement homomorphism from the free *-algebra $P(E)$ to B, sending $x_i \in P(E)$ to $y_i \in B$, for all $i \in I$, induces a *-homomorphism $\varphi_0 : A(E \mid R) \to B$ sending $x_i \in A(E \mid R)$ to $y_i \in B$, for all $i \in I$. For $x \in A(E \mid R)$, put $p(x) := \|\varphi_0(x)\|_B$. This is a C^*-seminorm and we conclude $\|\varphi_0(x)\|_B \leq \|x\|$, by Def. 6.5. Hence, φ_0 is continuous and we may extend it to a *-homomorphism $\varphi : C^*(E \mid R) \to B$. Uniqueness is by Lemma 3.26. □

From the proof above, we understand in retrospective, why we defined the norm in Def. 6.5 exactly this way: any *-homomorphism $\varphi : A \to B$ between C^*-algebras yields a C^*-seminorm $p(x) := \|\varphi(x)\|$ on A, where $x \in A$. So, since *-homomorphisms on C^*-algebras always satisfy $\|\varphi(x)\| \leq \|x\|$, see Lemma 3.8, the only way to obtain a "universal" norm on a *-algebra is by taking the supremum of all C^*-seminorms.

Example 6.8. Let us revisit Exm. 6.3 and consider the universal C^*-algebra generated by $E = \{x\}$ and the relations $R = \{x^2, xx^*x - x\}$. We write shorthand $C^*(x \mid x^2 = 0, xx^*x = x)$. Since $p(x)^2 = p(x^*x) = p(x^*xx^*x) = p(x^*x)^2 = p(x)^4$ for any C^*-seminorm p on $A(x \mid x^2 = 0, xx^*x = x)$ – and hence $p(x) \in \{0, 1\}$ – we infer that $C^*(x \mid x^2 = 0, xx^*x = x)$ exists, by Lemma 6.6(b).

Now, consider $y := E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{C})$. Since $y^2 = 0$ and $yy^*y = y$, there is a *-homomorphism $\varphi : C^*(x \mid x^2 = 0, xx^*x = x) \to M_2(\mathbb{C})$ mapping x to y, by the universal property (Prop. 6.7). Moreover, φ is surjective, as it hits all matrix units (see Exc 2.5): $\varphi(x) = E_{12}$, $\varphi(x^*) = E_{21}$, $\varphi(x^*x) = E_{22}$ and $\varphi(xx^*) = E_{11}$. But as $C^*(x \mid x^2 = 0, xx^*x = x)$ is (at most) four-dimensional, see Exm. 6.3, we conclude that φ is an isomorphism, i.e. $M_2(\mathbb{C})$ may be written as the universal C^*-algebra $C^*(x \mid x^2 = 0, xx^*x = x)$. In particular, this shows $C^*(x \mid x^2 = 0, xx^*x = x) \neq 0$.

Let us interpret this example a bit. Suppose we are interested in (order 2) nilpotent partial isometries for some reason. We compute the universal C^*-algebra of a single order 2 nilpotent partial isometry: it is $M_2(\mathbb{C})$. What does this tell us? Well, note that $M_2(\mathbb{C})$ is simple (Exc. 2.5) – so, the universal C^*-algebra generated by such a nilpotent partial isometry is simple. This means: We may not add any relations to a order 2 nilpotent partial isometry – whenever we have a order 2 nilpotent partial isometry x satisfying further relations in x (which are not implied by the relations $x^2 = 0$ and $xx^*x = x$), it must be trivial, see also Exm. 6.10. We conclude: the knowledge of the universal C^*-algebra tells us which additional relations may (or may not) hold, encoded in the ideal structure of the C^*-algebra; if the C^*-algebra is simple, we reached the end – no further relations may be added.

We now come to two non-examples.

Example 6.9. The C^*-algebra $C^*(x \mid x = x^*)$ does *not* exist: For any $\lambda > 0$, we find a C^*-algebra B and a selfadjoint element $y \in B$ with $\|y\|_B = \lambda$. Then $p(\sum_{k=1}^N \alpha_k x^k) := \|\sum_{k=1}^N \alpha_k y^k\|_B$, $N \in \mathbb{N}$, $\alpha_k \in \mathbb{C}$, defines a C^*-seminorm on $A(x \mid x = x^*)$ with $p(x) = \lambda$. Hence $\|x\| = \infty$ in Def. 6.5.

Example 6.10. The C^*-algebra $C^*(x \mid x^2 = 0, xx^*x = x, x = x^*)$ exists (all C^*-seminorms are bounded, see Exm. 6.8) – but $C^*(x \mid x^2 = 0, xx^*x = x, x = x^*) = 0$, since $\|x\|^2 = \|x^*x\| = \|x^2\| = 0$.

6.3. **Example: matrix algebras.** The remainder of this lecture is devoted to the study of further examples of universal C^*-algebras. Our goal is to write well-known C^*-algebras as universal C^*-algebras. We have seen that $M_2(\mathbb{C})$ can be written as a universal C^*-algebra, see Exm. 6.8. How about $M_N(\mathbb{C})$ in general?

As before, denote by $E_{ij} \in M_N(\mathbb{C})$, $i, j = 1, \ldots, N$ the matrix units, i.e. the i-j-th entry of E_{ij} is one and it is zero otherwise.

Proposition 6.11. *Let $N \geq 2$. The following C^*-algebras are isomorphic.*

(i) $M_N(\mathbb{C})$
(ii) $C^*(e_{ij}, i, j = 1, \ldots, N \mid e_{ij}^* = e_{ji},\ e_{ij}e_{kl} = \delta_{jk}e_{il}$ for all $i, j, k, l)$
(iii) $C^*(x_i, i = 1, \ldots, N \mid x_i^*x_j = \delta_{ij}x_1$ for all $i, j)$

Proof. Denote the C^*-algebra in (ii) by A_1 and the one in (iii) by A_2. We first check that A_1 exists: We have $p(e_{jj})^2 = p(e_{jj}^*e_{jj}) = p(e_{jj}) \in \{0, 1\}$ for all j and thus $p(e_{ij})^2 = p(e_{ji}e_{ij}) = p(e_{jj}) \leq 1$ for all i, j and all C^*-seminorms p. Let us now show $M_N(\mathbb{C}) \cong A_1$. It is easy to check that the matrix units $E_{ij} \in M_N(\mathbb{C})$ satisfy the relations of A_1; by the universal property (Prop. 6.7) we thus obtain a surjective *-homomorphism $\varphi : A_1 \to M_N(\mathbb{C})$ sending e_{ij} to E_{ij}, for all i, j. The monomials in

A_1 are exactly the elements e_{ij}, hence A_1 is N^2-dimensional and we conclude that φ is also injective.

As for $A_1 \cong A_2$, see Exc. 6.1 for details: the universal property of A_1 yields a *-homomorphism $\varphi : A_1 \to A_2$ sending e_{ij} to $x_i x_j^*$ while the universal property of A_2 yields a *-homomorphism $\psi : A_2 \to A_1$ sending x_i to e_{i1}. The homomorphisms are inverse to each other, which shows the isomorphism. In particular, the elements x_i correspond to $E_{i1} \in M_N(\mathbb{C})$. □

Corollary 6.12. *Let B be any C^*-algebra with $f_{ij} \in B$, $i, j = 1, \ldots, N$ satisfying $f_{ij}^* = f_{ji} \neq 0$ and $f_{ij} f_{kl} = \delta_{jk} f_{il}$ for all i, j, k, l. Let $B' := C^*(f_{ij}, i, j = 1, \ldots, N) \subseteq B$ be the C^*-subalgebra generated by the elements f_{ij}. Then $B' \cong M_N(\mathbb{C})$.*

Likewise, if D is a C^*-algebra containing elements $y_1, \ldots, y_N \in D$ with $y_i^* y_j = \delta_{ij} y_1 \neq 0$ for all i, j, then $C^*(y_1, \ldots, y_N) \cong M_N(\mathbb{C})$.

Proof. In Exc. 2.5, we have seen that $M_N(\mathbb{C})$ is simple, i.e. the only closed ideals in $M_N(\mathbb{C})$ are $\{0\}$ and $M_N(\mathbb{C})$ itself. Now, given a C^*-algebra B with the asserted properties, we find a *-homomorphism $\varphi : M_N(\mathbb{C}) \to B$ sending E_{ij} to f_{ij}, for all i, j, by Prop. 6.11 and Prop. 6.7. The kernel $\ker \varphi$ is an ideal in $M_N(\mathbb{C})$, hence $\ker \varphi = \{0\}$. Likewise for the statement on D. □

6.4. **Example: algebra of compact operators on a separable Hilbert space.** There is an infinite analog of Prop. 6.11.

Proposition 6.13. *The following C^*-algebras are isomorphic.*

(i) *The algebra of compact operators $\mathcal{K}(H)$ on a separable Hilbert space H.*
(ii) $C^*(e_{ij}, i, j \in \mathbb{N} \mid e_{ij}^* = e_{ji},\ e_{ij} e_{kl} = \delta_{jk} e_{il} \text{ for all } i, j, k, l)$
(iii) $C^*(x_i, i \in \mathbb{N} \mid x_i^* x_j = \delta_{ij} x_1 \text{ for all } i, j)$

Proof. We leave the verification of the existence of the universal C^*-algebras as an exercise, and the isomorphism with the C^*-algebra in item (iii) as well, see Exc. 6.1. Denote the C^*-algebra in (ii) by A. We prove $\mathcal{K}(H) \cong A$.

Let $(e_n)_{\mathbb{N}}$ be an orthonormal basis of H. For $i, j \in \mathbb{N}$, let $f_{ij} \in B(H)$ be the operator given by $f_{ij} e_n := \delta_{jn} e_i$, for $n \in \mathbb{N}$. (Observe that f_{ij} is an infinite analog of a matrix unit: while E_{ij} maps the j-th basis vector of $\mathbb{C}^N$ to the i-th basis vector, f_{ij} does the same in H.) Then $f_{ij} \in \mathcal{K}(H)$, since its range is one-dimensional. By the universal property, there is a *-homomorphism $\varphi : A \to \mathcal{K}(H)$ mapping e_{ij} to f_{ij}, for all i, j. Let us show that φ is an isomorphism.

The image of φ contains all linear combinations of the maps f_{ij} – in fact, even all limits of such linear combinations, by Prop. 4.24. Now, any compact operator may be approximated by limits of linear combinations of the maps f_{ij}, see Exc. 6.2. We conclude that φ is surjective.

As for injectivity, put $M_N := C^*(e_{ij}, i, j = 1, \ldots, N) \subseteq A$, for $N \in \mathbb{N}$. Then, $M_N \cong M_N(\mathbb{C})$, by Cor. 6.12. Let φ_N be the restriction of φ to $M_N \subseteq A$. Then $\varphi_N \neq 0$ is injective, since its kernel is an ideal in $M_N \cong M_N(\mathbb{C})$ and $M_N(\mathbb{C})$ is simple. Thus, φ_N is isometric, by Prop. 4.15. Hence, φ is isometric on the dense subset $\cup_{N\in\mathbb{N}} M_N \subseteq A$. This implies that φ is also isometric on all of A, i.e. φ is injective. $\square$

Similarly to $M_N(\mathbb{C})$, one can show that $\mathcal{K}(H)$ is simple, given H is separable, see Exc. 6.3. Thus, an analog of Cor. 6.12 holds true, as follows. Note that we allow ourselves to consider any countable, infinite set I as an indexing set, since such a set is in bijection with the indexing set $\mathbb{N}$ of Prop. 6.13.

Corollary 6.14. *If B is a C^*-algebra with $f_{ij} \in B$, $i,j \in I$, where I is a countable and infinite set, and if $f_{ij}^* = f_{ji} \neq 0$ and $f_{ij}f_{kl} = \delta_{jk}f_{il}$ for all i,j,k,l, then $C^*(f_{ij}, i,j \in I) \subseteq B$ is isomorphic to $\mathcal{K}(H)$.*

Proof. Similar to the proof of Cor. 6.12. $\square$

6.5. **Example: algebra of functions on the circle.** In Exm. 6.8 we were wondering about the universal C^*-algebra generated by a single nilpotent partial isometry, so about a "universal single nilpotent partial isometry", if you want. In the same sense we now ask: what is the "universal unitary" (in the sense of Def. 1.33)? Before thinking about this question in precise terms, let us think intuitively.

We know that the spectrum of a unitary is a subset of the circle S^1, see Prop. 3.30. So, the universal unitary shall allow for all possible spectra of unitaries. Hence, the universal unitary shall have full spectrum: it shall be all of S^1. A unitary with spectrum S^1 is also called a Haar unitary, by the way. As the C^*-algebra generated by a unitary is commutative, our guess is that the universal C^*-algebra generated by a unitary is isomorphic to the functions on its spectrum, i.e. to $C(S^1)$. This is indeed the case, as we will see soon.

Coming back to precise math, we consider $C^*(u, 1 \mid u^*u = uu^* = 1)$, the universal C^*-algebra generated by a unitary. In fact, it has two generators: u and 1. The relations are $u^*u = uu^* = 1$, but also the relations that turn 1 into the unit, so $1u = u1 = u$ and $1^2 = 1^* = 1$. We usually omit to write down these relations regarding the unit, and we sometimes even omit to write down 1 as a generator, when its existence is clear from the relations.

Checking that $C^*(u, 1 \mid u^*u = uu^* = 1)$ exists is easy: we have $p(1)^2 = p(1^*1) = p(1) \in \{0,1\}$ and $p(u)^2 = p(u^*u) = p(1) \in \{0,1\}$ for any C^*-seminorm p; so $C^*(u, 1 \mid u^*u = uu^* = 1)$ exists by Lemma 6.6.

Proposition 6.15. *Let A be a unital C^*-algebra and $z \in A$ be a unitary with $\mathrm{sp}(z) = S^1$. Then $C^*(u, 1 \mid u^*u = uu^* = 1) \cong C^*(z) \subseteq A$.*

Proof. We denote $C^*(u) := C^*(u, 1 \mid u^*u = uu^* = 1)$. Let A be a unital C^*-algebra, $z \in A$ a unitary and $\mathrm{sp}(z) = S^1$. By the universal property (Prop. 6.7), there is a *-homomorphism $\varphi : C^*(u) \to A$ sending u to z; it is surjective onto $C^*(z)$.

On the other hand, $C^*(z) \subseteq A$ is a commutative C^*-algebra and we have the isomorphism $\Psi_z : C(\mathrm{sp}(z)) \to C^*(z) \subseteq A$ from functional calculus, see Thm. 3.28. By the same argument, there is an isomorphism $\Psi_u : C(\mathrm{sp}(u)) \to C^*(u)$. Let $\Phi : C(S^1) \to C(\mathrm{sp}(u))$ be the restriction map $f \mapsto f_{|\mathrm{sp}(u)}$, $f \in C(S^1)$. Observe that

$C(\mathrm{sp}(z)) = C(S^1)$, by assumption. Putting everything together, we obtain a map $\psi : C^*(z) \to C^*(u)$ as the composition $\psi := \Psi_u \circ \Phi \circ \Psi_z^{-1}$. It maps z to u. Hence $\psi \circ \varphi$ maps u to u, so $\psi \circ \varphi$ coincides with the identity homomorphism $\mathrm{id} : C^*(u) \to C^*(u)$, by Lemma 3.26. This shows that φ is injective. □

Corollary 6.16. *We have $C^*(u, 1 \mid u^*u = uu^* = 1) \cong C(S^1)$.*

Proof. Denote the identity map on S^1 by z, so $z(t) = t$ for all $t \in S^1$ and $z \in C(S^1)$. Then z is a unitary: $z^*z = zz^* = 1$. Here, 1 denotes the constant function $1(t) = 1$, $t \in S^1$ on S^1. We have $\mathrm{sp}(z) = S^1$. Moreover, $C^*(z) = C(S^1)$ by the Stone-Weierstrass Theorem (Thm. 3.3). □

6.6. **The bilateral shift.** We learned from Cor. 6.16 that the identity function on S^1 is a "universal unitary". Another such unitary is the bilateral shift. Recall it from Exc. 1.7: let H be a separable Hilbert space with orthonormal basis $(e_n)_{n\in\mathbb{Z}}$. The bilateral shift $\tilde{S} \in B(H)$, given by $\tilde{S}e_n = e_{n+1}$, $n \in \mathbb{Z}$, is a unitary. We want to compute its spectrum. In order to do so, let $\lambda \in S^1$ and denote by $d(\lambda) \in B(H)$ the diagonal operator given by $d(\lambda)e_n = \lambda^n e_n$, $n \in \mathbb{Z}$.

Lemma 6.17. *On the Hilbert space H with orthonormal basis $(e_n)_{n\in\mathbb{Z}}$, we have:*

(a) *$d(\lambda)d(\lambda') = d(\lambda\lambda')$ and $d(\lambda)^* = d(\bar{\lambda})$, for all $\lambda, \lambda' \in S^1$.*
(b) *$d(\lambda)$ is a unitary and we have $d(\lambda)\tilde{S} = \lambda\tilde{S}d(\lambda)$, for all $\lambda \in S^1$.*
(c) *The map $\beta_\lambda : C^*(\tilde{S}) \to C^*(\tilde{S})$ given by $T \mapsto d(\lambda)Td(\lambda)^*$ is a *-isomorphism.*
(d) *The spectrum of the bilateral shift is $\mathrm{sp}(\tilde{S}) = S^1$.*

Proof. Item (a) is straightforward. Thus, $d(\lambda)$ is a unitary in the sense of Def. 1.33. Moreover, we check for $n \in \mathbb{Z}$:

$$d(\lambda)\tilde{S}e_n = d(\lambda)e_{n+1} = \lambda^{n+1}e_{n+1} = \lambda\tilde{S}d(\lambda)e_n$$

Regarding (c), consider the map $\alpha_\lambda : B(H) \to B(H)$ given by $T \mapsto d(\lambda)Td(\lambda)^*$. It can be verified directly that it is a *-homomorphism. As it maps $\tilde{S}$ to $\lambda\tilde{S}$, by (b), we infer that its restriction β_λ to $C^*(\tilde{S})$ yields a *-homomorphism from $C^*(\tilde{S})$ to itself. Its inverse is given by $\beta_{\bar{\lambda}}$, so β_λ is a *-isomorphism.

For (d), we use (c) and Lemma 3.8(a):

$$\mathrm{sp}(\tilde{S}) = \mathrm{sp}(\beta_\lambda(\tilde{S})) = \lambda\,\mathrm{sp}(\tilde{S})$$

holds for all $\lambda \in S^1$. This shows $\mathrm{sp}(\tilde{S}) = S^1$. □

Proposition 6.18. *We have $C^*(u, 1 \mid u^*u = uu^* = 1) \cong C^*(\tilde{S}) \subseteq B(H)$, where $\tilde{S} \in B(\ell^2(\mathbb{Z}))$ is the bilateral shift operator as in Exc. 1.7.*

Proof. This follows from Prop. 6.15 and Lemma 6.17. □

6.7. **The unilateral shift.** We consider an analog of the preceding subsection for the unilateral shift, see Exc. 1.7. Let H be a separable Hilbert space with orthonormal basis $(e_n)_{n\in\mathbb{N}}$. The unilateral shift $S \in B(H)$, given by $Se_n = e_{n+1}$, $n \in \mathbb{N}$, is an isometry ($S^*S = 1$) which is not a unitary ($SS^* \neq 1$). Denote again by $d(\lambda) \in B(H)$ the diagonal operator given by $d(\lambda)e_n = \lambda^n e_n$, $n \in \mathbb{N}$, given some $\lambda \in S^1$.

Recall from Prop. 1.38 that $\mathcal{K}(H)$ is a closed ideal in $B(H)$. It can be written as the closure of the span of all rank one operators f_{ij}, $i, j \in \mathbb{N}$, see Exc. 6.2.

Lemma 6.19. *We have $f_{ij} = S^{i-1}(1 - SS^*)(S^*)^{j-1}$ for all $i, j \in \mathbb{N}$. Hence, the compact operators $\mathcal{K}(H)$ form an ideal in $C^*(S)$.*

Proof. The formula $f_{ij} = S^{i-1}(1 - SS^*)(S^*)^{j-1}$ is a direct verification. Hence $f_{ij} \in C^*(S)$ for all $i, j \in \mathbb{N}$. Thus $\mathcal{K}(H) \subseteq C^*(S)$ by Exc. 6.2. Since $\mathcal{K}(H)$ is an ideal in $B(H)$, so it is in $C^*(S)$. □

A slight modification of Lemma 6.17 holds true. Denote the quotient map by $\sigma : B(H) \to B(H)/\mathcal{K}(H)$.

Lemma 6.20. *On the Hilbert space H with orthonormal basis $(e_n)_{n\in\mathbb{N}}$, we have:*

(a) *$d(\lambda)d(\lambda') = d(\lambda\lambda')$ and $d(\lambda)^* = d(\bar{\lambda})$, for all $\lambda, \lambda' \in S^1$.*
(b) *$d(\lambda)$ is a unitary and we have $d(\lambda)S = \lambda S d(\lambda)$, for all $\lambda \in S^1$.*
(c) *The map $\beta_\lambda : C^*(S) \to C^*(S)$ given by $T \mapsto d(\lambda)Td(\lambda)^*$ is a *-isomorphism with $\beta_\lambda(\mathcal{K}(H)) \subseteq \mathcal{K}(H)$.*
(d) *$\sigma(S) \in B(H)/\mathcal{K}(H)$ is a unitary and its spectrum is $\mathrm{sp}(\sigma(S)) = S^1$.*
(e) *The quotient of $C^*(S)$ by $\mathcal{K}(H)$ is isomorphic to $C(S^1)$. Hence, we have the following short exact sequence (see also Rem. 4.25 and Exc. 4.6):*

$$0 \to \mathcal{K}(H) \to C^*(S) \to C(S^1) \to 0$$

Proof. Items (a) and (b) are as in Lemma 6.17. Also, the fact that β_λ is a *-isomorphism is analogous. It maps S to λS; thus it maps f_{ij} to $\lambda^{i-j} f_{ij}$, by Lemma 6.19. We infer $\beta_\lambda(\mathcal{K}(H)) \subseteq \mathcal{K}(H)$. Hence, β_λ induces a *-isomorphism

$$\dot{\beta}_\lambda : C^*(S)/\mathcal{K}(H) \to C^*(S)/\mathcal{K}(H), \qquad \sigma(T) \mapsto \sigma(\beta_\lambda(T)).$$

As in Lemma 6.17, we conclude $\mathrm{sp}(\sigma(S)) = S^1$. Note that $\sigma(1 - SS^*) = \sigma(f_{11}) = 0$, i.e. $\sigma(S)\sigma(S)^* = 1$ and hence the isometry $\sigma(S)$ is actually a unitary.

For (e), note that $\mathcal{K}(H)$ is an ideal in $C^*(S) \subseteq B(H)$, by Lemma 6.19. By (d) and Prop. 6.15, the quotient $C^*(\sigma(S)) = C^*(S)/\mathcal{K}(H)$ is isomorphic to $C(S^1)$. □

6.8. **The Toeplitz algebra.** In Sect. 6.5, we considered the universal C^*-algebra generated by a unitary. How about an isometry?

Definition 6.21. The *Toeplitz algebra* $\mathcal{T}$ is the universal C^*-algebra generated by an isometry:

$$\mathcal{T} := C^*(v, 1 \mid v^*v = 1)$$

Reflecting upon our discussion in Sect. 6.2 on universal C^*-algebras in general, we may ask: what is the difference between an isometry v and a unitary in terms of relations? It is the relation $vv^* = 1$. So, if we add the relation $vv^* = 1$ to the Toeplitz algebra, we obtain the universal C^*-algebra generated by a unitary. In other words: taking the quotient by the ideal $\langle 1 - vv^* \rangle \lhd \mathcal{T}$ generated by $1 - vv^*$, we obtain $C(S^1)$; we write $\langle 1 - vv^* \rangle \lhd \mathcal{T}$ for the ideal generated by $1 - vv^* \in \mathcal{T}$, i.e. $\langle 1 - vv^* \rangle$ is the smallest closed (two-sided) ideal in $\mathcal{T}$ containing $1 - vv^*$. How to describe this ideal, can we name a well-known C^*-algebra to which this ideal is isomorphic? We prepare an answer. We use the notation $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Lemma 6.22. *Consider the generator $v \in \mathcal{T}$.*

(a) *The element $1 - vv^* \in \mathcal{T}$ is a projection in the sense of Def. 1.33. It is called the defect projection of v.*
(b) *We have $v^*(1 - vv^*) = (1 - vv^*)v = 0$.*
(c) *The ideal $\langle 1 - vv^* \rangle$ coincides with the closed linear span I of all elements $g_{ij} := v^i(1 - vv^*)(v^*)^j$, for $i, j \in \mathbb{N}_0$.*

Proof. From the relation $v^*v = 1$, the assertions in (a) and (b) follow immediately. As for (c), clearly, $I \subseteq \langle 1 - vv^* \rangle$, since all elements g_{ij} lie in $\langle 1 - vv^* \rangle$.

In order to show $I \supseteq \langle 1 - vv^* \rangle$, we need to convince ourselves that I is an ideal. Firstly, note that $vg_{ij} = g_{i+1,j}$ and $v^*g_{i+1,j} = g_{ij}$ for all $i, j \in \mathbb{N}_0$; moreover, $v^*g_{0j} = 0$ for all $j \in \mathbb{N}_0$, by (b). We conclude $vI, v^*I \subseteq I$. Thus, $v^k(v^*)^l I \subseteq I$ for all $k, l \in \mathbb{N}_0$. Since elements in $\mathcal{T}$ may be approximated by linear combinations of element $v^k(v^*)^l$, this shows $xI \subseteq I$ for all $x \in \mathcal{T}$. Now, as I is closed under taking adjoints, we infer $Ix = (x^*I)^* \subseteq I$ for all $x \in \mathcal{T}$. This proves that I is a two-sided ideal. Of course, it is also a closed linear subspace, so we conclude $I = \langle 1 - vv^* \rangle$. □

Proposition 6.23. *The ideal $\langle 1 - vv^* \rangle \lhd \mathcal{T}$ is isomorphic to $\mathcal{K}(H)$, where H is a separable Hilbert space. The quotient by this ideal is isomorphic to $C(S^1)$. Hence, we have the following short exact sequence:*

$$0 \to \mathcal{K}(H) \to \mathcal{T} \to C(S^1) \to 0$$

Proof. We first describe the ideal $\langle 1 - vv^* \rangle$. Consider the elements g_{ij} from Lemma 6.22. It is clear that $g_{ij}^* = g_{ji}$ holds. Furthermore, for $j > k$, we have $(v^*)^j v^k (1 - vv^*) = (v^*)^{j-k}(1 - vv^*) = 0$ by Lemma 6.22(b). Likewise, $j < k$ implies $(1 - vv^*)(v^*)^j v^k = 0$. Hence, for $j \neq k$,

$$g_{ij}g_{kl} = v^i(1 - vv^*)(v^*)^j v^k (1 - vv^*)(v^*)^l = 0,$$

whereas $j = k$ implies $g_{ij}g_{kl} = g_{il}$. Thus, $g_{ij}g_{kl} = \delta_{jk}g_{il}$ for all i, j, k, l. By Cor. 6.14, $C^*(g_{ij}, i, j \in \mathbb{N}_0) \cong \mathcal{K}(H)$. Note that $C^*(g_{ij}, i, j \in \mathbb{N}_0) = I$, where I is as in Lemma 6.22(c), so $\langle 1 - vv^* \rangle \cong \mathcal{K}(H)$ by Lemma 6.22(c).

Finally, recall that $C(S^1)$ is isomorphic to the universal C^*-algebra $C^*(u) := C^*(u, 1 \mid u^*u = uu^* = 1)$, by Cor. 6.16. As u is in particular an isometry, there is a *-homomorphism $\varphi : \mathcal{T} \to C^*(u)$ sending v to u, by the universal property (Prop.

6.7). As $\varphi(1-vv^*) = 0$, we have $\varphi(\langle 1-vv^*\rangle) = 0$. Thus, there is a *-homomorphism $\dot{\varphi} : \mathcal{T}/\langle 1 - vv^*\rangle \to C^*(u)$ sending $\dot{v}$ to u, where $\dot{v} \in \mathcal{T}/\langle 1 - vv^*\rangle$ denotes the image of v under the quotient map.

On the other hand, $\dot{v}$ is a unitary. Thus $\psi : C^*(u) \to \mathcal{T}/\langle 1-vv^*\rangle$ mapping u to $\dot{v}$ exists by the universal property. We conclude that φ and ψ are inverse to another finishing the proof. For the short exact sequence, see Rem. 4.25 and Exc. 4.6. □

We can interpret the above proposition by saying that the Toeplitz algebra is very close to $C(S^1)$ – or rather that the universal isometry is almost a universal unitary – up to a small defect: the compacts.

Finally, let us consider an analog of Prop. 6.18. There, we saw that the bilateral shift is a model of a universal unitary. Shouldn't the unilateral shift be a model of a universal isometry then? Yes, that is the case.

Corollary 6.24. *The canonical *-homomorphism $\varphi : \mathcal{T} \to C^*(S) \subseteq B(H)$ mapping v to S, where S is the unilateral shift on a separable Hilbert space H, is an isomorphism.*

Proof. This is a classical diagram chase (Lemma 6.27) given the exact sequences

$$0 \to \mathcal{K}(H) \to \mathcal{T} \to C(S^1) \to 0$$

from Prop. 6.23 and

$$0 \to \mathcal{K}(H) \to C^*(S) \to C(S^1) \to 0$$

from Lemma 6.20; the short exact sequences are linked by the identity maps on $\mathcal{K}(H)$ and $C(S^1)$ respectively, as well as $\varphi : \mathcal{T} \to C^*(S)$ producing a commutative diagram. See Lemma 6.27 for finishing the proof. □

Remark 6.25. The Toeplitz algebra is often introduced in a different form: as the algebra of Toeplitz operators, see for instance [13, Sect. V.1]. The idea is as follows. Consider the Hilbert space $L^2(S^1)$ with orthonormal basis $(e_n)_{n\in\mathbb{Z}}$ given by $e_n = z^n$, where z is the identity function on S^1. Let $H^2 \subseteq L^2(S^1)$ be the space spanned by $(e_n)_{n\geq 0}$, called the Hardy space, and let P_{H^2} be the projection onto this space. For $g \in L^\infty(S^1)$, let $M_g \in B(L^2(S^1))$ be the multiplication operator given by $M_g(f) := fg$, $f \in L^2(S^1)$. The Toeplitz operator $T_g \in B(H^2)$ is defined as $T_g := P_{H^2} M_g$, for $g \in L^\infty(S^1)$.

Observe that $T_z e_n = T_z z^n = z^{n+1} = e_{n+1}$ for $n \geq 0$, so T_z is the unilateral shift on the Hardy space. One can show:

$$\mathcal{T} \cong C^*(T_z) = \{T_g + K \mid g \in C(S^1), K \in \mathcal{K}(H^2)\} \subseteq B(H^2)$$

In a way, this is just a reformulation of Cor. 6.24 and Prop. 6.23: the Toeplitz algebra is an extension of $C(S^1)$ by the compacts.

Remark 6.26. There is the famous Wold decomposition of isometries [13, Sect. V.2]: let w be an isometry on a Hilbert space H. Then, w is unitarily equivalent to

$(S\otimes 1)\oplus u$, where S is the unilateral shift, $S\otimes 1$ is an amplification of the shift and u is a unitary. In other words: the unilateral shift is basically the only isometry!

Coburn then showed [13, Sect. V.2]: if w is a proper isometry on some Hilbert space H, i.e. $w^*w = 1$ and $ww^* \neq 1$, then $C^*(w) \subseteq B(H)$ is isomorphic to the Toeplitz algebra $\mathcal{T}$. This is a generalization of Cor. 6.24.

6.9. **Diagram chase.** We finish this lecture by mentioning a classic in homological algebra and category theory: the (short) five lemma. We formulate it for C^*-algebras, but it holds in much wider generality.

Assume we have the following commutative diagram of two short exact sequences.

$$\begin{tikzcd}
0 \arrow[r] & I_1 \arrow[r, "\iota_1"] \arrow[d, "\alpha"] & A_1 \arrow[r, "\pi_1"] \arrow[d, "\varphi"] & B_1 \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & I_2 \arrow[r, "\iota_2"'] & A_2 \arrow[r, "\pi_2"'] & B_2 \arrow[r] & 0
\end{tikzcd}$$

Explicitly, this means (see also Rem. 4.25 and Exc. 4.6): for $j = 1, 2$, let I_j, A_j and B_j be C^*-algebras. Let $\iota_j : I_j \to A_j$ be injective maps, $\pi_j : A_j \to B_j$ be surjective maps, and assume $\ker \pi_j = \operatorname{ran} \iota_j$. In particular, $\iota_j(I_j)$ is a closed ideal in A_j, and $A_j/\iota_j(I_j) \cong B_j$, for $j = 1, 2$. For convenience, we may think of $I_j \lhd A_j$ and ι_j simply being the embeddings. Moreover, assume that there are *-homomorphisms $\alpha : I_1 \to I_2$ and $\varphi : A_1 \to A_2$ and $\beta : B_1 \to B_2$ such that $\pi_2 \circ \varphi = \beta \circ \pi_1$ and $\iota_2 \circ \alpha = \varphi \circ \iota_1$, i.e. the diagram is commutative.

Lemma 6.27 (Five Lemma)**.** *Assume we have the following commutative diagram of two short exact sequences.*

$$\begin{tikzcd}
0 \arrow[r] & I_1 \arrow[r, "\iota_1"] \arrow[d, "\alpha"] & A_1 \arrow[r, "\pi_1"] \arrow[d, "\varphi"] & B_1 \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & I_2 \arrow[r, "\iota_2"'] & A_2 \arrow[r, "\pi_2"'] & B_2 \arrow[r] & 0
\end{tikzcd}$$

If α and β are *-isomorphisms, then also φ is a *-isomorphism.

Proof. The proof is fun and you should do it yourself once in your life. Here it goes.

φ is injective: Let $x \in A_1$ and $\varphi(x) = 0$. Then $\beta \circ \pi_1(x) = \pi_2 \circ \varphi(x) = 0$. Then $\pi_1(x) = 0$ as β is injective. Then $x \in \ker \pi_1 = \operatorname{ran} \iota_1$, i.e. $x = \iota_1(y)$ for some $y \in I_1$. Then $\iota_2 \circ \alpha(y) = \varphi \circ \iota_1(y) = \varphi(x) = 0$. Then $y = 0$ as ι_2 and α are injective. Then $x = \iota_1(y) = 0$ and φ is injective.

φ is surjective: Let $y \in A_2$. Then $\pi_2(y) \in B_2$. There is an $x_0 \in A_1$ with $\beta \circ \pi_1(x_0) = \pi_2(y)$ since β and π_1 are surjective. Then $\pi_2(y - \varphi(x_0)) = \pi_2(y) - \beta \circ \pi_1(x_0) = 0$. Then $y - \varphi(x_0) \in \ker \pi_2 = \operatorname{ran} \iota_2$, i.e. $y - \varphi(x_0) = \iota_2(z)$ for some $z \in I_2$. Then $z = \alpha(w)$ for some $w \in I_1$. Put $x := \iota_1(w) + x_0 \in A_1$. Then

$$\varphi(x) = \varphi \circ \iota_1(w) + \varphi(x_0) = \iota_2 \circ \alpha(w) + \varphi(x_0) = (y - \varphi(x_0)) + \varphi(x_0) = y$$

and φ is surjective. $\square$

6.10. **Exercises.**

Exercise 6.1. Let $N \geq 2$. As in Prop. 6.11, consider the universal C^*-algebras

$$A_1 := C^*(e_{ij}, i,j = 1,\dots,N \mid e_{ij}^* = e_{ji}, e_{ij}e_{kl} = \delta_{jk}e_{il} \text{ for all } i,j,k,l),$$
$$A_2 := C^*(x_i, i = 1,\dots,N \mid x_i^* x_j = \delta_{ij}x_1 \text{ for all } i,j).$$

(a) Show that $x_1 \in A_2$ is a projection in the sense of Def. 1.33. Moreover, verify $(x_i x_1 - x_i)^*(x_i x_1 - x_i) = 0$ and conclude $x_i x_1 = x_i$ for all i.
(b) Show that A_1 and A_2 exist, using Lemma 6.6(b)
(c) Show that there is a *-homomorphism $\varphi : A_1 \to A_2$ sending e_{ij} to $x_i x_j^*$, for all $i,j = 1,\dots,N$.
(d) Show that there is a *-homomorphism $\psi : A_2 \to A_1$ sending x_i to e_{i1}, for all $i = 1,\dots,N$.
(e) Show that $\varphi \circ \psi = \mathrm{id}_{A_2}$ and $\psi \circ \varphi = \mathrm{id}_{A_1}$. Hint: You only need to check this on the generators, by Lemma 3.26.
(f) Conclude $A_1 \cong A_2$.
(g) Perform (a) to (d) also in the case of $N = \infty$.

Exercise 6.2. Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis of a separable Hilbert space H. For $i,j \in \mathbb{N}$, let $f_{ij} \in B(H)$ be the operator given by $f_{ij}e_n := \delta_{jn}e_i$, for $n \in \mathbb{N}$, as in the proof of Prop. 6.13.

(a) Convince yourself that f_{ij} is a rank one operator and $f_{ij} \in \mathcal{K}(H)$.
(b) Let F be the set of all linear combinations of the maps f_{ij}, $i,j \in \mathbb{N}$. Show that F is dense in $\mathcal{K}(H)$. Hint: Use Prop. 1.38 and the approximate unit given by $p_n = \sum_{i=1}^n f_{ii}$, $n \in \mathbb{N}$.

Exercise 6.3. Let H be a separable Hilbert space and let $0 \neq I \lhd B(H)$ be a closed ideal.

(a) Show that all rank one operators f_{ij} from Exc. 6.2 are contained in I.
(b) Use (a) and Exc. 6.2 to show that $\mathcal{K}(H) \subseteq I$.
(c) Deduce that $\mathcal{K}(H)$ is simple. (Use Lemma 4.22.)

Exercise 6.4. Let $N \in \mathbb{N}$. We view $\mathbb{C}^N$ as a C^*-algebra with pointwise operations. In other words, given the finite set $X_N = \{1,\dots,N\}$, we view $\mathbb{C}^N = C(X_N)$ in a natural way.

(a) Show that $C^*(p, 1 \mid p = p^2 = p^*) \cong \mathbb{C}^2$.
(b) Show that $C^*(p_1,\dots,p_N, 1 \mid p_j = p_j^2 = p_j^*, j = 1,\dots,N, \sum_{k=1}^N p_k = 1) \cong \mathbb{C}^N$.
(c) Show that $C^*(u, 1 \mid u^*u = uu^* = 1, u = u^*) \cong \mathbb{C}^2$.
(d) Show that $C^*(u, 1 \mid u^*u = uu^* = 1, u^N = 1) \cong \mathbb{C}^N$.
(e) Write down an explicit isomorphism between $C^*(p, 1 \mid p = p^2 = p^*)$ and $C^*(u, 1 \mid u^*u = uu^* = 1, u = u^*)$.

7. Universal C^*-algebras II: rotation algebra and Cuntz algebra

Abstract. We continue our investigation of examples of universal C^*-algebras. We introduce the famous irrational rotation algebra A_ϑ (also called the noncommutative torus), we construct two faithful conditional expectations of it and composing them we obtain a faithful tracial state. This enables us to show that A_ϑ is simple. We then turn to another famous C^*-algebra: the Cuntz algebra $\mathcal{O}_n$. Again, we find a conditional expectation and we show that $\mathcal{O}_n$ is purely infinite – which implies that it is simple, too. The proofs of simplicity for A_ϑ and $\mathcal{O}_n$ have some similiarities which we will point out.

7.1. Definition and existence of the rotation algebra A_ϑ.

Definition 7.1. Let $\vartheta \in \mathbb{R}$. The *rotation algebra* (also called the *noncommutative torus*) is defined as the universal C^*-algebra

$$A_\vartheta := C^*(u, v \mid u, v \text{ are unitaries}, uv = e^{2\pi i\vartheta} vu).$$

We often abbreviate $\lambda := e^{2\pi i\vartheta} \in S^1$. If $\vartheta \notin \mathbb{Q}$, then A_ϑ is called the *irrational rotation algebra*.

Later, we will see that the case $\vartheta \notin \mathbb{Q}$ behaves much nicer than $\vartheta \in \mathbb{Q}$. This is why the adjective "irrational" is sometimes dropped in the literature and "the rotation algebra" then refers to the irrational rotation algebra only.

As the generators u and v are unitaries, we have $p(u), p(v) \in \{0, 1\}$ for all C^*-seminorms p on the *-algebra generated by u and v. Hence, A_ϑ exists by Lemma 6.6. Note that we omit to write down the generator 1 corresponding to the unit; it is implicitely mentioned by the term "unitaries" – recall that u is a unitary, if $u^*u = uu^* = 1$.

So, A_ϑ exists in the sense that the underlying universal *-algebra admits a C^*-norm, by Lemma 6.6. How about non-triviality?

Lemma 7.2. *The rotation algebra A_ϑ may be represented on $\ell^2(\mathbb{Z})$ as follows. Let $\lambda := e^{2\pi i\vartheta}$. Consider the bilateral shift $\tilde{S}$ and the diagonal operator $d(\lambda)$ given by*

$$\tilde{S}e_n = e_{n+1}, \qquad d(\lambda)e_n = \lambda^n e_n, \qquad \text{for all } n \in \mathbb{Z}.$$

Then $\pi : A_\vartheta \to B(\ell^2(\mathbb{Z}))$ mapping $u \mapsto d(\lambda)$ and $v \mapsto \tilde{S}$ is a representation of A_ϑ.

Proof. This follows immediately from Lemma 6.17(b): we have $d(\lambda)\tilde{S} = \lambda\tilde{S}d(\lambda)$. □

An alternative representation is given in Exc. 7.1. We conclude that $A_\vartheta \neq 0$. Let us take a look at the elements in A_ϑ. The following is a technical lemma. Denote by $\mathcal{S}$ the set consisting of elements $\sum_{k,l\in\mathbb{Z}} a_{kl}u^k v^l$ with $a_{kl} \in \mathbb{C}$ such that only finitely many coefficients a_{kl} are non-zero. Here, $u^{-k} := (u^*)^k$ and $v^{-k} := (v^*)^k$ for $k > 0$.

Lemma 7.3. *In A_ϑ, we have $u^*v = \bar{\lambda}vu^*$, and more generally $u^k v^l = \lambda^{kl} v^l u^k$ for all $k, l \in \mathbb{Z}$. Moreover, the set $\mathcal{S}$ is a *-algebra which is dense in A_ϑ.*

Proof. Multiplying $uv = \lambda vu$ with u^* from both sides implies $u^*v = \bar{\lambda} vu^*$. Inductively we deduce $u^k v^l = \lambda^{kl} v^l u^k$ for all $k, l \in \mathbb{Z}$. It is then easy to check that $\mathcal{S}$ is a *-algebra containing u and v. By the construction of universal C^*-algebras, it is thus dense in A_ϑ. □

Let us justify the name "noncommutative torus" by looking at the case $\vartheta = 0$.

Lemma 7.4. *Let $\vartheta = 0$. Then $A_\vartheta \cong C(\mathbb{T}^2)$, where $\mathbb{T}^2 \subseteq \mathbb{C}^2$ is the 2-torus.*

Proof. Observe that A_ϑ is commutative in case $\vartheta = 0$. Indeed, it is the universal C^*-algebra generated by two commuting unitaries. Hence, by the Gelfand-Naimark Theorem (Thm. 3.23), it must be isomorphic to $C(\mathrm{Spec}(A_\vartheta))$. The spectrum $\mathrm{Spec}(A_\vartheta)$ is homeomorphic to the 2-torus in the case $\vartheta = 0$. Indeed, given a character $\varphi \in \mathrm{Spec}(A_\vartheta)$, it is uniquely determined by the values $(\varphi(u), \varphi(v)) \in S^1 \times S^1 = \mathbb{T}^2$, by Lemma 3.26. Conversely, every value in $(\mu_1, \mu_2) \in \mathbb{T}^2$ gives rise to a character in $\mathrm{Spec}(A_\vartheta)$, simply because μ_1 and μ_2 are commuting unitaries in $\mathbb{C}$; then use the universal property (Prop. 6.7). This shows the assertion.

Working out the isomorphism $A_\vartheta \cong C(\mathbb{T}^2)$, we infer that it is given as follows. Consider the functions $\tilde{u}, \tilde{v} \in C(\mathbb{T}^2)$ defined by $\tilde{u}(\mu_1, \mu_2) := \mu_1$ and $\tilde{v}(\mu_1, \mu_2) := \mu_2$, for $\mu_1, \mu_2 \in S^1$. The *-homomorphism from A_ϑ to $C(\mathbb{T}^2)$ sending $u \mapsto \tilde{u}$ and $v \mapsto \tilde{v}$ is the above isomorphism. □

So, if $\vartheta = 0$, the rotation algebra A_ϑ corresponds to the (algebra of functions on) the torus $\mathbb{T}^2$. Hence, A_ϑ can be viewed as a kind of "algebra of functions on the noncommutative torus $\mathbb{T}^2_\vartheta$" for general $\vartheta \in \mathbb{R}$. Let us be clear: the noncommutative torus $\mathbb{T}^2_\vartheta$ does not exist as such! However, its "algebra of functions" does exist: it is A_ϑ. So, in the philosophy of Gelfand duality (Sect. 3.12), we study $C(\mathbb{T}^2)$ instead of its underlying space $\mathbb{T}^2$ – and we study A_ϑ as if there was some underlying noncommutative space $\mathbb{T}^2_\vartheta$.

This way of thinking may appear to be very weird when being confronted with it for the first time. However, it is *very* instructive when working in the field: imagining an underlying object, we may develop questions about this object, we may get inspiration from related classical objects and we may express structural theorems in an intuitive way. We elaborate more on the noncommutative torus and its role in quantization in Sect. 7.9.

7.2. **Conditional expectations and a tracial state on A_ϑ.** We are now preparing the proof of an important property of A_ϑ: this C^*-algebra is simple. Our tools to prove this will be conditional expectations and tracial states.

Definition 7.5. Let A be a unital C^*-algebra, $B \subseteq A$ a C^*-subalgebra and $1 \in B$.

(a) A *(conditional) expectation* of A onto B is a positive, linear, surjective, unital map $\varphi : A \to B$ with $\varphi \circ \varphi = \varphi$ (which is equivalent to $\varphi(b) = b$ for all $b \in \varphi(A)$).
(b) A positive, linear map is *faithful*, if $\varphi(a) = 0$ and $a \geq 0$ imply $a = 0$.

If $B = \mathbb{C}1 \subseteq A$, we see that any state is an expectation. So, expectations can be seen as B-valued states. We prepare the construction of two conditional expectations of A_ϑ. Given $\zeta, \mu \in S^1$, we define

$$\rho_{\zeta,\mu} : A_\vartheta \to A_\vartheta, \qquad u \mapsto \zeta u,\, v \mapsto \mu v.$$

This map exists by the universal property (Prop. 6.7), since $u' := \zeta u$ and $v' := \mu v$ are unitaries with $u'v' = \lambda v'u'$. Note that $\rho_{\bar\zeta,\bar\mu}$ is inverse to $\rho_{\zeta,\mu}$, so $\rho_{\zeta,\mu}$ is in fact a *-isomorphism.

Lemma 7.6. *Let $\vartheta \in \mathbb{R}$ and let $x \in A_\vartheta$.*

(a) *The map $f_x : \mathbb{T}^2 \to A_\vartheta$, $(\zeta, \mu) \mapsto \rho_{\zeta,\mu}(x)$ is continuous in norm.*
(b) *The maps $g_x : [0,1] \to A_\vartheta$, $t \mapsto f_x(1, e^{2\pi it})$ and $h_x : [0,1] \to \mathbb{A}_\vartheta$, $t \mapsto f_x(e^{2\pi it}, 1)$ are continuous in norm.*
(c) *The Riemannian sums $\frac{1}{n}\sum_{j=1}^n \rho_{1,e^{2\pi it_j}}(x)$ and $\frac{1}{n}\sum_{j=1}^n \rho_{e^{2\pi it_j},1}(x)$ converge, for uniform partitions $0 = t_0 < t_1 < \ldots < t_n = 1$ of the unit interval. We denote their limits by $\int_0^1 \rho_{1,e^{2\pi it}}(x)\mathrm{d}t$ and $\int_0^1 \rho_{e^{2\pi it},1}(x)\mathrm{d}t$ respectively.*

Proof. Given $x = \sum_{k,l\in\mathbb{Z}} a_{kl}u^kv^l \in \mathcal{S} \subseteq A_\vartheta$, we compute

$$\|f_x(\zeta_1, \mu_1) - f_x(\zeta_2, \mu_2)\| = \|\sum_{k,l\in\mathbb{Z}} a_{kl}(\zeta_1^k\mu_1^l - \zeta_2^k\mu_2^l)u^kv^l\| \le \sum_{k,l\in\mathbb{Z}} |a_{kl}||\zeta_1^k\mu_1^l - \zeta_2^k\mu_2^l|.$$

This expression tends to zero as (ζ_1, μ_1) tends to (ζ_2, μ_2), which proves (a) for $x \in \mathcal{S}$. For general $x \in A_\vartheta$, we use Lemma 7.3. Now, also (b) follows immediately. Using $g_x(t) = \rho_{1,e^{2\pi it}}(x)$ and $h_x(t) = \rho_{e^{2\pi it},1}(x)$, we derive (c) just like in the classical case of Riemannian sums and intergrals for complex valued functions. □

We consider $\varphi_1, \varphi_2 : A_\vartheta \to A_\vartheta$ given by

$$\varphi_1(x) := \int_0^1 \rho_{1,e^{2\pi it}}(x)\mathrm{d}t, \qquad \varphi_2(x) := \int_0^1 \rho_{e^{2\pi it},1}(x)\mathrm{d}t, \qquad x \in A_\vartheta.$$

Lemma 7.7. *Let $\vartheta \in \mathbb{R}$.*

(a) *For all $\sum_{k,l\in\mathbb{Z}} a_{kl}u^kv^l \in \mathcal{S}$ we have $\varphi_1(\sum_{k,l\in\mathbb{Z}} a_{kl}u^kv^l) = \sum_{k\in\mathbb{Z}} a_{k0}u^k$ and $\varphi_2(\sum_{k,l\in\mathbb{Z}} a_{kl}u^kv^l) = \sum_{l\in\mathbb{Z}} a_{0l}v^l$.*
(b) *We have $(\varphi_1)_{|C^*(u)} = \mathrm{id}_{|C^*(u)}$ and $(\varphi_2)_{|C^*(v)} = \mathrm{id}_{|C^*(v)}$.*
(c) *We have $\varphi_1(A_\vartheta) = C^*(u) \subseteq A_\vartheta$ and $\varphi_2(A_\vartheta) = C^*(v) \subseteq A_\vartheta$.*
(d) *The maps φ_j, $j = 1, 2$, are faithful conditional expectations with $\|\varphi_j\| \le 1$.*

Proof. For (a), we first consider $l \in \mathbb{Z}$ and check, using Lemma 7.6(c):

$$\varphi_1(v^l) = \int_0^1 \rho_{1,e^{2\pi it}}(v^l)\mathrm{d}t = \int_0^1 e^{2\pi ilt}v^l\mathrm{d}t = \left(\int_0^1 e^{2\pi ilt}\mathrm{d}t\right)v^l = \delta_{l0}$$

Secondly, let $k, l \in \mathbb{Z}$ and $x = u^kv^l$. Note that $\rho_{1,e^{2\pi it}}(u^kv^l) = u^k\rho_{1,e^{2\pi it}}(v^l)$ for any $t \in [0,1]$. Hence, by Lemma 7.6(c):

$$\varphi_1(u^kv^l) = \int_0^1 \rho_{1,e^{2\pi it}}(u^kv^l)\mathrm{d}t = u^k\left(\int_0^1 \rho_{1,e^{2\pi it}}(v^l)\mathrm{d}t\right) = u^k\varphi_1(v^l) = \delta_{l0}u^k$$

Passing to linear combinations, we finish the proof of (a) for φ_1; similarly for φ_2.

As elements $\sum_{k\in\mathbb{Z}} a_k u^k$ with finitely many nonzero coefficients $a_k \in \mathbb{C}$ are dense in $C^*(u) \subseteq A_\vartheta$, item (b) follows from (a). For (c), we use Lemma 7.3. For both, (b) and (c), we used that φ_1 and φ_2 are continuous: let $x \in A_\vartheta$. Note that $\|\rho_{1,e^{2\pi it}}(x)\| = \|x\|$ since $\rho_{1,e^{2\pi it}}$ is a *-isomorphism. Then $\|\frac{1}{n}\sum_{j=1}^n \rho_{1,e^{2\pi it_j}}(x)\| \leq \|x\|$ for any $t \in [0,1]$ and hence $\|\varphi_1(x)\| \leq \|x\|$ by Lemma 7.6(c). This proves $\|\varphi_1\| \leq 1$ and φ_1 is continuous; similarly for φ_2.

As for (d), since $\rho_{1,e^{2\pi it}}$ is linear and unital, it is clear that φ_1 is linear and unital, by Lemma 7.6(c). Moreover, given $x \geq 0$, we have $\rho_{1,e^{2\pi it}}(x) \geq 0$ for any $t \in [0,1]$. Thus, $\varphi_1(x) \geq 0$ by Lemma 7.6(c) and Cor. 4.9, since the cone of positive elements is closed.

For showing that φ_1 is faithful, let us assume $x \neq 0$ in addition to $x \geq 0$. Hence, $\rho_{1,e^{2\pi it}}(x) \neq 0$ for any $t \in [0,1]$, since $\rho_{1,e^{2\pi it}}$ is a *-isomorphism. Now, let $\psi : A_\vartheta \to \mathbb{C}$ be a state with $\psi(\rho_{1,e^{2\pi it_0}}(x)) \neq 0$ for some fixed $t_0 \in (0,1)$. It exists by the Hahn-Banach Theorem (Thm. 5.13). Define $f : [0,1] \to \mathbb{C}$ by $f(t) := \psi(\rho_{1,e^{2\pi it}}(x))$, $t \in [0,1]$. Then f is continuous by Lemma 7.6, positive and non-zero. Thus,

$$\psi(\varphi_1(x)) \longleftarrow \frac{1}{n}\sum_{j=1}^n \psi(\rho_{1,e^{2\pi it_j}}(x)) = \frac{1}{n}\sum_{j=1}^n f(t_j) \longrightarrow \int_0^1 f(t)\mathrm{d}t \neq 0,$$

where we used Lemma 7.6(c) again. This shows $\varphi_1(x) \neq 0$.

Finally, $\varphi_1^2 = \varphi_1$ follows from (b) and (c), and the same for $\varphi_2^2 = \varphi_2$. □

The next lemma will be a key ingredient for the proof of simplicity of A_ϑ: we have an algebraic description of φ_1 and φ_2. It allows us to deduce that φ_1 and φ_2 map ideals to themselves.

Lemma 7.8. *Let $\vartheta \notin \mathbb{Q}$ and let $\lambda := e^{2\pi i\vartheta} \in S^1 \subseteq \mathbb{C}$ as before.*

(a) *Given $l \in \mathbb{Z}$, the sequence $\left(\frac{1}{2n+1}\sum_{j=-n}^n \lambda^{jl}\right)_{n\in\mathbb{N}}$ converges to δ_{l0}.*
(b) *For all $x \in A_\vartheta$, we have:*

$$\varphi_1(x) = \lim_{n\to\infty} \frac{1}{2n+1}\sum_{j=-n}^n u^j x u^{-j}, \qquad \varphi_2(x) = \lim_{n\to\infty} \frac{1}{2n+1}\sum_{j=-n}^n v^j x v^{-j}$$

(c) *If $I \lhd A_\vartheta$ is a closed ideal, then $\varphi_1(I) \subseteq I$ and $\varphi_2(I) \subseteq I$.*

Proof. The statement in (a) is easy analysis, see Exc. 7.2. As for (b), let $k, l \in \mathbb{Z}$ and $x = u^k v^l$. We check:

$$\begin{aligned}\frac{1}{2n+1}\sum_{j=-n}^{n} u^j(u^k v^l)u^{-j} &= \frac{1}{2n+1}\sum_{j=-n}^{n} u^{j+k}v^l u^{-j}\\ &= \frac{1}{2n+1}\sum_{j=-n}^{n} \lambda^{jl}u^{j+k}u^{-j}v^l\\ &= \left(\frac{1}{2n+1}\sum_{j=-n}^{n}\lambda^{jl}\right)u^k v^l\end{aligned}$$

Since $\left(\frac{1}{2n+1}\sum_{j=-n}^{n}\lambda^{jl}\right)_{n\in\mathbb{N}}$ converges to δ_{l0} by (a), this proves the assertion for $x = u^k v^l$ and φ_1, using also Lemma 7.7(a). By linearity, it holds true for elements $x \in \mathcal{S}$; by continuity and Lemma 7.3 also for all $x \in A_\vartheta$. Likewise for φ_2.

Finally, let $I \lhd A_\vartheta$ be a closed ideal and $x \in I$. Then $\frac{1}{2n+1}\sum_{j=-n}^{n} u^j x u^{-j} \in I$ and hence $\varphi_1(x) \in I$, by (b). Similarly, $\varphi_2(I) \subseteq I$. □

Note that (a) of the above lemma fails to be true, if $\vartheta \in \mathbb{Q}$, see Exc. 7.2. We see already here in Lemma 7.8, that the irrational case is nicer than the rational case.

7.3. **Simplicity of A_ϑ.**

We are now going to prove the main result on A_ϑ in this lecture: simplicity. We use the technique of faithful traces in order to do so.

Definition 7.9. Let A be a unital C^*-algebra. A *(normalized) trace* (or a *tracial state*) on A is a state $\tau : A \to \mathbb{C}$ with $\tau(xy) = \tau(yx)$ for all $x, y \in A$.

We constructed the expectations φ_1 and φ_2 in the last section. The map φ_1 reads out the part generated by u, whereas φ_2 reads out the one generated by v, see Lemma 7.7(c). Composing these two maps, we shall obtain: $\mathbb{C}$. This is how we construct our faithful trace.

Proposition 7.10. *Putting $\tau := \varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1 : A_\vartheta \to \mathbb{C}1 \subseteq A_\vartheta$ we obtain a unital faithful trace on A_ϑ. It satisfies*

$$\tau(\sum_{k,l\in\mathbb{Z}} a_{kl}u^k v^l) = a_{00}.$$

If $\vartheta \notin \mathbb{Q}$, it is the unique (normalized) trace on A_ϑ.

Proof. We first check $\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1$. Let $k, l \in \mathbb{Z}$. By Lemma 7.7(a):

$$\varphi_1(\varphi_2(u^k v^l)) = \delta_{k0}\varphi_1(v^l) = \delta_{k0}\delta_{l0} = \delta_{l0}\varphi_2(u^k) = \varphi_2(\varphi_1(u^k v^l))$$

Now, putting $\tau := \varphi_1 \circ \varphi_2$, we have $\tau(\sum_{k,l\in\mathbb{Z}} a_{kl}u^k v^l) = a_{00} \in \mathbb{C}1 \subseteq A_\vartheta$ by Lemma 7.7(a). Moreover, τ is positive, linear, and unital by Lemma 7.7(d), hence it is a state; it is faithful, again by Lemma 7.7(d). For showing traciality, let $x = u^k v^l$ and $y = u^m v^n$, for $k, l, m, n \in \mathbb{Z}$. Then:

$$\tau(xy) = \tau(u^k v^l u^m v^n) = \lambda^{-lm} \tau(u^{k+m} v^{l+n}) = \delta_{k+m,0} \delta_{l+n,0} \lambda^{-lm}$$
$$\tau(yx) = \tau(u^m v^n u^k v^l) = \lambda^{-nk} \tau(u^{k+m} v^{l+n}) = \delta_{k+m,0} \delta_{l+n,0} \lambda^{-nk}$$

Since $k+m = 0$ and $l+n = 0$ imply $(k+m)n = 0$ and $l = -n$, we infer $nk = -mn = lm$. Thus, $\tau(xy) = \tau(yx)$ for $x = u^k v^l$ and $y = u^m v^n$. Passing to linear combinations and using the continuity of τ, we derive $\tau(xy) = \tau(yx)$ for all $x, y \in A_\vartheta$; see also Lemma 7.3.

Finally, assume $\vartheta \notin \mathbb{Q}$ and let τ' be another normalized trace. Let $x \in A_\vartheta$. Note that $\tau'(u^j x u^{-j}) = \tau'(x)$ for all j, by traciality. Thus, by Lemma 7.8(b):

$$\tau'(x) = \lim_{n\to\infty} \frac{1}{2n+1} \sum_{j=-n}^{n} \tau'(u^j x u^{-j}) = \tau'(\varphi_1(x))$$
$$\tau'(x) = \lim_{n\to\infty} \frac{1}{2n+1} \sum_{j=-n}^{n} \tau'(v^j x v^{-j}) = \tau'(\varphi_2(x))$$

As τ' is unital, we conclude:

$$\tau'(x) = \tau'(\varphi_1(x)) = \tau'(\varphi_2(\varphi_1(x)) = \tau'(\tau(x)) = \tau(x) \qquad \square$$

We observe that the trace τ is given by Fourier analysis on A_ϑ: it is the Fourier coefficient a_{00}.

Theorem 7.11. *The irrational rotation algebra A_ϑ is simple.*

Proof. Let I be a non-zero closed ideal in A_ϑ. We thus find an element $x \neq 0$ in I. Hence, we have $0 \neq x^*x \in I$. Since τ is faithful, we infer $\tau(x^*x) \neq 0$. On the other hand, $\tau(x^*x) = \varphi_1(\varphi_2(x^*x)) \in I$, by Lemma 7.8(c). Since $\tau(x^*x)$ is a nonzero multiple of $1 \in A_\vartheta$, this shows $1 \in I$ and hence $I = A_\vartheta$. $\square$

As mentioned in the last lecture (Sect. 6.2), simplicity for the universal C^*-algebra A_ϑ means, that we may add no further relations to those of A_ϑ, if ϑ is irrational. Indeed, let p be a polynomial in u, u^*, v and v^* (i.e. p is a relation); by Thm. 7.11, the closed ideal I generated by p is either 0 (in which case the relation p is already implied by the relations of A_ϑ), or it is all of A_ϑ (in which case adding the relation p to those of A_ϑ – i.e. taking the quotient of A_ϑ by I – would yield a trivial C^*-algebra). This is kind of surprising, because we *do* may add additional relations to the universal C^*-algebra generated by two commuting unitaries (which is A_ϑ for $\vartheta = 0$, see Lemma 7.4), see also Exc. 6.4. By the way, we may also derive $A_\vartheta \cong C^*(\tilde{S}, d(\lambda)) \subseteq B(\ell^2(\mathbb{Z}))$ from the simplicity of A_ϑ and Lemma 6.17. This comment is in the spirit of Cor. 6.24 and Prop. 6.18.

We end our discussion on A_ϑ here although a lot more could be said about it; it might be one of the best studied examples of a C^*-algebra. See also Sect. 7.9.

7.4. **Definition of the Cuntz algebra $\mathcal{O}_n$.** We turn to another very prominent example of a C^*-algebra: the Cuntz algebra O_n, introduced by Cuntz in 1977 [12]. Recall from Def. 1.33, that an element S in a C^*-algebra is an isometry, if $S^*S = 1$.

Definition 7.12. Let $n \in \mathbb{N}$, $n \geq 2$. The *Cuntz algebra* is the universal C^*-algebra

$$\mathcal{O}_n := C^*(S_1, \dots, S_n \mid S_i \text{ is an isometry, for all } i = 1, \dots, n, \sum_{i=1}^{n} S_i S_i^* = 1).$$

We may easily check that the Cuntz algebra exists in the sense of Lemma 6.6 and we check $\mathcal{O}_n \neq 0$ by finding some non-trivial representation on a Hilbert space.

For instance, if $(e_k)_{k\in\mathbb{N}}$ is an orthonormal basis of a separable Hilbert space H, choose injective functions $f_1, \dots, f_n : \mathbb{N} \to \mathbb{N}$ such that $f_i(\mathbb{N}) \cap f_j(\mathbb{N}) = \emptyset$ for $i \neq j$ and $\cup_{i=1}^{n} f_i(\mathbb{N}) = \mathbb{N}$ for their ranges. Putting $T_i e_k := e_{f_i(k)}$, for $i = 1, \dots, n$ and $k \in \mathbb{N}$, we obtain isometries $T_1, \dots, T_n \in B(H)$ with $\sum_i T_i T_i^* = 1$.

Hence, there is a non-trivial representation of $\mathcal{O}_n$ proving $\mathcal{O}_n \neq 0$. Note that the elements $T_i T_i^*$ are projections onto subspaces $K_i \subseteq H$ such that $H = K_1 \oplus \cdots \oplus K_n$. Hence, the Cuntz algebra is generated by n isometries decomposing the space into n copies of the space.

7.5. **Words in $\mathcal{O}_n$.** The main technical tool for studying Cuntz algebras is a detailed investigation of words in the generators.

Definition 7.13. Let $n \in \mathbb{N}$. A *multi-index* is a tuple $\mu = (\mu_1, \dots, \mu_k) \in \{1, \dots, n\}^k$ and $|\mu| = k$ is its *length*. Denote by $\mathcal{M}(k)$ the set of all multi indices of length k. A *word* in $\mathcal{O}_n$ is an element $S_\mu := S_{\mu_1} \cdots S_{\mu_k} \in \mathcal{O}_n$.

Let us take a look at the arithmetics of such words.

Lemma 7.14. *The words in $\mathcal{O}_n$, $n \geq 2$ satisfy the following relations.*

(a) $S_i^* S_j = \delta_{ij}$ *for all* $i, j = 1, \dots, n$.
(b) *Given multi indices μ and ν, we have:*

$$\text{If } |\mu| = |\nu|, \text{ then} \qquad S_\mu^* S_\nu = \delta_{\mu\nu}.$$

$$\text{If } |\mu| < |\nu|, \text{ then} \qquad S_\mu^* S_\nu = \begin{cases} S_{\nu'} & \text{if } \nu = \mu\nu' \\ 0 & \text{otherwise} \end{cases}.$$

$$\text{If } |\mu| > |\nu|, \text{ then} \qquad S_\mu^* S_\nu = \begin{cases} S_{\mu'}^* & \text{if } \mu = \nu\mu' \\ 0 & \text{otherwise} \end{cases}.$$

(c) *Let $k \in \mathbb{N}$. Then* $\sum_{\alpha\in\mathcal{M}(k)} S_\alpha S_\alpha^* = 1$.
(d) *Let μ, ν be multi indices with $|\mu| \neq |\nu|$ and $|\mu|, |\nu| \leq k$. Let $\alpha, \beta \in \mathcal{M}(k)$. Let $S_\gamma := S_1^{2k} S_2$. Then* $S_\gamma^* S_\alpha^* (S_\mu S_\nu^*) S_\beta S_\gamma = 0$.

Proof. For (a) and $i = j$, the relation $S_i^* S_i = 1$ holds by definition, since S_i is an isometry. For $i \neq j$, note that $S_i S_i^* + S_j S_j^* \leq \sum_k S_k S_k^* = 1$. Hence:

$$1 + S_i^* S_j S_j^* S_i = S_i^*(S_i S_i^* + S_j S_j^*) S_i \leq S_i^* S_i = 1$$

This shows $0 \leq -S_i^*S_jS_j^*S_i = -(S_j^*S_i)^*(S_j^*S_i)$. Thus $S_j^*S_i = 0$ by Lemma 4.7. The other relations in (b), (c) and (d) follow by direct algebraic manipulations and are left as an exercise, see Exc. 7.4. □

As a consequence, we find matrix units in $\mathcal{O}_n$ – and hence copies of matrix algebras. This is a crucial technical observation on the structure of $\mathcal{O}_n$. We define:

$$\begin{aligned}\mathcal{F}_n^k &:= \operatorname{span}\{S_\mu S_\nu^* \mid |\mu| = |\nu| = k\} \subseteq \mathcal{O}_n\\ \mathcal{F}_n &:= \overline{\operatorname{span}}\{S_\mu S_\nu^* \mid |\mu| = |\nu|\} \subseteq \mathcal{O}_n\\ \mathcal{S} &:= \operatorname{span}\{S_\mu S_\nu^* \mid \mu, \nu \text{ arbitrary multi indices}\} \subseteq \mathcal{O}_n\end{aligned}$$

Lemma 7.15. *For $n \geq 2$, we have the following.*

(a) *For $l \leq k$, we have $\mathcal{F}_n^l \subseteq \mathcal{F}_n^k$. Hence, $\mathcal{F}_n^k = \cup_{l\leq k}\mathcal{F}_n^l$.*
(b) *$\mathcal{S}$ is dense in $\mathcal{O}_n$.*
(c) *$\mathcal{F}_n^k$ is isomorphic to $M_{n^k}(\mathbb{C})$. The isomorphism may be chosen such that $S_1^k(S_1^*)^k$ corresponds to the matrix unit E_{11}.*

Proof. For (a), let $\mu, \nu \in \mathcal{M}(l)$. Then $S_\mu S_\nu^* = \sum_{\delta\in\mathcal{M}(k-l)} S_\mu S_\delta S_\delta^* S_\nu^* \in \mathcal{F}_n^k$.

Item (b) follows directly from Lemma 7.14(b), since all monomials in $\mathcal{O}_n$ are of the form $S_\mu S_\nu^*$ for μ, ν arbitrary multi indices (cf. also the proof of Lemma 7.3).

As for (c), put $e_{\mu\nu} := S_\mu S_\nu^* \in \mathcal{F}_n^k$. Using the relations from Lemma 7.14, we infer $e_{\mu\nu}^* = e_{\nu\mu}$ and $e_{\mu\nu}e_{\rho\sigma} = \delta_{\nu\rho}e_{\mu\sigma}$ for all $\mu, \nu, \rho, \sigma \in \mathcal{M}(k)$. Thus, the elements $e_{\mu\nu}$ are matrix units indexed by $|\mathcal{M}(k)| = n^k$ indices. By Cor. 6.12, $\mathcal{F}_n^k \cong M_{n^k}(\mathbb{C})$. We may arrange that the isomorphism maps $S_1^k(S_1^*)^k \in \mathcal{F}_n^k$ to the matrix unit $E_{11} \in M_{n^k}(\mathbb{C})$. □

7.6. **Conditional expectation of $\mathcal{O}_n$.** In analogy to Lemma 7.7 and Lemma 7.8, we will now construct a conditional expectation of $\mathcal{O}_n$. Given $\zeta \in S^1$, consider

$$\rho_\zeta : \mathcal{O}_n \to \mathcal{O}_n, \qquad S_i \mapsto \zeta S_i, \text{ for all } i = 1, \ldots, n.$$

This map exists by the universal property of $\mathcal{O}_n$ and it is a *-isomorphism with inverse $\rho_{\bar\zeta}$. We prove an analogue of Lemma 7.6.

Lemma 7.16. *Let $n \geq 2$ and $x \in \mathcal{O}_n$.*

(a) *The map $f_x : S^1 \to \mathcal{O}_n$, $\zeta \mapsto \rho_\zeta(x)$ is continuous in norm.*
(b) *The Riemannian sums $\frac{1}{n}\sum_{j=1}^n \rho_{e^{2\pi i t_j}}(x)$ converge, for uniform partitions $0 = t_0 < t_1 < \ldots < t_n = 1$ of the unit interval. We denote their limit by $\int_0^1 \rho_{e^{2\pi i t}}(x)\mathrm{d}t$.*

Proof. The proof is similar to the one for Lemma 7.6. We use $\rho_\zeta(S_\mu S_\nu^*) = \zeta^{|\mu|-|\nu|}S_\mu S_\nu^*$ and Lemma 7.15(b) for (a) and we argue as for classical Riemannian sums for (b). □

We then put

$$\varphi : \mathcal{O}_n \to \mathcal{O}_n, \qquad x \mapsto \int_0^1 \rho_{e^{2\pi i t}}(x)\mathrm{d}t.$$

Lemma 7.17. *The map $\varphi : \mathcal{O}_n \to \mathcal{O}_n$ is a faithful conditional expectation with $\|\varphi\| \leq 1$, $\varphi(S_\mu S_\nu^*) = \delta_{|\mu|,|\nu|} S_\mu S_\nu^*$ for all multi indices μ and ν, and $\varphi(\mathcal{O}_n) = \mathcal{F}_n$.*

Proof. As in the proof of Lemma 7.7(d), we derive that φ is a faithful conditional expectation with $\|\varphi\| \leq 1$. Moreover, let μ and ν be multi indices. Then:

$$\varphi(S_\mu S_\nu^*) = \int_0^1 \rho_{e^{2\pi it}}(S_\mu S_\nu^*)\mathrm{d}t = \left(\int_0^1 e^{2\pi it(|\mu|-|\nu|)}\mathrm{d}t\right) S_\mu S_\nu^* = \delta_{|\mu|,|\nu|} S_\mu S_\nu^* \qquad \square$$

As in the case of A_ϑ, we have an "algebraic description" of φ – at least locally.

Lemma 7.18. *Let $n \geq 2$ and $k \in \mathbb{N}$. There is an isometry $w \in \mathcal{O}_n$ such that the following holds true.*

(a) *We have $wx = xw$ for all $x \in \mathcal{F}_n^k$.*
(b) *We have $w^* S_\mu S_\nu^* w = \delta_{|\mu|,|\nu|} S_\mu S_\nu^*$ for all $\mu, \nu \in \cup_{l \leq k} \mathcal{M}(l)$.*
(c) *We have $\varphi(x) = w^* x w \in \mathcal{F}_n^k$ for all $x \in \operatorname{span}\{S_\mu S_\nu^* \mid \mu, \nu \in \cup_{l \leq k} \mathcal{M}(l)\}$.*

Proof. Let $S_\gamma := S_1^{2k} S_2$ and put

$$w := \sum_{\alpha \in \mathcal{M}(k)} S_\alpha S_\gamma S_\alpha^*.$$

We use Lemma 7.14 and check:

$$w^* w = \sum_{\alpha,\beta \in \mathcal{M}(k)} S_\alpha S_\gamma^* S_\alpha^* S_\beta S_\gamma S_\beta^* = \sum_{\alpha \in \mathcal{M}(k)} S_\alpha S_\gamma^* S_\gamma S_\alpha^* = \sum_{\alpha \in \mathcal{M}(k)} S_\alpha S_\alpha^* = 1$$

For (a), let $x = S_\mu S_\nu^* \in \mathcal{F}_n^k$ (i.e. $|\mu| = |\nu| = k$). We have

$$w S_\mu = \sum_{\alpha \in \mathcal{M}(k)} S_\alpha S_\gamma S_\alpha^* S_\mu = S_\mu S_\gamma \qquad \text{and} \qquad S_\nu^* w = \sum_{\alpha \in \mathcal{M}(k)} S_\nu^* S_\alpha S_\gamma S_\alpha^* = S_\gamma S_\nu^*$$

and hence $w S_\mu S_\nu^* = S_\mu S_\gamma S_\nu^* = S_\mu S_\nu^* w$. Passing to linear combinations, we infer $wx = xw$ for all $x \in \mathcal{F}_n^k$.

For (b), let $\mu, \nu \in \cup_{l \leq k} \mathcal{M}(l)$. In the first case, assume $|\mu| = |\nu| = k$. Then $w S_\mu S_\nu^* = S_\mu S_\nu^* w$ by (a) and hence $w^* S_\mu S_\nu^* w = S_\mu S_\nu^*$.

In the second case, assume $|\mu| = |\nu| = l < k$. Then $S_\mu S_\nu^* = \sum_{\alpha \in \mathcal{M}(k-l)} S_\mu S_\alpha S_\alpha^* S_\nu^*$ by Lemma 7.14(c). Then, $w^* S_\mu S_\alpha S_\alpha^* S_\nu^* w = S_\mu S_\alpha S_\alpha^* S_\nu^*$ by the first case and hence $w^* S_\mu S_\nu^* w = S_\mu S_\nu^*$.

In the third case, assume $|\mu| \neq |\nu|$. By Lemma 7.14:

$$w^* S_\mu S_\nu^* w = \sum_{\alpha,\beta \in \mathcal{M}(k)} S_\alpha S_\gamma^* S_\alpha^* S_\mu S_\nu^* S_\beta S_\gamma S_\beta^* = 0$$

Finally, for (c), let $x = S_\mu S_\nu^*$ with $\mu, \nu \in \cup_{l \leq k} \mathcal{M}(l)$. If $|\mu| = |\nu|$, then $\varphi(S_\mu S_\nu^*) = S_\mu S_\nu^* = w^* S_\mu S_\nu^* w$ by Lemma 7.17 and (b). On the other hand, if $|\mu| \neq |\nu|$, then $\varphi(S_\mu S_\nu^*) = 0$ by Lemma 7.17; moreover, $w^* S_\mu S_\nu^* w = 0$ by (b). Passing to linear combinations, we conclude $\varphi(x) = w^* x w$ for $x \in \operatorname{span}\{S_\mu S_\nu^* \mid \mu, \nu \in \cup_{l \leq k} \mathcal{M}(l)\}$; furthermore, $\varphi(x) \in \mathcal{F}_n^k$, by Lemma 7.15(a). $\square$

7.7. **Pure infiniteness of $\mathcal{O}_n$.** We have all ingredients to prove that $\mathcal{O}_n$ is simple. We may even show a stronger property: it is purely infinite.

Definition 7.19. A unital C^*-algebra is *purely infinite*, if for any non-zero element $x \in A$, there are $a, b \in A$ with $axb = 1$.

Note that any unital, purely infinite C^*-algebra A is simple. Indeed, let $I \lhd A$ be a nonzero closed ideal and choose $0 \neq x \in I$. We then find $a, b \in A$ such that $1 = axb \in I$, which shows $I = A$.

Theorem 7.20. *The Cuntz algebra $\mathcal{O}_n$ is purely infinite, for all $n \geq 2$.*

Proof. The proof is really cool: given $0 \neq x \in \mathcal{O}_n$, we shift it (or rather an element which is close to it) via the expectation φ to the matrix algebras $\mathcal{F}_n^k \cong M_{n^k}(\mathbb{C})$; we view it as a matrix and we project onto the eigenspace of its largest eigenvalue; we then rescale this small space to 1 which we can do since φ is locally algebraic in the sense of Lemma 7.18. Let us make this more precise.

Let $0 \neq x \in \mathcal{O}_n$. Since $x^*x \geq 0$ and $x^*x \neq 0$, we infer $\varphi(x^*x) \neq 0$, since the expectation φ is faithful by Lemma 7.17. Hence, we may assume $\|\varphi(x^*x)\| = 1$ after possibly normalizing.

(1) We find some $y \in \mathcal{S}$ with $y = y^*$, $\|x^*x - y\| < \frac{1}{4}$ and $\|\varphi(y)\| > \frac{3}{4}$.

Since $\mathcal{S}$ is dense in $\mathcal{O}_n$, see Lemma 7.15, we find $y_0 \in \mathcal{S}$ with $\|x^*x - y_0\| < \frac{1}{4}$. Putting $y := \frac{1}{2}(y_0 + y_0^*) \in \mathcal{S}$, we have $\|x^*x - y\| \leq \frac{1}{2}\|x^*x - y_0\| + \frac{1}{2}\|x^*x - y_0^*\| < \frac{1}{4}$. Moreover:

$$1 = \|\varphi(x^*x)\| \leq \|\varphi(x^*x - y)\| + \|\varphi(y)\| < \frac{1}{4} + \|\varphi(y)\|$$

This shows $\|\varphi(y)\| > \frac{3}{4}$.

(2) There is an isometry $w \in \mathcal{O}_n$ with $w^*yw = \varphi(y)$.

Since $y \in \mathcal{S}$, it is of the form $y = \sum_{i=1}^m \alpha_i S_{\mu^{(i)}} S^*_{\nu^{(i)}}$ for some $m \in \mathbb{N}$, $\alpha_i \in \mathbb{C}$ and multi indices $\mu^{(i)}$ and $\nu^{(i)}$, $i = 1, \dots, m$. Denote by $k \in \mathbb{N}$ the maximal length of these multi indices. By Lemma 7.18, there is an isometry $w \in \mathcal{O}_n$ with $w^*yw = \varphi(y)$.

(3) There are a unitary $u \in \mathcal{F}_n^k$ and $\beta \in \{-1, 1\}$ with $(S_1^*)^k u\varphi(y)u^* = \beta\|\varphi(y)\|(S_1^*)^k$.

This is the crucial step in the proof. We have $\varphi(y) \in \mathcal{F}_n^k$ by Lemma 7.18. By Lemma 7.15, $\mathcal{F}_n^k \cong M_{n^k}(\mathbb{C})$. Hence, $\varphi(y)$ corresponds to a matrix $Y \in M_{n^k}(\mathbb{C})$. It satisfies $r(Y) = r(\varphi(y)) = \|\varphi(y)\|$, by Cor. 2.14. Moreover, Y is selfadjoint, since $\varphi(y)$ is selfadjoint. Hence, all eigenvalues of Y are real, and $\beta\|\varphi(y)\|$ is an eigenvalue with $\beta = -1$ or $\beta = 1$.

Next, we firstly work in $M_{n^k}(\mathbb{C})$. Since Y is selfadjoint, we may diagonalise it. Thus, there is a unitary $U \in M_{n^k}(\mathbb{C})$ such that UYU^* is a diagonal matrix with the upper left entry being $\beta\|\varphi(y)\|$. Recall that $E_{11} \in M_{n^k}(\mathbb{C})$ is the projection with 1 in the upper left entry and zero elsewhere. We thus have:

$$E_{11}UYU^* = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} \beta\|\varphi(y)\| & 0 & \cdots & 0 \\ 0 & * & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & * \end{pmatrix} = \beta\|\varphi(y)\|E_{11}$$

Secondly, we transfer this equation to $\mathcal{F}_n^k$ using the isomorphism $\mathcal{F}_n^k \cong M_{n^k}(\mathbb{C})$. As mentioned in Lemma 7.15, the projection $E_{11} \in M_{n^k}(\mathbb{C})$ corresponds to the projection $S_1^k(S_1^*)^k \in \mathcal{F}_n^k$; the unitary $U \in M_{n^k}(\mathbb{C})$ corresponds to some unitary $u \in \mathcal{F}_n^k$; and $Y \in M_{n^k}(\mathbb{C})$ corresponds to $\varphi(y) \in \mathcal{F}_n^k$. We thus have:

$$S_1^k(S_1^*)^k u\varphi(y)u^* = \beta\|\varphi(y)\|S_1^k(S_1^*)^k$$

Multiplying with $(S_1^*)^k$ from the left yields $(S_1^*)^k u\varphi(y)u^* = \beta\|\varphi(y)\|(S_1^*)^k$.

(4) For $z := \|\varphi(y)\|^{-\frac{1}{2}}(S_1^*)^k uw^*$, we have $zyz^* = \beta 1$ and zx^*xz^* is invertible.

Building on (2) and (3), we compute:

$$\begin{aligned}
zyz^* &= \|\varphi(y)\|^{-1}(S_1^*)^k uw^*ywu^*S_1^k \\
&= \|\varphi(y)\|^{-1}(S_1^*)^k u\varphi(y)u^*S_1^k \\
&= \|\varphi(y)\|^{-1}\beta\|\varphi(y)\|(S_1^*)^k S_1^k \\
&= \beta 1
\end{aligned}$$

Next, let us compute the norm of z. Using (1), we have:

$$\|z\| \le \|\varphi(y)\|^{-\frac{1}{2}}\|(S_1^*)^k\|\|u\|\|w^*\| \le \|\varphi(y)\|^{-\frac{1}{2}} < \frac{2}{\sqrt{3}}$$

Again, using (1), we have:

$$\|1 - \beta zx^*xz^*\| = \|\beta z(y - x^*x)z^*\| \le \|z\|^2\|y - x^*x\| \le \frac{4}{3}\cdot\frac{1}{4} = \frac{1}{3} < 1$$

This shows that βzx^*xz^* is invertible, by Lemma 2.6, and so is zx^*xz^*.

(5) Putting $b := z^*(zx^*xz^*)^{-1}$ and $a := zx^*$, we have $axb = 1$.

Finally, compute:

$$axb = zx^*xz^*(zx^*xz^*)^{-1} = 1 \qquad \square$$

7.8. **Exercises.**

Exercise 7.1. Let $\vartheta \in \mathbb{R}$, $\lambda := e^{2\pi i\vartheta} \in S^1$ and $L^2(S^1)$ be the Hilbert space of L^2-integrable functions on the circle S^1. We define $\tilde{u} : L^2(S^1) \to L^2(S^1)$ and $\tilde{v} : L^2(S^1) \to L^2(S^1)$ by:

$$(\tilde{u}f)(z) := f(\lambda z), \qquad (\tilde{v}f)(z) := zf(z), \qquad f \in L^2(S^1), z \in S^1$$

Show that $\pi : A_\vartheta \to B(L^2(S^1))$ mapping $u \mapsto \tilde{u}$ and $v \mapsto \tilde{v}$ is a representation of A_ϑ. Considering the orthonormal basis $(e_n)_{n\in\mathbb{Z}}$, with $e_n(z) := z^n$ for $z \in S^1$, $n \in \mathbb{Z}$ – how does this representation relate to the one given in Lemma 7.2?

Exercise 7.2. Let $\zeta \in S^1 \subseteq \mathbb{C}$ and put:

$$a_n(\zeta) := \frac{1}{2n+1}\sum_{j=-n}^{n} \zeta^j, \qquad n \in \mathbb{N}$$

Let $\vartheta \in \mathbb{R}$ and put $\lambda := e^{2\pi i\vartheta} \in S^1$. Let $l \in \mathbb{Z}$.

(a) Show that $(a_n(\zeta))_{n\in\mathbb{N}}$ converges to zero, if $\zeta \neq 1$. *Hint:* Show that $\sum_{j=-n}^{n} \zeta^j$ is bounded using the formulas from geometric progression.
(b) Assume $\vartheta \notin \mathbb{Q}$. Show that $(a_n(\lambda^l))_{n\in\mathbb{N}}$ converges to δ_{l0}.
(c) Assume $\vartheta = \frac{p}{q} \in \mathbb{Q}$. Show that $(a_n(\lambda^l))_{n\in\mathbb{N}}$ converges to $\delta_{l\in q\mathbb{Z}}$. In particular, the sequence $(a_n(\lambda^l))_{n\in\mathbb{N}}$ converges to 1 for infinitely many powers λ^l, $l \in \mathbb{Z}$. Thus, in the rational case, the map $x \mapsto \lim_{n\to\infty} \frac{1}{2n+1} \sum_{j=-n}^{n} u^j x u^{-j}$ is way different from φ_1, see Lemma 7.8.

Exercise 7.3. Let $\vartheta = \frac{p}{q} \in \mathbb{Q}$.

(a) Find a representation $\pi : A_\vartheta \to M_q(\mathbb{C})$.
(b) Find unital C^*-algebras B and D as well as unital *-homomorphisms $\varphi : A_\vartheta \to B$ and $\psi : A_\vartheta \to D$ such that $\varphi(v^q) = 1$ and $\psi(v^q) \neq 1$.
(c) Conclude that A_ϑ is not simple.
(d) Show that there is a *-homomorphism $\sigma : C(\mathbb{T}^2) \to C^*(u^q, v^q) \subseteq A_\vartheta$ mapping the generators $\tilde{u}$ and $\tilde{v}$ of $C(\mathbb{T}^2)$ (see Lemma 7.4) to u^q and v^q. (In fact, it is even a *-isomorphism.)
(e) Convince yourself that none of these statements holds true for A_ϑ if $\vartheta \notin \mathbb{Q}$.

Exercise 7.4. Show the relations in Lemma 7.14, for $n \geq 2$.

(a) Convince yourself that $S_i S_i^*$ are projections in the sense of Def. 1.33.
(b) Given multi indices μ and ν, show:

$$\text{If } |\mu| = |\nu|, \text{ then} \qquad S_\mu^* S_\nu = \delta_{\mu\nu}.$$

$$\text{If } |\mu| < |\nu|, \text{ then} \qquad S_\mu^* S_\nu = \begin{cases} S_{\nu'} & \text{if } \nu = \mu\nu' \\ 0 & \text{otherwise} \end{cases}.$$

$$\text{If } |\mu| > |\nu|, \text{ then} \qquad S_\mu^* S_\nu = \begin{cases} S_{\mu'}^* & \text{if } \mu = \nu\mu' \\ 0 & \text{otherwise} \end{cases}.$$

(c) Let $k \in \mathbb{N}$. Show $\sum_{\alpha\in\mathcal{M}(k)} S_\alpha S_\alpha^* = 1$.
(d) Let μ, ν be multi indices with $|\mu| \neq |\nu|$ and $|\mu|, |\nu| \leq k$. Let $\alpha, \beta \in \mathcal{M}(k)$. Let $S_\gamma := S_1^{2k} S_2$. Show $S_\gamma^* S_\alpha^* (S_\mu S_\nu^*) S_\beta S_\gamma = 0$.

7.9. **Comments on A_ϑ.** The rotation algebra – in particular the irrational one – is one of the most studied examples of C^*-algebras. See for instance [13] or [23, Ch. 12] for more on A_ϑ. It is a key example of a noncommutative manifold in Connes's noncommutative geometry [10] and this is also a place where we study the noncommutative torus $\mathbb{T}_\vartheta^2$ in close analogy to the classical torus $\mathbb{T}^2$. This noncommutative geometry is very much inspired from physics and we quote from [23, Ch. 12]:

> At a deep and perhaps fundamental level, quantum field theory and noncommutative geometry are made of the same stuff.

See [23, Ch. 12] and [10] for more on this link to physics. Also, the noncommutative torus has been used to describe some mathematics behind the famous quantum Hall effect, see [2].

Back to mathematics, we may wonder whether the rotation algebra A_ϑ depends on the parameter $\vartheta \in \mathbb{R}$: do we obtain distinct C^*-algebras given distinct parameters? The answer is: A_ϑ and $A_{\vartheta'}$ are isomorphic if and only if $\vartheta' = \pm\vartheta \bmod \mathbb{Z}$. For the proof, we need the so called Powers-Rieffel projections and methods from K-theory, which is a homological theory of invariants for C^*-algebras. See [13] for this characterization of parameter dependence.

Along the lines of deforming the torus $\mathbb{T}^2$ to $\mathbb{T}^2_\vartheta$, Rieffel developed a program of the so called Rieffel deformation [47]. The basic idea is to use cocycle twists in order to deform the Cartesian product $S^1 \times S^1 = \mathbb{T}^2$. The Cartesian product corresponds to the tensor product of C^*-algebras on the C^*-algebraic side (note that there is no unique way to equip an algebraic tensor product of C^*-algebras with a C^*-norm in general [8]), so we may view A_ϑ as a deformation of the tensor product $C(S^1) \otimes C(S^1)$.

And in the upcoming lectures, we will see that the rotation algebra is actually an example of a crossed product: we will show that $A_\vartheta = C(S^1) \rtimes_\vartheta \mathbb{Z}$. So, A_ϑ arises from a dynamical system on the circle S^1, which is a rotation.

Besides, in case you are wondering about an "isometry version" of A_ϑ: we studied one "universal unitary" and one "universal isometry" in Lecture 6, and we studied two "universal commuting unitaries" as well as a scalar deformation of this commutation relation in the present lecture; how about two "universal commuting isometries" as well as scalar deformations of the commutation relation? This has actually been the content of the lecturer's PhD thesis; you may take a look at [56].

7.10. **Comments on $\mathcal{O}_n$.** The Cuntz algebra $\mathcal{O}_n$ is a very important example of a C^*-algebra, too. It has been studied extensively since its introduction by Cuntz in 1977 [12]. A remarkable feature is, that it is not only an example, but also a building block in the theory of C^*-algebras. For instance, there are statements of the form: "a separable C^*-algebra is exact if and only if it embeds into $\mathcal{O}_2$" or "a C^*-algebra A is unital, separable, simple and nuclear if and only if the tensor product $A \otimes \mathcal{O}_2$ is isomorphic to $\mathcal{O}_2$"; here, exactness and nuclearity are important approximation properties of C^*-algebras (actually, nuclearity corresponds to amenability of groups) [8]. So, amazingly, we may characterize certain properties with the help of the Cuntz algebra.

As for the term purely infinite, this comes from the theory of von Neumann algebras. A von Neumann algebra is of type III (or purely infinite), if it does not possess any finite projections. One can show that a similar feature holds for purely infinite C^*-algebras. For instance, any $S_\mu S_\mu^*$ is a projection; it can be decomposed into arbitrarily many subprojections, since

$$S_\mu S_\mu^* = \sum_{\alpha \in \mathcal{M}(k)} S_\alpha S_\mu S_\mu^* S_\alpha^*$$

for all k. Thus, $S_\mu S_\mu^*$ cannot be finite. Let us also mention that purely infiniteness plays an important role in the classification of C^*-algebras. There is the concept of a Kirchberg algebra, also called pi-sun algebras, where pi-sun stands for purely infinite, separable, unital, nuclear. Such C^*-algebras are classifiable via K-theory. See [54] for more on the classification of C^*-algebras.

Actually, K-theory (or Ext-theory) also plays a role when distinguishing the Cuntz algebras: again, we may ask whether $\mathcal{O}_n$ depends on the parameter $n \in \mathbb{N}$. The answer is yes: we have $\mathcal{O}_n \ncong \mathcal{O}_m$ for $n \neq m$, see [13].

Let us mention a couple of generalizations of Cuntz algebras. Firstly, what is $\mathcal{O}_n$ for $n = 1$, by the way? Well, you could say it is the C^*-algebra generated by one single isometry S_1 such that $S_1 S_1^* = 1$. Hence, S_1 is in fact a unitary and we conclude $\mathcal{O}_1 = C(S^1)$, by Cor. 6.16.

How about $n = \infty$? Yes, this C^*-algebra exists. But, $\sum_{i=1}^{\infty} S_i S_i^* = 1$ would not be a reasonable relation, since $a_n := \sum_{i=1}^{n} S_i S_i^*$ cannot converge in norm (check that!). The way out is to use the seemingly weaker relation $S_i^* S_j = \delta_{ij}$ from Lemma 7.14. The Cuntz algebra $\mathcal{O}_\infty$ is defined as the universal C^*-algebra generated by isometries S_k, $k \in \mathbb{N}$ with $S_i^* S_j = \delta_{ij}$ for all $i, j \in \mathbb{N}$. One can show that also $\mathcal{O}_\infty$ is purely infinite, so it is simple in particular. Hence, the relations $S_i^* S_j = \delta_{ij}$ are already the best we can do – we may not add further relations.

Now, we could wonder: okay, if $S_i^* S_j = \delta_{ij}$ is good enough for defining a simple C^*-algebra in the case $n = \infty$ – how about for $n < \infty$? Do the relations $S_i^* S_j = \delta_{ij}$ imply the relations $\sum_{i=1}^{n} S_i S_i^* = 1$? The answer is no, obviously – otherwise we would have $\sum_{i=1}^{n} S_i S_i^* = 1$ and $\sum_{i=1}^{n+1} S_i S_i^* = 1$ in $\mathcal{O}_{n+1}$, so S_{n+1} would be zero. In fact, the C^*-algebra $\mathcal{E}_n$ generated by isometries $S_1, \ldots, S_n$ and relations $S_i^* S_j = \delta_{ij}$ is called the extended Cuntz algebra. It contains the algebra of compact operators as an ideal and $\mathcal{O}_n$ as the corresponding quotient. So, in some sense, the difference between the relations $S_i^* S_j = \delta_{ij}$ and $\sum_i S_i S_i^*$ is $\mathcal{K}(H)$, see also Prop. 6.23.

Finally, let us briefly mention that Cuntz algebras have been generalized to Cuntz-Krieger algebras. They are given by partial isometries $S_1, \ldots, S_n$ (not necessarily isometries) with mutually orthogonal ranges and relations $S_i^* S_i = \sum_j a_{ij} S_j S_j^*$, where $a_{ij} \in \{0, 1\}$. Cuntz-Krieger algebras in turn have been generalized to graph C^*-algebras, where we assign a C^*-algebra $C^*(\Gamma)$ to a graph Γ. Graph C^*-algebras include the matrix algebras $M_N(\mathbb{C})$, the algebra of compact operators $\mathcal{K}(H)$ on a separable Hilbert space, the function algebra $C(S^1)$, the Toeplitz algebra $\mathcal{T}$, the Cuntz algebras $\mathcal{O}_n$ and many more. See [46] for more.

8. Inductive limits and AF algebras

Abstract. We introduce the concept of inductive limits of C^*-algebras. We then show that finite-dimensional C^*-algebras are exactly given by direct sums of matrix algebras. This is followed by a brief introduction to AF algebras (approximately finite-dimensional C^*-algebras) and the main tool to study them: Bratteli diagrams. We discuss how to read off the ideal structure of an AF algebra from its Bratteli diagram.

8.1. **Inductive limits.** In mathematics, the concept of approximation is everywhere: we want to understand a complicated object by approximating it with simpler ones. In the language of category theory, we may formulate this idea in terms of inductive limits. We present this concept adapted to the theory of C^*-algebras, but it is a principle of much wider generality.

Definition 8.1. An *inductive system (of C^*-algebras)* $(A_n, \varphi_n)_{n\in\mathbb{N}}$ is given by C^*-algebras A_n and *-homomorphisms $\varphi_n : A_n \to A_{n+1}$, for all $n \in \mathbb{N}$, i.e. we have:

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \cdots \qquad\qquad \xrightarrow{\varphi_{n-1}} A_n \xrightarrow{\varphi_n} A_{n+1} \xrightarrow{\varphi_{n+1}} \cdots$$

More generally, in category theory, an inductive system is given by objects $(A_n)_{n\in\mathbb{N}}$ and morphisms $(\varphi_n)_{n\in\mathbb{N}}$ between them, in a diagram as above. We may then ask for the existence of a limit object for such a sequence.

Proposition 8.2. *Let $(A_n, \varphi_n)_{n\in\mathbb{N}}$ be an inductive system of C^*-algebras. There are a C^*-algebra $\lim_{\overrightarrow{\varphi_n}} A_n$ and *-homomorphisms $\overline{\varphi_n} : A_n \to \lim_{\overrightarrow{\varphi_n}} A_n$ with $\overline{\varphi}_{n+1} \circ \varphi_n = \overline{\varphi}_n$ satisfying the following universal property: Given any C^*-algebra B and *-homomorphisms $\beta_n : A_n \to B$, $n \in \mathbb{N}$ with $\beta_{n+1} \circ \varphi_n = \beta_n$ there is a unique *-homomorphism $\beta : \lim_{\overrightarrow{\varphi_n}} A_n \to B$ such that the following diagram commutes.*

$$\begin{tikzcd}
 & & & & \lim_{\overrightarrow{\varphi_n}} A_n \arrow[dd, "\beta"] \\
\cdots \arrow[r] & A_n \arrow[urrr, "\overline{\varphi_n}"] \arrow[r, "\varphi_n"] \arrow[drrr, "\beta_n"'] & A_{n+1} \arrow[urr, "\overline{\varphi_{n+1}}"] \arrow[r, "\cdots"] \arrow[drr, "\beta_{n+1}"] & {} & \\
 & & & & B
\end{tikzcd}$$

The C^*-algebra $\lim_{\overrightarrow{\varphi_n}} A_n$ is unique up to isomorphism; it is called the inductive limit.

Proof. The idea is to define the inductive limit C^*-algebra as the set of all eventually stationary sequences. We are going to develop this algebra step by step.

(1) Construction of $\mathcal{A}$.

Let $\mathcal{A}_0$ be the set of all sequences $(x_n)_{n\in\mathbb{N}}$ with $x_n \in A_n$ for $n \in \mathbb{N}$ and the additional requirement: there exists some $N \in \mathbb{N}$ such that we have $x_{n+1} = \varphi_n(x_n)$ for all $n \geq N$. We say that $(x_n), (y_n) \in \mathcal{A}_0$ are equivalent, if they coincide on their tails, i.e. if there is some $N \in \mathbb{N}$ such that $x_n = y_n$ for all $n \geq N$. It is easy to check that this is an equivalence relation indeed; we denote by $[(x_n)_{n\in\mathbb{N}}]$ the equivalence classes. Let $\mathcal{A}$ be the quotient of $\mathcal{A}_0$ by this equivalence relation.

(2) Construction of a C^*-seminorm on $\mathcal{A}$ and definition of $\lim_{\overrightarrow{\varphi_n}} A_n$.

Given a sequence $(x_n)_{n\in\mathbb{N}} \in \mathcal{A}_0$, there is some $N \in \mathbb{N}$ with $x_{n+1} = \varphi_n(x_n)$ for all $n \geq N$, by definition. We then have for all $n \geq N$:

$$\|x_{n+1}\|_{A_{n+1}} = \|\varphi_n(x_n)\|_{A_{n+1}} \leq \|x_n\|_{A_n}$$

Thus, the sequence $(\|x_n\|_{A_n})_{n\in\mathbb{N}} \in \mathbb{C}$ converges and we may define

$$\|[(x_n)_{n\in\mathbb{N}}]\| := \lim_{n\to\infty} \|x_n\|_{A_n}$$

for $[(x_n)_{n\in\mathbb{N}}] \in \mathcal{A}$; one may check that this is a C^*-seminorm on $\mathcal{A}$. We mod out its null space and define $\lim_{\overrightarrow{\varphi_n}} A_n$ as the completion:

$$\lim_{\overrightarrow{\varphi_n}} A_n := \overline{\mathcal{A}/\{[(x_n)_{n\in\mathbb{N}}] \mid \|[(x_n)_{n\in\mathbb{N}}]\| = 0\}}^{\|\cdot\|}$$

One may then check that $\lim_{\overrightarrow{\varphi_n}} A_n$ is a C^*-algebra with the canonical entrywise operations. The elements in $\mathcal{A}/\{[(x_n)_{n\in\mathbb{N}}] \mid \|[(x_n)_{n\in\mathbb{N}}]\| = 0\}$ are again denoted by $[(x_n)_{n\in\mathbb{N}}]$.

(3) Construction of the maps $\overline{\varphi_n} : A_n \to \lim_{\overrightarrow{\varphi_n}} A_n$ with $\lim_{\overrightarrow{\varphi_n}} A_n = \overline{\cup_{n\in\mathbb{N}}\overline{\varphi_n}(A_n)}$.

For $x \in A_n$ we put:

$$\overline{\varphi_n}(x) := [(0, \ldots, 0, x, \varphi_n(x), \varphi_{n+1}(\varphi_n(x)), \ldots)]$$

Here, the entry x is at the n-th position. We may check that this is a *-homomorphism. Moreover, it satisfies:

$$\begin{aligned}
\overline{\varphi_{n+1}}(\varphi_n(x)) &= [(0, \ldots, 0, 0, \varphi_n(x), \varphi_{n+1}(\varphi_n(x)), \ldots)] \\
&= [(0, \ldots, 0, x, \varphi_n(x), \varphi_{n+1}(\varphi_n(x)), \ldots)] \\
&= \overline{\varphi_n}(x)
\end{aligned}$$

Here, we used the fact that two sequences are in the same equivalence class, if they coincide on their tails. Let us now prove that $\lim_{\overrightarrow{\varphi_n}} A_n = \overline{\cup_{n\in\mathbb{N}}\overline{\varphi_n}(A_n)}$ holds. Indeed, given $[(x_n)_{n\in\mathbb{N}}] \in \mathcal{A}/\{[(x_n)_{n\in\mathbb{N}}] \mid \|[(x_n)_{n\in\mathbb{N}}]\| = 0\}$, we may find some $N \in \mathbb{N}$ with $x_{n+1} = \varphi_n(x_n)$ for all $n \geq N$. Thus, $[(x_n)_{n\in\mathbb{N}}] = \overline{\varphi_N}(x_N) \in \cup_{n\in\mathbb{N}}\overline{\varphi_n}(A_n)$.

(4) The universal property of $\lim_{\overrightarrow{\varphi_n}} A_n$.

Let B be a C^*-algebra and let $\beta_n : A_n \to B$ be *-homomorphisms with $\beta_{n+1} \circ \varphi_n = \beta_n$, for all $n \in \mathbb{N}$. We construct $\beta : \lim_{\overrightarrow{\varphi_n}} A_n \to B$. Given $[(x_n)_{n\in\mathbb{N}}] \in \mathcal{A}/\{[(x_n)_{n\in\mathbb{N}}] \mid \|[(x_n)_{n\in\mathbb{N}}]\| = 0\}$, we find some $N \in \mathbb{N}$ such that $x_{n+1} = \varphi_n(x_n)$ for all $n \geq N$.

Thus, $\beta_n(x_n) = \beta_{n+1}(\varphi_n(x_n)) = \beta_{n+1}(x_{n+1})$ for all $n \geq N$. We may thus define

$$\beta([(x_n)_{n\in\mathbb{N}}]) := \lim_{n\to\infty} \beta_n(x_n)$$

and check that it is a *-homomorphism whose norm is bounded by 1; we then extend it to $\lim A_n$. Moreover:

$$\beta(\overline{\varphi_n}(x)) = \beta([(0,\ldots,0,x,\varphi_n(x),\varphi_{n+1}(\varphi_n(x)),\ldots)]) = \beta_n(x)$$

Hence, the diagram of the assertion is commutative. Moreover, β is unique, since any other map $\beta' : \lim_{\overrightarrow{\varphi_n}} A_n \to B$ with $\beta' \circ \overline{\varphi_n} = \beta_n$, for all $n \in \mathbb{N}$, coincides with β on $\cup_{n\in\mathbb{N}}\overline{\varphi_n}(A_n)$ and hence also on $\lim_{\overrightarrow{\varphi_n}} A_n = \overline{\cup_{n\in\mathbb{N}}\overline{\varphi_n}(A_n)}$.

As a side note, observe that putting $B = \lim_{\overrightarrow{\varphi_n}} A_n$, the map $\mathrm{id}_{\lim A_n} : \lim_{\overrightarrow{\varphi_n}} A_n \to \lim_{\overrightarrow{\varphi_n}} A_n$ is the unique map with the property $\mathrm{id}_{\lim A_n} \circ \overline{\varphi_n} = \overline{\varphi_n}$, for all $n \in \mathbb{N}$, by the universal property of $\lim_{\overrightarrow{\varphi_n}} A_n$.

(5) Uniqueness of $\lim_{\overrightarrow{\varphi_n}} A_n$.

Finally, let us show that $\lim_{\overrightarrow{\varphi_n}} A_n$ is unique up to isomorphism. So, let A' be another C^*-algebra with *-homomorphisms $\overline{\varphi_n}' : A_n \to A'$ and $\overline{\varphi}'_{n+1} \circ \varphi_n = \overline{\varphi}'_n$, for all $n \in \mathbb{N}$, and assume that A' satisfies the above universal property. Then, there is a map $\beta : \lim_{\overrightarrow{\varphi_n}} A_n \to A'$ by the universal property of $\lim_{\overrightarrow{\varphi_n}} A_n$ satisfying $\beta \circ \overline{\varphi_n} = \overline{\varphi_n}'$ for all $n \in \mathbb{N}$; likewise, we have a map $\beta' : A' \to \lim_{\overrightarrow{\varphi_n}} A_n$ by the universal property of A' satisfying $\beta' \circ \overline{\varphi_n}' = \overline{\varphi_n}$ for all $n \in \mathbb{N}$. This implies $\beta' \circ \beta \circ \overline{\varphi_n} = \overline{\varphi_n}$, for all $n \in \mathbb{N}$. Hence, $\beta' \circ \beta = \mathrm{id}_{\lim A_n}$ by the side note in step (4). Likewise $\beta \circ \beta' = \mathrm{id}_{A'}$. We conclude that $\lim_{\overrightarrow{\varphi_n}} A_n$ and A' are isomorphic. $\square$

Remark 8.3. We comment on some facts, omitting the proofs.

(a) In step (3) of the above proof, we have seen that $\lim_{\overrightarrow{\varphi_n}} A_n = \overline{\cup_{n\in\mathbb{N}}\overline{\varphi_n}(A_n)}$ holds. In the special case of a sequence of C^*-subalgebras $A_1 \subseteq A_2 \subseteq \ldots \subseteq A$, we have $\lim_{\overrightarrow{\iota_n}} A_n = \overline{\cup_{n\in\mathbb{N}} A_n} \subseteq A$, where $\iota_n : A_n \to A_{n+1}$ is the identity map.

(b) If all maps $\beta_n : A_n \to B$ in the above proposition are injective, then so is $\beta : \lim_{\overrightarrow{\varphi_n}} A_n \to B$. Likewise, if $\cup\beta_n(A_n)$ is dense in B, then β is surjective.

(c) If all C^*-algebras A_n are simple, then so is $\lim_{\overrightarrow{\varphi_n}} A_n$.

We will come to examples of inductive limits soon.

8.2. **Finite-dimensional C^*-algebras.** We briefly discuss finite-dimensional C^*-algebras. In fact, the topology does not play any role in finite dimensions, so we are actually dealing with purely algebraic objects.

We have seen examples of finite-dimensional C^*-algebras before: there are the matrix algebras $M_N(\mathbb{C})$, and there is $\mathbb{C}^N$, see Exc. 6.4. Actually, we can write the latter one also as $\mathbb{C}\oplus\cdots\oplus\mathbb{C}$, where we used the direct sum of C^*-algebras as defined in Def. 1.8. Generalizing these considerations, we infer that we know many examples of finite-dimensional C^*-algebras, namely $\bigoplus_{i=k}^m M_{N_k}(\mathbb{C})$ with $N_1,\ldots,N_m \in \mathbb{N}$ and $m \in \mathbb{N}$. And in fact: these are already all finite-dimensional C^*-algebras! This is known as Wedderburn's Theorem, or Artin-Wedderburn Theorem, which holds true more generally, for semisimple rings. It predates the theory of C^*-algebras by a number of decades.

Lemma 8.4. *Any simple finite-dimensional C^*-algebra A is isomorphic to some $M_N(\mathbb{C})$.*

Proof. We do not give a proof here, but the idea is as follows. We represent A on some $B(H)$ by an irreducible representation π. Irreducibility will help us to deduce that H is finite-dimensional, so $B(H) \cong M_N(\mathbb{C})$ for some $N \in \mathbb{N}$. Since A is simple, we obtain $A \cong \pi(A) \subseteq M_N(\mathbb{C})$. Further work shows $\pi(A) = M_N(\mathbb{C})$. $\square$

Proposition 8.5 (Wedderburn's Theorem)**.** *Let A be a finite-dimensional C^*-algebra. There are $m \in \mathbb{N}$ and $N_1, \ldots, N_m \in \mathbb{N}$ such that:*

$$A \cong \bigoplus_{k=1}^{m} M_{N_k}(\mathbb{C})$$

Proof. We give the proof in the language of C^*-algebras.

(1) A is unital.

Let (u_λ) be an approximate unit of A. Since A is finite-dimensional, the set $\{x \in A \mid \|x\| \leq 1\}$ is compact; hence (u_λ) possesses a convergent subsequence. Denoting its limit by 1, we may check that 1 is a unit of A indeed.

(2) For any ideal $I \lhd A$, there is a central projection $p \in A$ (i.e. $ap = pa$ for all $a \in A$) with $I = Ap$.

Since A is finite-dimensional, I is closed. Hence, I is a C^*-algebra. By (1) it is unital, i.e. there is some $p \in I$ with $p = p^* = p^2$ and $pa = a$ for all $a \in I$. This shows $I \subseteq pA$. The converse, $I \supseteq pA$ follows from $p \in I$. Finally, p is central, i.e. $ap = pa$ for all $a \in A$. Indeed, $ap = pap$ for all $a \in A$, since $ap \in I$ and p is a unit for I. Thus, $pa = (a^*p)^* = (pa^*p)^* = pap = ap$ for all $a \in A$.

(3) There exist $m \in \mathbb{N}$ and central projections $p_1, \ldots, p_m \in A$ with $p_i p_j = 0$ for $i \neq j$, $\sum_k p_k = 1$ and $A = \sum_k Ap_k$, where all Ap_k are simple.

The center $Z(A) := \{a \in A \mid ab = ba \text{ for all } b \in A\} \subseteq A$ is a commutative C^*-algebra. Hence $Z(A) \cong C(X)$ for some compact space X. As A is finite-dimensional, X is finite, so $X = \{1, \ldots, m\}$ for some $m \in \mathbb{N}$. The characteristic functions $p'_k := \chi_{\{k\}} \in C(X)$ are continuous and they are projections with $p'_i p'_j = 0$ for $i \neq j$ and $\sum_k p'_k = 1$; see also Exc. 6.4. We thus find corresponding projections $p_1, \ldots, p_m \in Z(A)$ with $p_i p_j = 0$ for $i \neq j$. Since the unit of A is in $Z(A)$, the unit of $Z(A)$ and the one of A coincide and we obtain $\sum_k p_k = 1$ and $A = \sum_k Ap_k$. Checking $\mathbb{C}p_k \subseteq Z(Ap_k) \subseteq Z(A)p_k = \mathbb{C}p_k$, we may easily derive that Ap_k is simple, for $k = 1, \ldots, m$, by (2).

(4) We have $A \cong \oplus_{k=1}^m M_{N_k}(\mathbb{C})$.

By Lemma 8.4 and (3), we obtain (4). $\square$

We are now going to study homomorphisms between matrix algebras. Given a finite-dimensional C^*-algebra $A = \bigoplus_{k=1}^m M_{N_k}(\mathbb{C})$ with $k \in \{1, \ldots, m\}$, denote by $e_{ij}^{(k)} \in M_{N_k}(\mathbb{C})$, $i,j = 1,\dots,N_k$ the matrix units in $M_{N_k}(\mathbb{C})$; let us use small letters for matrices from now on. Note that the elements $e_{ii}^{(k)}$ are projections in the sense of Def. 1.33, i.e. $e_{ii}^{(k)} = (e_{ii}^{(k)})^* = (e_{ii}^{(k)})^2$. They form a partition of unity: $\sum_{k=1}^m \sum_{i=1}^{N_k} e_{ii}^{(k)} = 1 \in A$.

Recall that $\mathrm{Tr} : M_N(\mathbb{C}) \to \mathbb{C}$ denotes the (unnormalized) trace, i.e. $\mathrm{Tr}((a_{ij})_{i,j}) = \sum_{i=1}^N a_{ii}$, see Exm. 5.2. If $p \in M_N(\mathbb{C})$ is a projection, then $\mathrm{Tr}(p) \in \mathbb{N}$ is the dimension of the subspace onto which p projects.

Given A as above, let $B = \bigoplus_{l=1}^n M_{K_l}(\mathbb{C})$ be another finite-dimensional C^*-algebra and let $\varphi : A \to B$ be a *-homomorphism. We denote by $\varphi_1,\dots,\varphi_n$ its components, i.e. we have $\varphi_l : A \to M_{K_l}(\mathbb{C})$, for $l = 1,\dots,n$. We put

$$\Phi_{lk} := \mathrm{Tr}_{M_{K_l}}(\varphi_l(e_{11}^{(k)})) \in \mathbb{N}_0$$

for $l = 1,\dots,n$ and $k = 1,\dots,m$. Let us call the matrix $\Phi = (\Phi_{lk})_{l,k} \in M_{n\times m}(\mathbb{N}_0)$ the coefficient matrix of φ. By the way, note that

$$\mathrm{Tr}_{M_{K_l}}(\varphi_l(e_{ii}^{(k)})) = \mathrm{Tr}_{M_{K_l}}(\varphi_l(e_{i1}^{(k)}e_{1i}^{(k)})) = \mathrm{Tr}_{M_{K_l}}(\varphi_l(e_{1i}^{(k)}e_{i1}^{(k)})) = \mathrm{Tr}_{M_{K_l}}(\varphi_l(e_{11}^{(k)})).$$

So, Φ contains the information about the whole partition of unity $\sum_k \sum_i e_{ii}^{(k)} = 1$.

Lemma 8.6. *The coefficient matrix determines a map between finite-dimensional C^*-algebras completely. More precisely, let A and B be finite-dimensional C^*-algebras and let $\varphi,\psi : A \to B$ be *-homomorphisms. If $\Phi = \Psi$ for their coefficient matrices, then there is some unitary $u \in B$ such that $\varphi(x) = u\psi(x)u^*$ for all $x \in A$.*

Proof. The details of the proof are shifted to Exc. 8.1. By comparison of the coefficients Φ and Ψ, we know that given $k \in \{1,\dots,m\}$ and $l \in \{1,\dots,n\}$, the subspaces $\varphi_l(e_{11}^{(k)})\mathbb{C}^{K_l}$ and $\psi_l(e_{11}^{(k)})\mathbb{C}^{K_l}$ have the same dimensions. We may thus find a partial isometry $v_{lk} \in M_{K_l}(\mathbb{C})$ mapping $\varphi_l(e_{11}^{(k)})\mathbb{C}^{K_l}$ to $\psi_l(e_{11}^{(k)})\mathbb{C}^{K_l}$. Putting these partial isometries v_{lk} together in a clever way, we obtain the unitary u. □

As a consequence, we may present a *-homomorphism $\varphi : A \to B$ between finite-dimensional C^*-algebras $A = \bigoplus_{k=1}^m M_{N_k}(\mathbb{C})$ and $B = \bigoplus_{l=1}^n M_{K_l}(\mathbb{C})$ by a diagram of the following form, where we write Φ_{lk} many arrows between N_k and K_l.

$$
\begin{tikzcd}
N_1 \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & K_1 \\
N_2 \arrow[ur] \arrow[r] \arrow[dddr] & K_2 \\
\dots & \dots \\
N_m \arrow[dr, shift left] \arrow[dr] \arrow[dr, shift right] & \dots \\
 & K_n
\end{tikzcd}
$$

Example 8.7. Let $n \in \mathbb{N}$, $A = M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ and $B = M_{2n}(\mathbb{C}) \oplus M_{2n}(\mathbb{C})$. Consider $\varphi : A \to B$ given by:

$$\varphi(x, y) = \left(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \right) \in M_{2n}(\mathbb{C}) \oplus M_{2n}(\mathbb{C}), \quad (x, y) \in M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$$

Then, the corresponding diagram is of the form:

$$
\begin{tikzcd}
n \arrow[r] \arrow[dr] & 2n \\
n \arrow[ur] \arrow[r] & 2n
\end{tikzcd}
$$

Note that the map

$$\varphi(x, y) = \left(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix} \right) \in M_{2n}(\mathbb{C}) \oplus M_{2n}(\mathbb{C}), \quad (x, y) \in M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$$

yields the same diagram as above.

8.3. **AF algebras and Bratteli diagrams.**

Definition 8.8. A C^*-algebra A is an *AF-algebra* (or *approximately finite-dimensional C^*-algebra*) if it is the inductive limit $\lim_{\overrightarrow{\varphi_n}} A_n$ of an inductive system $(A_n, \varphi_n)_{n \in \mathbb{N}}$, where all C^*-algebras A_n are finite-dimensional.

Let $(A_n, \varphi_n)_{n \in \mathbb{N}}$ be an inductive system underlying an AF algebra A. According to Lemma 8.6 and the subsequent discussion, we may represent the maps $\varphi_n : A_n \to A_{n+1}$ by a diagram. So, the whole inductive system yields an infinite diagram, where the vertices represent the matrix algebras of the finite-dimensional

C^*-algebras A_n and the arrows represent the maps φ_n in terms of their coefficient matrices. Such a diagram is called a Bratteli diagram, named after the Norwegian mathematician Ola Bratteli[5] [5]. It completely determines an AF algebra.

Lemma 8.9. *Two AF algebras with the same Bratteli diagrams are isomorphic.*

Proof. The details of the proof are contained in Exc. 8.2. Let A and B be AF algebras arising from inductive systems $(A_n, \varphi_n)_{n\in\mathbb{N}}$ and $(B_n, \psi_n)_{n\in\mathbb{N}}$ respectively. Assume that they have the same Bratteli diagrams.

From the numbers in the Bratteli diagrams, we infer that $A_n = B_n$ for all $n \in \mathbb{N}$ (up to reordering of direct sums). By Lemma 8.6, for any $n \in \mathbb{N}$, we may find unitaries $v_{n+1} \in B_{n+1}$ with $\varphi_n(x) = v_{n+1}\psi_n(x)v_{n+1}^*$ for all $x \in A_n$.

Putting $u_1 := 1$ and $u_{n+1} := v_{n+1}\psi_n(u_n) \in B_{n+1}$ in case $\psi_n(1) = 1$ (and some extension to a unitary otherwise), we obtain

$$u_{n+1}^*\varphi_n(x)u_{n+1} = \psi_n(u_n^*)v_{n+1}^*v_{n+1}\psi_n(x)v_{n+1}^*v_{n+1}\psi_n(u_n) = \psi_n(u_n^*xu_n).$$

One then checks that the maps $\alpha_n : A_n \to B_n$, $x \mapsto u_n^*xu_n$ induce $A \cong B$. $\square$

Example 8.10. Let us take a look at a number of examples of AF algebras.

(a) The Bratteli diagram

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \dots$$

yields the algebra of compact operators $\mathcal{K}(H)$ on a separable Hilbert space H. Indeed, this diagram may be translated to

$$\mathbb{C} \xrightarrow{\iota_1} M_2(\mathbb{C}) \xrightarrow{\iota_2} M_3(\mathbb{C}) \xrightarrow{\iota_3} \dots$$

where the homomorphisms $\iota_n : M_n(\mathbb{C}) \to M_{n+1}(\mathbb{C})$ are given by

$$x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}.$$

So, we are in the situation of Rem. 8.3(a) with $\mathbb{C} \subseteq M_2(\mathbb{C}) \subseteq \dots \subseteq \mathcal{K}(H)$ (see also the proof of Prop. 6.13) and $\lim_{\overrightarrow{\iota_n}} M_n(\mathbb{C}) = \overline{\cup_{n\in\mathbb{N}} M_n(\mathbb{C})} = \mathcal{K}(H)$.

(b) The Bratteli diagram

$$\begin{tikzcd} 1 \arrow[r] \arrow[dr] & 1 \arrow[r] \arrow[dr] & 1 \arrow[r] \arrow[dr] & \dots \\ 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & \dots \end{tikzcd}$$

yields the unitization $\widetilde{\mathcal{K}(H)}$ as in Exm. 2.23.

[5] Don't pronounce it like an Italian name – the stress is on the first syllable.

(c) The Bratteli diagram

$$1 \rightrightarrows 2 \rightrightarrows 4 \rightrightarrows 8 \rightrightarrows \dots$$

yields the so called CAR (canonical anticommutation relations) algebra[6] denoted by M_{2^∞}, the arrows being interpreted as

$$x \mapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}.$$

In fact, the following Bratteli diagram also yields the CAR algebra:

$$\begin{tikzcd}
1 \arrow[r] \arrow[dr] & 2 \arrow[r] \arrow[dr] & 4 \arrow[r] \arrow[dr] & \dots \\
1 \arrow[r] \arrow[ur] & 2 \arrow[r] \arrow[ur] & 4 \arrow[r] \arrow[ur] & \dots
\end{tikzcd}$$

Hence, the converse of Lemma 8.9 is not true: Different Bratteli diagrams may produce the same AF algebra. In the present case, consider a third Bratteli diagram

$$\begin{tikzcd}
 & 1 \arrow[dr] & & 2 \arrow[dr] & & 4 \arrow[dr] & & \dots \\
1 \arrow[ur] \arrow[dr] & & 2 \arrow[ur] \arrow[dr] & & 4 \arrow[ur] \arrow[dr] & & 8 \arrow[ur] \arrow[dr] & \\
 & 1 \arrow[ur] & & 2 \arrow[ur] & & 4 \arrow[ur] & & \dots
\end{tikzcd}$$

Check that $(A_{2n+1}, \varphi'_{2n+1})_{n\in\mathbb{N}}$ yields the first diagram of (c), whereas $(A_{2n}, \varphi''_{2n})_{n\in\mathbb{N}}$ produces the second one; here φ'_{2n+1} and φ''_{2n} are appropriate compositions of the maps in the diagram. Since

$$\lim(A_n, \varphi_n) = \lim(A_{2n}, \varphi'_{2n}) = \lim(A_{2n+1}, \varphi''_{2n+1}),$$

all these three Bratteli diagrams yield the same AF algebra.

(d) Given a sequence $(n_k)_{k\in\mathbb{N}}$ of natural numbers $n_k \geq 2$, the Bratteli diagram

$$1 \overset{(n_1\text{ many})}{\rightrightarrows} n_1 \overset{(n_2\text{ many})}{\rightrightarrows} n_1 n_2 \overset{(n_3\text{ many})}{\rightrightarrows} n_1 n_2 n_3 \overset{(n_4\text{ many})}{\rightrightarrows} \dots$$

yields a so called UHF algebra generalizing M_{2^∞}.

[6]By the way, the CAR algebra may also be represented by creation and annihilation operators on the antisymmetric Fock space.

8.4. **Ideals in AF algebras.** One of the main avantages of Bratteli diagrams is that we can read the ideal structure of an AF algebra from its diagram as follows. We view a Bratteli diagram as a directed graph in the canonical way.

Definition 8.11. Let a Bratteli diagram be given.

(a) If ζ is a vertex in the diagram and there is an arrow pointing to another vertex ξ, then ξ is called a *successor* of ζ.
(b) A subdiagram of a Bratteli diagram is called *directed*, if it is closed under all successors of its vertices, i.e. if ζ is a vertex in the subdiagram and ξ is its successor, then also ξ must be in the subdiagram.
(c) A subdiagram is called *hereditary*, if it is closed under predecessors in the following way: if ζ is a vertex in the Bratteli diagram such that all of its successors lie in the subdiagram, then also ζ must be in the subdiagram.

Theorem 8.12. *Given an AF algebra A with some Bratteli diagram, there is a bijection between the closed ideals in A and the directed, hereditary subdiagrams of this Bratteli diagram. Such a subdiagram gives rise to a Bratteli diagram of the corresponding ideal I; the complement of this subdiagram in turn is a Brattelli diagram of the quotient A/I. In particular, ideals and quotients of AF algebras are AF algebras.*

Proof. We omit the proof; see [13, Sect. III.4]. $\square$

Example 8.13. Let us take a look at the AF algebras in Exm. 8.10 and their ideals.

(a) The Bratteli diagram in Exm. 8.10(a) has no directed, hereditary subdiagram apart from itself. Hence, the corresponding AF algebra $\mathcal{K}(H)$ is simple. We knew this already from Cor. 6.14.
(b) The lower line of the diagram in Exm. 8.10(b) is the only non-trivial directed, hereditary subdiagram. By Exm. 8.10(a), it corresponds to the ideal $\mathcal{K}(H)$ in $\widetilde{\mathcal{K}(H)}$ (cf. also Prop. 2.20). The quotient of $\widetilde{\mathcal{K}(H)}$ by $\mathcal{K}(H)$ has the upper line of the diagram in Exm. 8.10(a) as its Bratteli diagram. It is thus $\mathbb{C}$.
(c) UHF algebras are simple, in particular the CAR algebra is simple.

One may give a converse of Thm. 8.12 as follows, combining results by Bratteli [5], Brown and Elliott [7, 17].

Theorem 8.14. *Let A be a separable C^*-algebra and $I \lhd A$ be a closed ideal. The algebra A is an AF algebra, if and only if I and A/I are AF algebras.*

Proof. We omit the proof; see [13, Sect. III.6]. $\square$

For those who like short exact sequences, we can say that AF algebras satisfy the "two out of three property": let

$$0 \to I \to A \to B \to 0$$

be a short exact sequence; if two of the three C^*-algebras I, A and B are AF algebras, then so is the third, by Thm. 8.14; see also Rem. 4.25.

Just for completeness, we shall mention Bratteli's [5] local characterization of AF algebras (which is an ingredient of the proof of Thm. 8.12).

Proposition 8.15. *A separable C^*-algebra A is an AF algebra, if and only if for all $\varepsilon > 0$, all $n \in \mathbb{N}$ and all $x_1, \ldots, x_n \in A$ there is a finite-dimensional C^*-subalgebra $B \subseteq A$ and elements $y_1, \ldots, y_n \in B$ such that $\|x_i - y_i\| < \varepsilon$ for all $i = 1, \ldots, n$.*

Proof. We omit the proof; see [5, Thm. 2.2]. □

8.5. **Some remarks on AF algebras.** AF algebras were the first class of C^*-algebras that was classified (by Elliott) via K-theory, the latter being a homological theory of invariants. This was the starting point of Elliott's classification program for C^*-algebras which found a climax in [54].

In 1980, Pimsner and Voiculescu [38] constructed an AF algebra into which they embedded the rotation algebra A_ϑ. This was a key step towards the classification of A_ϑ by its parameter ϑ, see Sect. 7.9. Their AF algebra is constructed as follows. Let $\vartheta \in \mathbb{R}\backslash\mathbb{Q}$. We may write ϑ as a continued fraction

$$\vartheta = \lim_{n\to\infty} \frac{p_n}{q_n},$$

where $\frac{p_n}{q_n}$ is given as

$$\frac{p_n}{q_n} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\cdots + \frac{1}{a_n}}}}.$$

There is a recursion formula for p_n and q_n given by

$$\begin{pmatrix} p_{n+1} & q_{n+1} \\ p_n & q_n \end{pmatrix} = \begin{pmatrix} a_{n+1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_n & q_n \\ p_{n-1} & q_{n-1} \end{pmatrix}$$

The AF algebra constructed in [38] is then given by $A_n := M_{q_n}(\mathbb{C}) \oplus M_{q_{n-1}}(\mathbb{C})$ and $\varphi_n : A_n \to A_{n+1}$ with the diagram

$$\begin{tikzcd} q_n \arrow[r, shift left, "(a_{n+1}\text{ many})"] \arrow[r, shift right] \arrow[dr] & q_{n+1} \\ q_{n-1} \arrow[ur] & q_n \end{tikzcd}$$

One may then show that there is an injective *-homomorphism from A_ϑ to $\lim_{\overrightarrow{\varphi_n}} A_n$.

8.6. **Exercises.**

Exercise 8.1. We prove some details needed in the proof of Lemma 8.6. Let $A = \bigoplus_{k=1}^m M_{N_k}(\mathbb{C})$ and $B = \bigoplus_{l=1}^n M_{K_l}(\mathbb{C})$ and let $\varphi, \psi : A \to B$ be two *-homomorphisms. Assume $\Phi = \Psi$, i.e. assume for all $k = 1, \ldots, m$ and $l = 1, \ldots, n$:

$$\mathrm{Tr}_{M_{K_l}}(\varphi_l(e_{11}^{(k)})) = \Phi_{lk} = \Psi_{lk} = \mathrm{Tr}_{M_{K_l}}(\psi_l(e_{11}^{(k)})) \in \mathbb{N}_0$$

(a) Let $k \in \{1, \dots, m\}$ and $l \in \{1, \dots, n\}$. Show that there is a partial isometry $v_{lk} \in M_{K_l}(\mathbb{C})$ such that $v_{lk}^* v_{lk} = \varphi_l(e_{11}^{(k)})$ and $v_{lk} v_{lk}^* = \psi_l(e_{11}^{(k)})$. (cf. Exc. 1.8)
(b) Put $v_l := \sum_{k=1}^m \sum_{i=1}^{N_k} \psi_l(e_{i1}^{(k)}) v_{lk} \varphi_l(e_{1i}^{(k)}) \in M_{K_l}(\mathbb{C})$. Show that we have $v_l^* v_l = \sum_{k=1}^m \sum_{i=1}^{N_k} \varphi_l(e_{ii}^{(k)})$.
(c) Deduce from (b) that $v := \sum_{l=1}^n v_l \in B$ is a partial isometry with $v^*v = \varphi(1)$ and $vv^* = \psi(1)$.
(d) Use $\Phi = \Psi$ to find a partial isometry $w \in B$ with $w^*w = 1 - \varphi(1)$ and $ww^* = 1 - \psi(1)$.
(e) Put $u := v + w \in B$. Show that u is a unitary with $u^*\psi(e_{ij}^{(k)})u = \varphi(e_{ij}^{(k)})$ for all $i, j = 1, \dots, N_k$ and $k = 1, \dots, m$.
(f) Deduce $u^*\psi(x)u = \varphi(x)$ for all $x \in A$.

Exercise 8.2. We investigate the details of the proof of Lemma 8.9. Let A and B be AF algebras arising from inductive systems $(A_n, \varphi_n)_{n\in\mathbb{N}}$ and $(B_n, \psi_n)_{n\in\mathbb{N}}$ respectively. Assume that they have the same Bratteli diagrams.

(a) Convince yourself: $A_n = B_n$ for all $n \in \mathbb{N}$, up to reordering of direct sums.
(b) Let $n \in \mathbb{N}$. Convince yourself that there is a unitary $v_{n+1} \in B_{n+1}$ with $\varphi_n(x) = v_{n+1}\psi_n(x)v_{n+1}^*$ for all $x \in A_n$, by Lemma 8.6.
(c) Put $u_1 := 1 \in A_1$. For $n \geq 1$, put $u_{n+1}^0 := v_{n+1}\psi_n(u_n) \in B_{n+1}$ and $u_{n+1} := v_{n+1}\psi_n(u_n) + w_n \in B_{n+1}$, where w_n is a partial isometry with $w_n^* w_n = 1 - (u_{n+1}^0)^* u_{n+1}^0$ and $w_n w_n^* = 1 - u_{n+1}^0 (u_{n+1}^0)^*$. Show that such a partial isometry w_n exists. Show that u_{n+1} is a unitary.
(d) Define $\alpha_n : A_n \to B_n$ via $\alpha_n(x) := u_n^* x u_n$ for $x \in A_n$. Show that the following diagram is commutative.

$$
\begin{tikzcd}
A_1 \arrow[r, "\varphi_1"] \arrow[d, "\alpha_1"] & A_2 \arrow[r, "\varphi_2"] \arrow[d, "\alpha_2"] & A_3 \arrow[r, "\varphi_3"] \arrow[d, "\alpha_3"] & \dots \\
B_1 \arrow[r, "\psi_1"] & B_2 \arrow[r, "\psi_2"] & B_3 \arrow[r, "\psi_3"] & \dots
\end{tikzcd}
$$

(e) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence with $x_n \in A_n$ for $n \in \mathbb{N}$ and assume that we have some $N \in \mathbb{N}$ with $x_{n+1} = \varphi_n(x_n)$ for all $n \geq N$. Use (d) to show that $\alpha_{n+1}(x_{n+1}) = \psi_n(\alpha_n(x_n))$ for all $n \geq N$. Deduce that we may define $\alpha : A \to B$ by $[(x_n)_{n\in\mathbb{N}}] \mapsto [(\alpha_n(x_n))_{n\in\mathbb{N}}]$. Show that it is a *-isomorphism.

Exercise 8.3. Consider the Bratteli diagram:

$$\begin{tikzcd}
1 \arrow[r] \arrow[ddddr] & 1 \arrow[r] \arrow[ddr] & 1 \arrow[r] \arrow[dr] & 1 \arrow[r] & \dots \\
 & & & 1 & \dots \\
 & & 1 \arrow[r] \arrow[dr] & 1 & \dots \\
 & & & 1 & \\
 & 1 \arrow[r] \arrow[ddr] & 1 \arrow[r] \arrow[dr] & 1 & \\
 & & & 1 & \\
 & & 1 \arrow[r] \arrow[dr] & 1 & \\
 & & & 1 &
\end{tikzcd}$$

(a) Determine all ideals of the corresponding AF algebra A.
(b) Show that A is isomorphic to $C(X)$, where X is the Cantor set.

9. C^*-DYNAMICAL SYSTEMS AND CROSSED PRODUCTS

ABSTRACT. We introduce C^*-dynamical systems and their associated crossed product C^*-algebras. The crossed product construction produces a wealth of examples of C^*-algebras, and allows one to link the study of C^*-algebras with dynamical systems. We verify the universal property of full crossed products, and establish some basic properties of reduced crossed products. As a special case of these general considerations we discuss the full and reduced group C^*-algebras of a discrete group.

9.1. **C^*-dynamical systems.** As explained in the introduction, the aim of the remainder of this course is to explain links between dynamical systems and C^*-algebras. In the present lecture we prepare the ground for this by introducing the basic concepts and constructions providing a bridge between these two subjects. The scope of the theory is broad, but for expository reasons we will restrict ourselves to discrete dynamical systems, and only consider actions on compact spaces. On the C^*-algebraic side this means that we will work with actions of discrete groups on unital C^*-algebras. In particular, we will not consider systems with continuous time evolution. Both our assumptions – requiring the group to be discrete and the algebra to be unital – can be lifted, but they simplify some of the arguments and, we hope, will allow us to make the basic ideas more transparent. At the same time, already in this setting there exist plenty of interesting examples which exhibit key features of the theory.

So what kind of dynamical systems will we study? Assume that we are given a topological space X and a homeomorphism $\alpha_1 : X \to X$. The points of X can be thought of as states of a physical system, and we may view the action of α_1 as a discrete time evolution, mapping a state x at time n to $\alpha_1(x)$ at time $n+1$. Here our discrete time variable n can be viewed as an element of $\mathbb{Z}$; note that we can also go "backwards in time" using α_1^{-1}. In fact, the map α_1 determines a group homomorphism $\alpha : \mathbb{Z} \to \mathrm{Homeo}(X)$ into the group of homeomorphisms of X such that $\alpha_n(x) = \alpha_1^n(x)$ is given by iterated application of α_1 or α_1^{-1}. Note that we have $\alpha_0 = \mathrm{id}$, and the definition of α_n is compatible with our original notation for $n = 1$.

Allowing more general (discrete) groups than $\mathbb{Z}$ to enter the picture we arrive at the following definition:

Definition 9.1. A *classical dynamical system* is a triple (X, G, α) consisting of a group G, a topological space X, and a group homomorphism $\alpha : G \to \mathrm{Homeo}(X)$.

In the sequel we will only consider classical dynamical systems (X, G, α) for which the underlying topological space X is a compact Hausdorff space. As indicated above, this simplifies some arguments and constructions. Just to reiterate we will only consider discrete groups G in the sequel.

How does the concept of a dynamical system extend to the noncommutative world? Let A be a unital C^*-algebra. Recall that a $*$-automorphism of A is a bijective $*$-homomorphism $\alpha : A \to A$. By Proposition 4.15 every $*$-automorphism is isometric, and the inverse of α is again a $*$-automorphism. It follows that the $*$-automorphisms of A form a group which we denote by $\mathrm{Aut}(A)$. By an *action* of a group G on A we mean a group homomorphism $\alpha : G \to \mathrm{Aut}(A)$.

Definition 9.2. Let G be a group and let A be a unital C^*-algebra. A *C^*-dynamical system* is a triple (A, G, α) consisting of a C^*-algebra A, a group G, and an action $\alpha : G \to \mathrm{Aut}(A)$.

When dealing with a C^*-dynamical system (A, G, α), we shall usually write $\alpha_t(a)$, or simply $t \cdot a$, for the action of $t \in G$ on an element $a \in A$ implemented by α. Note that we have $t \cdot 1 = 1$ for all $t \in G$, that is, the unit element of A is fixed by the action of G, because every $*$-automorphism of a unital C^*-algebra is automatically unital.

Example 9.3. Let us take a look at some examples of C^*-dynamical systems.

(a) Let A be an arbitrary unital C^*-algebra and let G be a group. Defining $\tau_t(a) = a$ for all $t \in G$ and $a \in A$ defines an action $\tau : G \to \mathrm{Aut}(A)$. For obvious reasons, this action is called the *trivial action* of G on A.

(b) Let A be any unital C^*-algebra and let $\alpha_1 \in \mathrm{Aut}(A)$ be a fixed automorphism. Then $\alpha_n(a) = \alpha_1^n(a)$ defines an action of $\mathbb{Z}$ on A. Conversely, every action of $\mathbb{Z}$ arises from an automorphism in this way, so that a dynamical system $(A, \mathbb{Z}, \alpha)$ is the same thing as a unital C^*-algebra A together with a single $*$-automorphism.

(c) Let X be a compact space and let (X, G, α) be a classical dynamical system, with the action of $t \in G$ on $x \in X$ written $t \cdot x$. Then we obtain an action on $A = C(X)$ by setting $\alpha_t(f)(x) = (t \cdot f)(x) = f(t^{-1} \cdot x)$. The only reason to take the inverse of t on the right hand side of this formula is to make α a group homomorphism, and not a group anti-homomorphism.

(d) As a special case of the above examples consider the algebra $A = C(\mathbb{T})$ of continuous functions on the torus $\mathbb{T} = S^1$, and let α be the homeomorphism of $\mathbb{T}$ given by $\alpha(z) = e^{2\pi i\vartheta} z$ for some $\vartheta \in \mathbb{R}$. This homeomorphism induces an automorphism of A as in (c), so an action of $\mathbb{Z}$ on A as in (b). For $\vartheta = 0$ this becomes the trivial action of $\mathbb{Z}$ on A as in (a). The most interesting case is that ϑ is irrational, and we will come back to this further below.

Two C^*-dynamical systems (A, G, α) and (B, H, β) are called *conjugate*, if there exists a group isomorphism $f : G \to H$ and a $*$-isomorphism $\varphi : A \to B$ such that $\beta_{f(t)}(\varphi(a)) = \varphi(\alpha_t(a))$ for all $a \in A$ and $t \in G$. A basic question in the theory is to understand when two given systems are conjugate.

9.2. **Covariant representations and the convolution algebra.** Let A be a unital C^*-algebra. A $*$-automorphism $\alpha \in \mathrm{Aut}(A)$ is called *inner* if there exists a unitary element $u \in A$ such that $\alpha(a) = uau^*$ for all $a \in A$. It is easy to check that the set $\mathrm{Inn}(A)$ of all inner automorphisms forms a normal subgroup of $\mathrm{Aut}(A)$.

We say that an action of a group G on A is *inner* if there is a group homomorphism $u : G \to U(A)$ from G into the group $U(A)$ of unitary elements in A such that $\alpha_t(a) = u_t a u_t^*$ for all $t \in G$ and all $a \in A$. Clearly, there are no interesting inner actions on commutative C^*-algebras, because commutativity implies that any such action is trivial.

The basic idea behind the construction of crossed products of C^*-dynamical systems is to turn an arbitrary action into an inner action, in a certain prescribed way. There are actually *two* natural ways in which this can be done, leading to the notion of full and reduced crossed products. It may actually be surprising at first sight that these two constructions really lead to different results in general.

In order to define crossed products we need some preparations. By a unitary representation of a group G on a Hilbert space H we mean a group homomorphism from G into the group $U(H)$ of unitary elements in the algebra $B(H)$ of bounded operators on H.

Definition 9.4. A *covariant representation* of a C^*-dynamical system (A, G, α) consists of a Hilbert space H, a unital $*$-representation $\pi : A \to B(H)$, and a unitary representation $U : G \to U(H)$ such that

$$\pi(\alpha_t(a)) = U_t \pi(a) U_t^*$$

for all $a \in A$ and $t \in G$.

Note that a covariant representation (π, U) represents the underlying C^*-algebra of a C^*-dynamical system on a Hilbert space in such a way that the action of G has the correct form to become inner. However, the implementing unitaries are not necessarily contained in the algebra we started with, or rather in the image $\pi(A) \subseteq B(H)$ of A. In order to resolve this issue we shall form a new algebra, the convolution algebra of (A, G, α), by "adding" these unitaries to A.

The underlying vector space of the convolution algebra of a C^*-dynamical system (A, G, α) is the space

$$C_c(G, A) = \{f : G \to A \mid f(t) \neq 0 \text{ only for finitely many } t \in G\}$$

of finitely supported maps from G with values in A. We define a multiplication and $*$-structure on $C_c(G, A)$ by setting

$$(f * g)(t) = \sum_{s \in G} f(s) s \cdot g(s^{-1} t)$$

and

$$f^*(t) = t \cdot f(t^{-1})^* = (t \cdot f(t^{-1}))^*$$

for $f, g \in C_c(G, A)$. Note that the summation in the definition of $f * g$ runs only over finitely many group elements since both f and g are finitely supported. The product $f * g$ is also called the convolution of f and g. Since the $*$-algebra structure on $C_c(G, A)$ depends not only on G and A but also on α we shall write $C_c(G, A, \alpha)$ instead of $C_c(G, A)$ in the sequel. We should first check, of course, that we really obtain a $*$-algebra in this way:

Lemma 9.5. *With the above operations, $C_c(G, A, \alpha)$ becomes a $*$-algebra.*

Proof. Let us verify that convolution is associative. We compute

$$\begin{aligned}((f * g) * h)(t) &= \sum_{r \in G} (f * g)(r) r \cdot h(r^{-1}t) \\ &= \sum_{r \in G} \sum_{s \in G} f(s)(s \cdot g(s^{-1}r)) r \cdot h(r^{-1}t) \\ &= \sum_{r \in G} \sum_{s \in G} f(s) s \cdot (g(r) r \cdot h(r^{-1}s^{-1}t)) \\ &= \sum_{s \in G} f(s) s \cdot (g * h)(s^{-1}t) \\ &= (f * (g * h))(t),\end{aligned}$$

shifting summation from r to sr in the third step. The $*$-operation is clearly a complex antilinear map, and since

$$f^{**}(t) = t \cdot f^*(t^{-1})^* = t \cdot (t^{-1} \cdot f(t)^*)^* = f(t)^{**} = f(t)$$

we have $f^{**} = f$ for all $f \in C_c(G, A, \alpha)$. Moreover

$$\begin{aligned}(f * g)^*(t) &= \sum_{s \in G} t \cdot (f(s) s \cdot g(s^{-1}t^{-1}))^* \\ &= \sum_{s \in G} t \cdot (s \cdot g(s^{-1}t^{-1})^* f(s)^*) \\ &= \sum_{s \in G} s \cdot g(s^{-1})^* t \cdot f(t^{-1}s)^* \\ &= \sum_{s \in G} g^*(s) s \cdot f^*(s^{-1}t) \\ &= (g^* * f^*)(t),\end{aligned}$$

shifting summation from s to $t^{-1}s$ in the third step. This shows $(f * g)^* = g^* * f^*$ as required. □

As already indicated above, we call $C_c(G, A, \alpha)$ the convolution algebra of (A, G, α). It is important to keep in mind that the algebra structure relevant for us is *not* the one given by pointwise multiplication – although the latter may seem to be the more obvious choice at first sight.

A good way to understand the structure of the convolution algebra is to write elements of $C_c(G, A, \alpha)$ as finite linear combinations of elements $a\delta_t$, where $a \in A$ and $t \in G$, as follows. We identify elements of A with "Dirac" functions supported at the identity element e of G, that is, using the embedding $A \subseteq C_c(G, A, \alpha)$ implemented by the $*$-homomorphism $i_A : A \to C_c(G, A, \alpha)$ given by

$$i_A(a)(t) = \begin{cases} a & t = e \\ 0 & \text{else.} \end{cases}$$

Similarly, we consider group elements $t \in G$ as elements of $C_c(G, A, \alpha)$ using the map $i_G : G \to C_c(G, A, \alpha)$ given by

$$i_G(s)(t) = \delta_s(t) = \begin{cases} 1 & t = s \\ 0 & \text{else.} \end{cases}$$

With this notation in place we calculate

$$(a\delta_s)(b\delta_t) = a(s \cdot b)\delta_{st},$$

that is, the multiplication of $C_c(G, A, \alpha)$ is determined by the multiplication of A, together with the formula $\delta_s\delta_t = \delta_{st}$ and the "twisted" commutation relation

$$\delta_s a = (s \cdot a)\delta_s,$$

involving the action of G on A. Note that every element of $C_c(G, A, \alpha)$ can indeed be written as a linear combination $\sum_{s \in G} a_s\delta_s$ for elements $a_s \in A$, with $a_s \neq 0$ for only finitely many group elements $s \in G$. We also get

$$(a\delta_s)^* = (s^{-1} \cdot a^*)\delta_{s^{-1}},$$

which can be understood by writing $(a\delta_s)^* = \delta_{s^{-1}}a^*$ and then reordering factors using the above twisted commutation relation. That is, the $*$-structure of $C_c(G, A, \alpha)$ is uniquely determined by the given $*$-structure on A and the formula $\delta_t^* = \delta_{t^{-1}}$ for $t \in G$. We note also that the above description shows that $C_c(G, A, \alpha)$ is in fact a unital $*$-algebra with unit element δ_e.

Let us now explain the connection between covariant representations and the convolution algebra.

Definition 9.6. If (π, U) is a covariant representation of (A, G, α) on a Hilbert space H then we define the *integrated form* of (π, U) as the $*$-representation $\pi \rtimes U : C_c(G, A, \alpha) \to B(H)$ of the convolution algebra $C_c(G, A, \alpha)$ by

$$(\pi \rtimes U)(f)(\xi) = \sum_{t \in G} \pi(f(t))U_t(\xi).$$

It is straightforward to check that the integrated form of a covariant representation is a unital $*$-representation of the convolution algebra.

9.3. **Reduced crossed products.** Our next aim is to describe the construction of the reduced crossed product of a C^*-dynamical system. However, before we come to this we need an additional preparation regarding tensor products of Hilbert spaces.

Let H, K be Hilbert spaces. Then the algebraic tensor product $H \otimes_{alg} K$ is equipped with a natural sequilinear inner product such that

$$\langle x \otimes y, x' \otimes y' \rangle_{H \otimes K} = \langle x, x' \rangle_H \langle y, y' \rangle_K$$

for $x, x' \in H, y, y' \in K$. However, unless H or K are finite dimensional the algebraic tensor product is not complete with respect to this inner product. The completion of $H \otimes_{alg} K$ with respect to the norm associated with $\langle\ ,\ \rangle_{H\otimes K}$ is again a Hilbert space, which we will denote by $H \otimes K$ in the sequel.

Actually we will only need a very special case of this construction, where one of the Hilbert spaces is $l^2(G)$, the Hilbert space of all square summable functions $f : G \to \mathbb{C}$. In this case one has an isometric isomorphism

$$l^2(G) \otimes H \cong l^2(G, H) = \{f : G \to H \mid \sum_{t\in G} \|f(t)\|^2 < \infty\}$$

for any Hilbert space H, with respect to Hilbert space structure on $l^2(G, H)$ defined by

$$\langle f, g\rangle = \left(\sum_{t\in G} \langle f(t), g(t)\rangle\right)^{1/2}.$$

In this picture, an element of the form $\delta_s \otimes \xi \in l^2(G) \otimes H$ for $s \in G$ and $\xi \in H$ corresponds to the function in $l^2(G, H)$ which takes the value ξ at $t = s$, and is zero for $t \neq s$.

Let (A, G, α) be a C^*-dynamical system and let $\pi^u : A \to B(H_u)$ be the universal representation of A, that is, the direct sum of the GNS-representations for all states of A as in the proof of Theorem 5.19. We define a covariant representation (Π^u, λ^u) of (A, G, α) on $H_u \otimes l^2(G)$ by

$$\Pi^u(a)(\xi \otimes \delta_s) = \pi^u(s^{-1} \cdot a)(\xi) \otimes \delta_s$$

and

$$\lambda^u_t(\xi \otimes \delta_s) = \xi \otimes \delta_{ts}.$$

To verify the covariance condition we compute

$$\begin{aligned}
\Pi^u(\alpha_t(a))(\xi \otimes \delta_s) &= \pi^u(s^{-1}t \cdot a)(\xi) \otimes \delta_s \\
&= \lambda^u_t(\pi^u(s^{-1}t \cdot a)(\xi) \otimes \delta_{t^{-1}s})) \\
&= \lambda^u_t \Pi^u(a)(\xi \otimes \delta_{t^{-1}s}) \\
&= \lambda^u_t \Pi^u(a)(\lambda^u_t)^*(\xi \otimes \delta_s)
\end{aligned}$$

as required. The integrated form $\Pi^u \rtimes \lambda^u$ satisfies

$$(\Pi^u \rtimes \lambda^u)(a\delta_t)(\xi \otimes \delta_s) = \pi^u((ts)^{-1} \cdot a)(\xi) \otimes \delta_{ts} = (\pi^u((ts)^{-1} \cdot a) \otimes \lambda^u_t)(\xi \otimes \delta_s).$$

We call $\Pi^u \rtimes \lambda^u$ the *regular representation* of (A, G, α). Using the above formula for $(\Pi^u \rtimes \lambda^u)(a\delta_t)$ one can check that $\Pi^u \rtimes \lambda^u : C_c(G, A, \alpha) \to B(H)$ is injective, see Exercise 9.2. In particular, the reduced crossed product norm, defined by

$$\|f\|_{\mathrm{r}} = \|(\Pi^u \rtimes \lambda^u)(f)\|,$$

is indeed a norm on $C_c(G, A, \alpha)$, and not just a seminorm.

Definition 9.7. Let (A, G, α) be a C^*-dynamical system. The *reduced crossed product* $A \rtimes_{\alpha,\mathrm{r}} G$ is the completion of $C_c(G, A, \alpha)$ with respect to the reduced crossed product norm.

In other words, the reduced crossed product $A \rtimes_{\alpha,\mathrm{r}} G$ is the norm closure of the image of $C_c(G, A, \alpha)$ in $B(H \otimes l^2(G))$ under the map $\Pi^u \rtimes \lambda^u$. If there is no risk of confusion we will also write $A \rtimes_{\mathrm{r}} G$ instead of $A \rtimes_{\alpha,\mathrm{r}} G$.

In practice it is not very convenient to work with the universal representation of A when studying the structure of $A \rtimes_{\alpha,\mathrm{r}} G$. Note that if $\pi : A \to B(H)$ is an arbitrary unital representation of A we can define a covariant representation (Π, λ) of (A, G, α) on $H \otimes l^2(G)$ in the same way as we did for the regular representation. That is, we just need to leave out the superscript u in the formulas discussed above. In this way we obtain a unital $*$-representation $(\Pi \rtimes \lambda) : C_c(G, A, \alpha) \to B(H \otimes l^2(G))$ of the convolution algebra.

Proposition 9.8. *Let (A, G, α) be a C^*-dynamical system. If $\pi : A \to B(H)$ is a faithful unital $*$-representation then, with the notation introduced above, we have*

$$\|(\Pi \rtimes \lambda)(f)\| = \|f\|_{\mathrm{r}}$$

for all $f \in C_c(G, A, \alpha)$. That is, we can use any faithful $*$-representation of A to define the minimal crossed product norm.

Proof. Our argument follows [8] Chapter 4, Proposition 1.5. Let $F \subseteq G$ be a finite set and denote by $\mathbb{C}[F] \subseteq l^2(G)$ the linear span of the standard orthonormal basis vectors δ_t for $t \in F$. Let $p_F \in B(H \otimes l^2(G))$ be the orthogonal projection onto the closed subspace $H \otimes \mathbb{C}[F]$ of $H \otimes l^2(G)$.

We will first show that the operator norm of $p_F(\Pi \rtimes \lambda)(h)p_F$ is independent of the choice of π for any element $h \in C_c(G, A, \alpha)$. To this end we write $h = \sum_{r \in G} a_r \delta_r$ as a finite linear combination of "Dirac" functions with values in A, and let $e_{r,t} \in B(\mathbb{C}[F])$ for $r, t \in F$ be the standard matrix units. We calculate

$$\begin{aligned} p_F(\Pi \rtimes \lambda)(h)p_F &= \sum_{r \in G} \sum_{s \in F} p_F(\pi((rs)^{-1} \cdot a_r) \otimes e_{rs,s}) \\ &= \sum_{r \in G} \sum_{s \in F \cap r^{-1}F} \pi((rs)^{-1} \cdot a_r) \otimes e_{rs,s}. \end{aligned}$$

The right hand side of this formula can be viewed as the image of an operator in $A \otimes B(\mathbb{C}[F])$ under the faithful representation of $A \otimes B(\mathbb{C}[F])$ on $H \otimes \mathbb{C}[F]$ induced by π. The claim follows from the fact that the C^*-norm on the $*$-algebra $A \otimes B(\mathbb{C}[F])$, which is nothing but a matrix algebra with entries in A, is unique.

To elaborate on this write $F = \{t_1, \dots, t_n\}$ and view $H \otimes \mathbb{C}[F] \cong \bigoplus_{i=1}^n H$ as an orthogonal direct sum of copies of H, by identifying $H \otimes \delta_{t_i}$ with the i-th summand. Then $B(H \otimes \mathbb{C}[F])$ identifies with $M_n(B(H))$ in a canonical way, by associating the operator $T \in B(H \otimes \mathbb{C}[F])$ given by

$$T(x_1, \ldots, x_n) = (\sum_j T_1 j(x_j), \ldots, \sum_j T_{nj}(x_j))$$

to the matrix $(T_{ij}) \in M_n(B(H))$. If we view A as a $*$-subalgebra of $B(H)$ via π, then these considerations allows us to view $M_n(A)$ as a $*$-subalgebra of $B(H \otimes \mathbb{C}[F])$. Using that $A \subseteq B(H)$ is closed one checks that the same is true for $M_n(A) \subseteq B(H \otimes \mathbb{C}[F])$. In other words, the $*$-algebra $M_n(A)$ of $n \times n$ matrices with entries in the C^*-algebra A is again a C^*-algebra. In particular, the C^*-norm on this algebra is uniquely determined.

Now let $f \in C_c(G, A, \alpha)$ be arbitrary. Consider the set $\mathbb{F}$ of all finite subsets of G, directed by inclusion. Then we clearly have $\lim_{F \in \mathbb{F}} p_F(\xi) = \xi$ for every $\xi \in H \otimes l^2(G)$. Hence

$$\begin{aligned}
\|(\Pi \rtimes \lambda)(f)\| &= \sup_{\|\xi\|=1} \|(\Pi \rtimes \lambda)(f)(\xi)\| \\
&= \sup_{\|\xi\|=1} \sup_{F \in \mathbb{F}} \|(\Pi \rtimes \lambda)(f) p_F(\xi)\| \\
&= \sup_{\|\xi\|=1} \sup_{F \in \mathbb{F}} \langle (\Pi \rtimes \lambda)(f) p_F(\xi), (\Pi \rtimes \lambda)(f) p_F(\xi) \rangle^{1/2} \\
&= \sup_{\|\xi\|=1} \sup_{F \in \mathbb{F}} \langle p_F(\xi), p_F(\Pi \rtimes \lambda)(f^* f) p_F(\xi) \rangle^{1/2} \\
&= \sup_{F \in \mathbb{F}} \|p_F(\Pi \rtimes \lambda)(f^* f) p_F\|^{1/2},
\end{aligned}$$

so that the first part of our proof applied to $h = f^* f$ yields the claim. $\square$

9.4. **Full crossed products.** If (A, G, α) is a C^*-dynamical system we define the *full crossed product norm* on the convolution algebra $C_c(G, A, \alpha)$ by

$$\|f\|_{\mathrm{f}} = \sup_{(\pi, U)} \|(\pi \rtimes U)(f)\|,$$

where the supremum runs over all possible covariant representations. For the careful reader, we note that there are no set-theoretical issues with this supremum: it is in fact enough to consider covariant representations on Hilbert spaces of sufficiently large cardinality.

Writing $f = \sum_{r \in G} a_r \delta_r$ as a finite linear combination of Dirac functions we obtain

$$\|(\pi \rtimes U)(f)\| = \|\sum_{r \in G} \pi(a_r) U_r\| \leq \sum_{r \in G} \|a_r\|$$

for all covariant representations (π, U), so that $\|f\|_{\mathrm{f}} < \infty$ for all $f \in C_c(G, A, \alpha)$. This implies that the full crossed product norm is a well-defined seminorm on $C_c(G, A, \alpha)$. Using the regular covariant representation, which was the basis for the construction of the reduced crossed product $A \rtimes_{\alpha, \mathrm{r}} G$, we see that the full crossed product norm is indeed a norm, and not only a seminorm. Indeed, we have $\|f\|_{\mathrm{f}} \geq \|f\|_{\mathrm{r}}$ for all $f \in C_c(G,A,\alpha)$ by construction, so that $\|f\|_{\mathrm{f}} \neq 0$ for $f \neq 0$, by the corresponding property of $\|f\|_{\mathrm{r}}$.

Definition 9.9. Let (A,G,α) be a C^*-dynamical system. The *full crossed product* $A \rtimes_{\alpha,\mathrm{f}} G$ is the completion of $C_c(G,A,\alpha)$ with respect to the full crossed product norm.

In the same way as for reduced crossed products we will write $A \rtimes_{\mathrm{f}} G$ instead of $A \rtimes_{\alpha,\mathrm{f}} G$ if there is no risk for confusion. We get maps $i_A : A \to A \rtimes_{\alpha,\mathrm{f}} G$ and $i_G : G \to A \rtimes_{\alpha,\mathrm{f}} G$ by extending the corresponding maps for the convolution algebra, using the embedding $C_c(G,A,\alpha) \to A \rtimes_{\alpha,\mathrm{f}} G$.

By construction of $A \rtimes_{\alpha,\mathrm{f}} G$, there is a canonical surjective $*$-homomorphism $\pi : A \rtimes_{\alpha,\mathrm{f}} G \to A \rtimes_{\alpha,\mathrm{r}} G$. From Definition 9.9 we also obtain the following universal property.

Proposition 9.10. *Let (A,G,α) be a C^*-dynamical system. The full crossed product $A \rtimes_{\alpha,\mathrm{f}} G$ is universal for covariant representations of (A,G,α). That is, if (π,U) is a covariant representation of (A,G,α) on a Hilbert space H, then there exists a unique unital $*$-homomorphism $\pi \rtimes U : A \rtimes_{\alpha,\mathrm{f}} G \to B(H)$ such that*

$$(\pi \rtimes U) \circ i_A = \pi, \qquad (\pi \rtimes U) \circ i_G = U.$$

Proof. The integrated form $(\pi \rtimes U) : C_c(G,A,\alpha) \to B(H)$ of the covariant representation (π,U) is norm-decreasing by definition of the full crossed product norm. Therefore it extends canonically to a unital $*$-homomorphism $A \rtimes_{\alpha,\mathrm{f}} G \to B(H)$, which we denote again by $\pi \rtimes U$. The relations $(\pi \rtimes U) \circ i_A = \pi$ and $(\pi \rtimes U) \circ i_G = U$ hold by construction.

Uniqueness of $\pi \rtimes U : A \rtimes_{\alpha,\mathrm{f}} G \to B(H)$ follows from the fact that $C_c(G,A,\alpha)$ is dense in $A \rtimes_{\alpha,\mathrm{f}} G$, and that the restriction of $\pi \rtimes U$ to $C_c(G,A,\alpha)$ is uniquely determined by $(\pi \rtimes U) \circ i_A = \pi$ and $(\pi \rtimes U) \circ i_G = U$. $\square$

Proposition 9.10 shows that the unital $*$-representations of $A \rtimes_{\alpha,\mathrm{f}} G$ encode precisely the covariant representations of (A,G,α). Moreover, the full crossed product $A \rtimes_{\alpha,\mathrm{f}} G$, and hence also the reduced crossed product $A \rtimes_{\alpha,\mathrm{r}} G$, contains unitaries $U_t = i_G(t)$ for $t \in G$ such that the original action on A becomes inner. In this way the crossed product construction turns an arbitrary action into an inner action.

Example 9.11. For $\vartheta \in \mathbb{R}$ the rotation algebra A_ϑ from Definition 7.1 is nothing but the full crossed product $C(\mathbb{T}) \rtimes_{\alpha,\mathrm{f}} \mathbb{Z}$ of the action α induced by rotations on the torus $\mathbb{T} = S^1$ as in Example 9.3. Indeed, recall from Proposition 6.15 that $C(\mathbb{T})$ is the universal C^*-algebra generated by a unitary element v. Using this fact it is straightforward to verify that a covariant representation of $(C(\mathbb{T}),\mathbb{Z},\alpha)$ on a Hilbert space H is the same thing as a pair of unitaries $u,v \in U(H)$ satisfying $uvu^* = e^{2\pi i\vartheta}v$. Therefore we obtain $C(\mathbb{T}) \rtimes_{\alpha,\mathrm{f}} \mathbb{Z} \cong A_\vartheta$ from Proposition 9.10.

According to Theorem 7.11 the C^*-algebra A_ϑ is simple if ϑ is irrational. We conclude that the canonical quotient map $\pi : C(\mathbb{T}) \rtimes_{\alpha,\mathrm{f}} \mathbb{Z} \to C(\mathbb{T}) \rtimes_{\alpha,\mathrm{r}} \mathbb{Z}$ is an isomorphism in this case. Indeed, the kernel of π is an ideal of $C(\mathbb{T}) \rtimes_{\alpha,\mathrm{f}} \mathbb{Z} \cong A_\vartheta$, but the only ideals in a simple C^*-algebra are the entire algebra and the zero ideal. Since the reduced crossed product is clearly nonzero, we conclude $\ker(\pi) = 0$ as required.

Remark 9.12. The quotient map $\pi : C(\mathbb{T}) \rtimes_{\alpha,\mathrm{f}} \mathbb{Z} \to C(\mathbb{T}) \rtimes_{\alpha,\mathrm{r}} \mathbb{Z}$ in example 9.11 is always an isomorphism, irrespective of whether ϑ is rational or irrational. We will not prove this fact, but we will make some additional comments on the relation between full and reduced crossed products in Remark 9.17 further below.

9.5. **Functoriality of crossed products.** Let us discuss functoriality of full and reduced crossed products. If we are given C^*-dynamical systems $(A, G, \alpha), (B, G, \beta)$ over the same group G, then we say that a $*$-homomorphism $\varphi : A \to B$ is G-equivariant if $\varphi(t \cdot a) = t \cdot \varphi(a)$ for all $a \in A$ and $t \in G$.

Theorem 9.13. *Let $(A, G, \alpha), (B, G, \beta)$ be C^*-dynamical systems and let $\varphi : A \to B$ be a unital G-equivariant $*$-homomorphism. Then the linear map $C_c(G, \varphi) : C_c(G, A, \alpha) \to C_c(G, B, \beta)$ given by $C_c(G, \varphi)(f)(t) = \varphi(f(t))$ extends uniquely to unital $*$-homomorphisms*

$$\varphi \rtimes_{\mathrm{f}} G : A \rtimes_{\mathrm{f},\alpha} G \to B \rtimes_{\mathrm{f},\beta} G$$
$$\varphi \rtimes_{\mathrm{r}} G : A \rtimes_{\mathrm{r},\alpha} G \to B \rtimes_{\mathrm{r},\beta} G$$

between the corresponding full and reduced crossed products.

Proof. Using equivariance it is straightforward to check that the linear map $C_c(G, \varphi) : C_c(G, A, \alpha) \to C_c(G, B, \beta)$ is a unital $*$-homomorphism. Uniqueness of a potential extension of $C_c(G, \varphi)$ to either the full or reduced crossed products is clear from the density of $C_c(G, A, \alpha)$ in $A \rtimes_{\mathrm{f},\alpha} G$ and $A \rtimes_{\mathrm{r},\alpha} G$, respectively.

Let us prove existence in the case of full crossed products. The seminorm on $C_c(G, A, \alpha)$ given by

$$\|f\|_{\mathrm{f},\varphi} = \sup_{(\pi,U)} \|(\pi \rtimes U) C_c(G, \varphi)(f)\|,$$

where (π, U) runs over all covariant representations of (B, G, β) clearly satisfies $\|f\|_{\mathrm{f},\varphi} \leq \|f\|_{\mathrm{f}}$ for all $f \in C_c(G, A, \alpha)$. It follows that $C_c(G, \varphi)$ extends to a unital $*$-homomorphism $\varphi \rtimes_{\mathrm{f}} G : A \rtimes_{\mathrm{f},\alpha} G \to B \rtimes_{\mathrm{f},\beta} G$ as required.

The argument for reduced crossed products is slightly more tricky. We observe first that every G-equivariant unital $*$-homomorphism $\varphi : A \to B$ can be factorized as a composition of a surjective and an injective equivariant unital $*$-homomorphism, namely $A \to A/\ker(\varphi) \to B$. Note here that $\ker(\varphi)$ is a G-invariant ideal, so that the action of G on A induces canonically an action of G on $A/\ker(\varphi)$. It therefore suffices to prove the claim only for injective G-equivariant $*$-homomorphisms and surjective G-equivariant $*$-homomorphisms, respectively.

If $\varphi : A \to B$ is an injective G-equivariant unital $*$-homomorphism then any faithful unital $*$-representation $\pi : B \to B(H)$ on a Hilbert space H induces a faithful unital $*$-representation $\pi \circ \varphi : A \to B(H)$ of A on the same Hilbert space.

Therefore, by Proposition 9.8 we obtain an injective $*$-homomorphism $A \rtimes_{\mathrm{r},\alpha} G \to B \rtimes_{\mathrm{r},\beta} G$ extending $C_c(G,\varphi)$. Note that this argument shows not only the existence of the desired extension, but also that the reduced crossed product construction preserves injectivity.

If $\varphi : A \to B$ is a surjective G-equivariant unital $*$-homomorphism then we can identify $B = A/I$ for some G-invariant ideal $I \subseteq A$. Consider the set $S_I(A)$ of all states of A which vanish on I. Then we obtain a direct sum decomposition

$$H_u = \bigoplus_{\varphi \in S_I(A)} H_\varphi \oplus \bigoplus_{\psi \in S(A) \setminus S_I(A)} H_\psi$$

of the universal representation of A. Since $S_I(A)$ identifies canonically with the set $S(B)$ of states of B, the orthogonal projection p onto the closed subspace $K_u = \bigoplus_{\varphi \in S_I(A)} H_\varphi$ induces a bounded linear map $P : B(H_u \otimes l^2(G)) \to B(K_u \otimes l^2(G))$ by defining $P(T) = (p \otimes \mathrm{id})T(p \otimes \mathrm{id})$. It is straightforward to verify that P restricts to $C_c(G,\varphi)$ on the image of $C_c(G,A,\alpha)$ in $B(H_u \otimes l^2(G))$ under the regular representation. Since P is bounded we conclude that $C_c(G,\varphi)$ extends continuously to a unital $*$-homomorphism $A \rtimes_{\mathrm{r},\alpha} G \to B \rtimes_{\mathrm{r},\beta} G$ as required. $\square$

9.6. **Full and reduced group C^*-algebras.** The simplest possible C^*-dynamical system for a group G is given by the trivial action on $A = \mathbb{C}$. The associated full and reduced crossed products only use the group G as input, and these C^*-algebras deserve special attention.

Let us start with the following definition.

Definition 9.14. Let G be a group. The *reduced group C^*-algebra* of G is $C^*_\mathrm{r}(G) = \mathbb{C} \rtimes_\mathrm{r} G$, the reduced crossed product of the trivial action of G on $\mathbb{C}$.

Observe that the convolution algebra $C_c(G,\mathbb{C},\tau)$ associated to the trivial action $\tau : G \to \mathrm{Aut}(\mathbb{C})$ can be identified with the complex group ring $\mathbb{C}[G]$. By definition, the latter has a linear basis given by elements δ_t for $t \in G$, the multiplication satisfies $\delta_s\delta_t = \delta_{st}$ for all $s,t \in G$, the $*$-structure is determined by $\delta_t^* = \delta_{t^{-1}}$ for $t \in G$, and the identity element is δ_e.

Due to Proposition 9.8 the reduced group C^*-algebra $C^*_\mathrm{r}(G)$ can be identified with the norm closure of the complex group ring $\mathbb{C}[G]$ under the regular representation $\lambda : G \to B(l^2(G))$, given by

$$\lambda_t(\delta_s) = \delta_{ts}$$

on the standard orthonormal basis of $l^2(G)$. Indeed, Proposition 9.8 shows that we may consider the obvious unital $*$-representation of $\mathbb{C}$ on a one-dimensional Hilbert space in the definition of the reduced crossed product, so that the image of the convolution algebra $C_c(G,\mathbb{C},\tau) = \mathbb{C}[G]$ in $B(\mathbb{C} \otimes l^2(G)) \cong B(l^2(G))$ looks precisely as described above. We write δ_t both for elements of $\mathbb{C}[G]$ and for vectors in the Hilbert space $l^2(G)$, but this should not lead to confusion.

Since $C^*_{\mathrm{r}}(G)$ is naturally a subalgebra of $B(l^2(G))$ we obtain a state $\tau : C^*_{\mathrm{r}}(G) \to \mathbb{C}$ by the formula

$$\tau(x) = \langle x\delta_e, \delta_e\rangle.$$

If $x = \sum_{t\in G} \alpha_t \delta_t \in \mathbb{C}[G] \subseteq C^*_{\mathrm{r}}(G)$ then we get

$$\tau(x) = \sum_{t\in G} \alpha_t \tau(\delta_t) = \alpha_e,$$

that is, the map τ picks up the coefficient at the identity element δ_e. Using this formula it is easy to verify that τ satisfies $\tau(xy) = \tau(yx)$ for all $x, y \in C^*_{\mathrm{r}}(G)$, which means that τ is a tracial state. The state τ is called the *canonical trace* on $C^*_{\mathrm{r}}(G)$.

Proposition 9.15. *For every group G the canonical trace $\tau : C^*_{\mathrm{r}}(G) \to \mathbb{C}$ is faithful.*

Proof. For $t \in G$ we consider the right translation action $\rho : G \to U(l^2(G))$ given by $\rho_t(\delta_s) = \delta_{st^{-1}}$. This defines a unitary representation of G on $l^2(G)$, and since left translation commutes with right translation we see that all elements $y \in \mathbb{C}[G] \subseteq C^*_{\mathrm{r}}(G)$ satisfy $y\rho_t = \rho_t y$ for $t \in G$. By continuity we see that this relation holds in fact for all $y \in C^*_{\mathrm{r}}(G)$.

Now assume $x \in C^*_{\mathrm{r}}(G)$ satisfies $\tau(x^*x) = \langle x^*x\delta_e, \delta_e\rangle = 0$. Then we get

$$0 = \langle x^*x\delta_e, \delta_e\rangle = \langle x^*x\rho_t\rho_t^*\delta_e, \delta_e\rangle = \langle \rho_t x^*x\rho_t^*\delta_e, \delta_e\rangle = \langle x^*x\delta_t, \delta_t\rangle$$

for all $t \in G$, or equivalently $\||x|^{1/2}\delta_t\| = 0$. This implies

$$|\langle x^*x\delta_t, \delta_s\rangle| = |\langle |x|^{1/2}\delta_t, |x|^{1/2}\delta_s\rangle| \leq \||x|^{1/2}\delta_t\|\||x|^{1/2}\delta_s\| = 0$$

for all $s, t \in G$ by the Cauchy-Schwarz inequality. Thus $x^*x = 0$ and therefore $x = 0$, which means that τ is faithful. $\square$

Let us next discuss the full group C^*-algebra of a group.

Definition 9.16. Let G be a group. The *full group C^*-algebra* of G is $C^*_{\mathrm{f}}(G) = \mathbb{C} \rtimes_{\mathrm{f}} G$, the full crossed product of the trivial action of G on $\mathbb{C}$.

From our general discussion of crossed products we know that there is a canonical surjective $*$-homomorphism $C^*_{\mathrm{f}}(G) \to C^*_{\mathrm{r}}(G)$.

Remark 9.17. The canonical $*$-homomorphism $C^*_{\mathrm{f}}(G) \to C^*_{\mathrm{r}}(G)$ is not injective in general. For a discrete group G, the case we have considered here, one can show that this map is an isomorphism if and only if the group G is *amenable*. We refer to [8] for more information. In fact, for amenable G the canonical $*$-homomorphism $A \rtimes_{\mathrm{f},\alpha} G \to A \rtimes_{\mathrm{r},\alpha} G$ is an isomorphism for every C^*-dynamical system (A, G, α).

A basic example of a nonamenable group is the free group $\mathbb{F}_2$ on two generators. By a result of Powers [39], the group C^*-algebra $C^*_{\mathrm{r}}(\mathbb{F}_2)$ is in fact simple. This implies in particular that the full and reduced group C^*-algebras of $\mathbb{F}_2$ are not isomorphic, in accordance with the above remark.

Remark 9.18. A group G is called C^*-*simple* if $C^*_{\mathrm{r}}(G)$ is a simple C^*-algebra. In recent years there has been remarkable progress on the task of determining which groups are C^*-simple, with a dynamical characterization obtained by Kalantar and Kennedy in [30]. More precisely, according to the main result of [30], a discrete group G is C^*-simple iff its action on the Furstenberg boundary $\partial_F(G)$ is topologically free. We refer to [30], [6], [15] for more information on this fascinating topic.

9.7. **Exercises.**

Exercise 9.1. Show that every C^*-dynamical system (A, G, α) over a commutative unital C^*-algebra A comes from a classical dynamical system. More precisely, show that there exists a compact space X and an action $G \to \mathrm{Homeo}(X)$ such that (A, G, α) is conjugate to $(C(X), G, \beta)$, where $\beta_t(f) = f(t^{-1} \cdot x)$ is the induced action on $C(X)$.

Exercise 9.2. Let (A, G, α) be a C^*-dynamical system. Show that the integrated form $\Pi^u \rtimes \lambda^u : C_c(G, A, \alpha) \to B(H_u \otimes l^2(G))$ of the regular representation of (A, G, α) is injective.

Exercise 9.3. Let (A, G, α) be a C^*-dynamical system such that the action α is inner. Show that the full or reduced crossed products of (A, G, α) are isomorphic to the corresponding crossed products for (A, G, τ), where τ denotes the trivial action of G on A.

Exercise 9.4. Let G be a group. Verify that the canonical trace $\tau : C^*_{\mathrm{r}}(G) \to \mathbb{C}$ on a reduced group C^*-algebra is a tracial state.

Exercise 9.5. Let $G = \mathbb{Z}$. Show that the full group C^*-algebra $C^*_{\mathrm{f}}(\mathbb{Z})$ is isomorphic to $C(\mathbb{T})$. Show also that the canonical $*$-homomorphism $C^*_{\mathrm{f}}(\mathbb{Z}) \to C^*_{\mathrm{r}}(\mathbb{Z})$ is an isomorphism.

Exercise 9.6. In this exercise we study the structure of group algebras of finite groups.

(a) Let G be a finite group. Show that $C^*_{\mathrm{f}}(G) \cong C^*_{\mathrm{r}}(G) \cong \mathbb{C}[G]$ is a finite direct sum of matrix algebras. This is also known as Maschke's Theorem.
(b) Let $G = S^3$ be the symmetric group on three elements. Describe the structure of $C^*_{\mathrm{f}}(S^3) = C^*_{\mathrm{r}}(S^3)$, that is, determine the number of matrix blocks and their sizes in the decomposition of $C^*_{\mathrm{f}}(S^3)$ as in part (a).
(c) For $G = \mathbb{Z}/n\mathbb{Z}$ use the Gelfand-Naimark theorem to show that $C^*_{\mathrm{f}}(G) \cong C(\mathbb{Z}/n\mathbb{Z})$. Can you find an explicit isomorphism? (Hint: Consider the discrete Fourier transform)

10. Traces, simplicity, and further examples

Abstract. We study the existence and uniqueness of traces on crossed products, and present criteria which guarantee that the crossed product of a C^*-dynamical system is simple. As in the previous lecture we restrict our attention to actions of discrete groups throughout. In order to enhance our supply of examples we discuss odometer actions on Cantor space. Applying the general results on simplicity and existence of traces presented in the first half of this lecture, we show that the crossed products of odometer actions yield simple C^*-algebras with unique trace.

10.1. **Conditional expectations.** We have already seen in our analysis of the rotation algebras A_ϑ that conditional expectations are a useful tool to understand the structure of C^*-algebras, see Lemma 7.7. Similarly, the canonical trace on $C^*_{\mathrm{r}}(\mathbb{F}_2)$, see the last lecture, plays an important role in showing that the reduced group C^*-algebra of the free group on two generators is simple.

These maps can be viewed as special cases of a general construction which we shall discuss now.

Proposition 10.1. *Let (A, G, α) be a C^*-dynamical system. Then the formula*

$$E(a\delta_t) = \begin{cases} a & \text{if } t = e \\ 0 & \text{else} \end{cases}$$

determines a conditional expectation $E : A \rtimes_{\mathrm{r},\alpha} G \to A$.

Proof. Clearly, the given formula defines a unital linear map $E : C_c(G, A, \alpha) \to A$, and since $C_c(G, A, \alpha)$ is dense in $A \rtimes_{\mathrm{r},\alpha} G$ there is at most one way in which this can be extended continuously to the crossed product. In order to verify that such an extension exists we shall describe the map E on the level of the convolution algebra in a different way.

Let $H = H_u \otimes l^2(G)$ be the underlying Hilbert space of the regular representation of (A, G, α), and let $p \in B(H)$ be the orthogonal projection onto the closed subspace $H_u \otimes \delta_e$. For $x = \sum_{s\in G} a_s\delta_s \in C_c(G, A, \alpha)$ we compute

$$p(\Pi^u \rtimes \lambda^u)(x)p(\xi \otimes \delta_t) = \delta_{e,t} \sum_{s\in G} p(\pi^u(s^{-1} \cdot a_s)(\xi) \otimes \delta_s) = \delta_{e,t}\pi^u(a_e)(\xi) \otimes \delta_e.$$

Hence the linear map $E : C_c(G, A, \alpha) \to A$ defined by $E(a\delta_s) = \delta_{s,e}a$ is implemented by cutting down with the projection p inside $B(H)$. From this observation it follows that this map extends canonically to a bounded positive linear map $A \rtimes_{\mathrm{r},\alpha} G \to A$, denoted again E.

We clearly have $E(a) = a$ for all $a \in A \subseteq A \rtimes_{\mathrm{r},\alpha} G$, and one easily checks $E(axb) = aE(x)b$ for $x \in A \rtimes_{\mathrm{r},\alpha} G$ and $a, b \in A$. According to Definition 7.5 this means that E is a conditional expectation. $\square$

We call the map $E : A \rtimes_{\mathrm{r},\alpha} G \to A$ obtained in Proposition 10.1 the *canonical conditional expection* of the crossed product. Note that for the trivial action of G

on A the map $E : C^*_{\mathrm{r}}(G) = \mathbb{C} \rtimes_{\mathrm{r},\alpha} G \to \mathbb{C}$ is nothing but the canonical trace. In fact, the map E can be viewed as a natural generalization of the canonical trace, in the sense that we have the following generalization of Proposition 9.15.

Proposition 10.2. *The canonical conditional expectation $E : A \rtimes_{\mathrm{r},\alpha} G \to A$ is faithful.*

Proof. Let $\pi^u : A \to B(H_u)$ be the universal representation of A, and let $\Pi^u \rtimes \lambda^u : A \rtimes_{\mathrm{r},\alpha} G \to B(H)$ be the defining representation of the reduced crossed product on $H = H_u \otimes l^2(G)$, see Definition 9.7. By construction this is a faithful unital $*$-representation. Therefore the $*$-representation $\Theta = (\Pi^u \rtimes \lambda^u) \otimes 1 : A \rtimes_{\mathrm{r},\alpha} G \to B(H \otimes l^2(G))$ given by

$$\Theta(x)(\xi \otimes \delta_t) = (\Pi^u \rtimes \lambda^u)(x)(\xi) \otimes \delta_t$$

is faithful as well. Consider the unitary operator $V \in U(H \otimes l^2(G))$ given by

$$V(\xi \otimes \delta_r \otimes \delta_s) = \xi \otimes \delta_r \otimes \delta_{rs}$$

for $\xi \otimes \delta_r \otimes \delta_s \in H_u \otimes l^2(G) \otimes l^2(G) = H \otimes l^2(G)$. We calculate

$$\begin{aligned} V(\Pi^u(a) \otimes 1)(\xi \otimes \delta_r \otimes \delta_s) &= V(\pi^u(r^{-1} \cdot a)(\xi) \otimes \delta_r \otimes \delta_s) \\ &= \pi^u(r^{-1} \cdot a)(\xi) \otimes \delta_r \otimes \delta_{rs} \\ &= (\Pi^u(a) \otimes 1)(\xi \otimes \delta_r \otimes \delta_{rs}) \\ &= (\Pi^u(a) \otimes 1)V(\xi \otimes \delta_r \otimes \delta_s) \end{aligned}$$

and, writing $\lambda^u(\delta_t) = \lambda^u_t$,

$$\begin{aligned} V(\lambda^u(\delta_t) \otimes 1)(\xi \otimes \delta_r \otimes \delta_s) &= V(\xi \otimes \delta_{tr} \otimes \delta_s) \\ &= (\xi \otimes \delta_{tr} \otimes \delta_{trs}) \\ &= (\lambda^u(\delta_t) \otimes \lambda_t)(\xi \otimes \delta_r \otimes \delta_{rs}) \\ &= (\lambda^u(\delta_t) \otimes \lambda_t)V(\xi \otimes \delta_r \otimes \delta_s) \end{aligned}$$

for all $\xi \in H_u$ and $r, s \in G$, which means

$$V\Theta(a)V^* = \Pi^u(a) \otimes 1, \qquad V\Theta(\delta_t)V^* = \lambda^u(\delta_t) \otimes \lambda_t$$

for $a \in A \subseteq A \rtimes_{\mathrm{r},\alpha} G$ and $t \in G \subseteq A \rtimes_{\mathrm{r},\alpha} G$.

Given a vector $v \in H$, let $\vartheta_v : l^2(G) \to H \otimes l^2(G)$ be the bounded linear operator given by $\vartheta_v(\delta_t) = v \otimes \delta_t$. Then $\vartheta_v^*(w \otimes \delta_t) = \langle w, v\rangle \delta_t$. The above formulas show

$$\vartheta_v^* V\Theta(a)V^*\vartheta_v = \langle \Pi^u(a)(v), v\rangle \,\mathrm{id}, \qquad \vartheta_v^* V\Theta(\delta_t)V^*\vartheta_v = \langle \lambda^u(\delta_t)(v), v\rangle \lambda_t$$

for $a \in A \subseteq A \rtimes_{\mathrm{r},\alpha} G$ and $t \in G \subseteq A \rtimes_{\mathrm{r},\alpha} G$, which implies that that $\vartheta_v^* V\Theta(y)V^*\vartheta_v$ is contained in $C^*_{\mathrm{r}}(G)$ for all $y \in A \rtimes_{\mathrm{r},\alpha} G$. Moreover, if $y = x^*x$ is a positive element in $A \rtimes_{\mathrm{r},\alpha} G$ then $\vartheta_v^* V\Theta(y)V^*\vartheta_v$ is a positive element in $C^*_{\mathrm{r}}(G)$.

Let us also write $\vartheta_e : H \to H \otimes l^2(G)$ for the bounded linear operator given by $\vartheta_e(v) = v \otimes \delta_e$. Then we compute

$$\begin{aligned}\langle \vartheta_e^* V\Theta(a\delta_t)V^*\vartheta_e(v), w\rangle &= \langle V\Theta(a\delta_t)V^*(v\otimes\delta_e), w\otimes\delta_e\rangle \\ &= \langle \Pi^u(a)\lambda^u(\delta_t)(v)\otimes\lambda_t(\delta_e), w\otimes\delta_e\rangle \\ &= \langle (\Pi^u \rtimes \lambda^u)(a\delta_t)(v)\otimes\delta_t, w\otimes\delta_e\rangle \\ &= \delta_{e,t}\langle (\Pi^u \rtimes \lambda^u)(a\delta_t)(v), w\rangle \\ &= \langle \Pi^u E(a\delta_t)(v), w\rangle\end{aligned}$$

for all $v, w \in H$, so that

$$\vartheta_e^* V\Theta(a\delta_t)V^*\vartheta_e = \Pi^u E(a\delta_t).$$

From linearity and continuity we get $\vartheta_e^* V\Theta(y)V^*\vartheta_e = \Pi^u E(y)$ for all $y \in A \rtimes_{\mathrm{r},\alpha} G$.

Now assume $x \in A \rtimes_{\mathrm{r},\alpha} G$ satisfies $E(x^*x) = 0$. Then

$$0 = \langle \Pi^u E(x^*x)v, v\rangle = \langle V\Theta(x^*x)V^*(v\otimes\delta_e), v\otimes\delta_e\rangle = \tau(\vartheta_v^* V\Theta(x^*x)V^*\vartheta_v)$$

for all $v \in H$. We know from Proposition 9.15 that the canonical trace $\tau : C^*_{\mathrm{r}}(G) \to \mathbb{C}$ is faithful, so that $\vartheta_v^* V\Theta(x^*x)V^*\vartheta_v = 0$. Since $v \in H$ was arbitrary we obtain $\langle w, V\Theta(x^*x)V^*(w)\rangle = 0$ for all $w \in H \otimes l^2(G)$. This means $\Theta(x^*x) = 0$, and using that Θ is faithful we conclude $x^*x = 0$, or equivalently, $x = 0$. $\square$

An important difference between the canonical conditional expectation on a reduced crossed product $A \rtimes_{\mathrm{r},\alpha} G$ and the canonical trace on the reduced group C^*-algebra $C^*_{\mathrm{r}}(G)$ is that the former is not a trace in general. Still, the canonical conditional expectation can be used to construct traces on crossed products. This is what we will discuss next.

10.2. **Invariant measures and traces.** Let (A, G, α) be a C^*-dynamical system. We say that a state φ on A is *G-invariant* if $\varphi(a) = \varphi(t \cdot a)$ for all $t \in G$ and $a \in A$. If $\tau : A \to \mathbb{C}$ is a G-invariant tracial state, then we obtain a tracial state $\tau_E : A \rtimes_{\mathrm{r},\alpha} G \to \mathbb{C}$ by the formula

$$\tau_E(x) = \tau(E(x)),$$

where $E : A \rtimes_{\mathrm{r},\alpha} G \to A$ is the canonical conditional expectation. Indeed, it is clear that τ_E is a positive unital map, and we compute

$$\begin{aligned}\tau_E(a\delta_t b\delta_s) &= \tau(E(a(t\cdot b)\delta_{ts})) \\ &= \delta_{t,s^{-1}}\tau(a(t\cdot b)) \\ &= \delta_{t,s^{-1}}\tau((t^{-1}\cdot a)b) \\ &= \delta_{t,s^{-1}}\tau(b(s\cdot a)) \\ &= \tau_E(b\delta_s a\delta_t)\end{aligned}$$

for $a, b \in A$ and $s, t \in G$, using both G-invariance and the trace property of τ. This implies that τ_E is a trace. We will call τ_E the trace induced from τ.

If $A = C(X)$ is a commutative unital C^*-algebra then every state on A is automatically a trace. Moreover, by the Riesz-Markov theorem, compare Example 5.2, states on A correspond bijectively to probability measures on X. Explicitly, given a probability measure μ, one associates the state $\varphi^\mu : C(X) \to \mathbb{C}$ given by

$$\varphi^\mu(f) = \int_X f(x)d\mu(x)$$

to μ. Assume that the group G acts on $A = C(X)$, corresponding to an action of G on X by homeomorphisms. Then the state φ_μ associated to μ is G-invariant if and only if μ is G-invariant, that is, if and only if $\mu(t \cdot T) = \mu(T)$ for all measurable subsets $T \subseteq X$ and $t \in G$. In particular, every G-invariant probability measure on X induces a tracial state on $\varphi^\mu_E : C(X) \rtimes_{\mathrm{r}} G \to \mathbb{C}$.

Definition 10.3. Let G be a group. An action $\alpha : G \to \mathrm{Homeo}(X)$ of G on a topological space X is called *free* if $t \cdot x = x$ for some $x \in X$ implies $t = e$. We say that a classical dynamical system (X, G, α) is free if α is free.

In other words, the action $\alpha : G \to \mathrm{Homeo}(X)$ is free if and only if the stabilizer groups $\mathrm{Stab}_x = \{t \in G \mid t \cdot x = x\}$ are trivial for all $x \in X$. This implies in particular that the group homomorphism $\alpha : G \to \mathrm{Homeo}(X)$ is injective, but note that freeness is a much stronger condition.

Example 10.4. If G is an arbitrary group then one obtains a free action of G by taking $X = G$ with the discrete topology, and the action given by left translations. Of course, unless G is finite the space X is not compact. An example of a free action of $\mathbb{Z}$ on a compact space is given by rotations by an irrational angle ϑ on $X = \mathbb{T}$, compare Example 9.3 (d).

Our next goal is to describe the structure of traces on reduced crossed products of free actions on compact spaces. First we need a lemma:

Lemma 10.5. *Let (X, G, α) be a free classical dynamical system on a compact space X. For any finite set $F \subseteq G$ there exists $n \in \mathbb{N}$ and elements $h_j \in C(X)$ for $1 \leq j \leq n$ such that $|h_j(x)| = 1$ for all $x \in X$ and*

$$\frac{1}{n}\sum_{j=1}^n h_j(x)\overline{h_j(t^{-1} \cdot x)} = 0$$

for all $x \in X$ and all nontrivial elements $t \in F$.

Proof. If $T \subseteq G$ is a finite set and $U \subseteq X$ is open then we say that (T, U) is inessential if there exists $n \in \mathbb{N}$ and functions $h_1, \dots, h_n \in C(X)$ such that $|h_j(x)| = 1$ for all $x \in X$ and

$$\frac{1}{n}\sum_{j=1}^n h_j(x)\overline{h_j(t^{-1} \cdot x)} = 0$$

for all $x \in U$ and all nontrivial elements $t \in T$. Our goal is to show that (F, X) is inessential.

This will be obtained in several steps as follows.

(1) *For any $x \in X$ and any $t \in G \setminus \{e\}$ there exists an open neighborhood U of x such that $(\{t\}, U)$ is inessential.*
To prove this claim note that $t \cdot x \neq x$ by freeness of the action, so that we find an open neighborhood U of x such that $t \cdot \overline{U} \cap \overline{U} = \emptyset$. Let $n = 2$ and set $h_1 = 1$. Use Urysohn's Lemma to find a continuous function $f : X \to \mathbb{R}$ such that $f(x) = 0$ for $x \in U$ and $f(x) = \pi$ for $x \in t^{-1} \cdot U$, and define $h_2(x) = \exp(if(x))$. Then we compute
$$\frac{1}{2}\sum_{j=1}^{2} h_j(x)\overline{h_j(t^{-1} \cdot x)} = \frac{1}{2}(1 \cdot 1 + 1 \cdot (-1)) = 0$$
for $x \in U$ as required.

(2) *If $F \subseteq G$ is a finite set and $U, V \subseteq X$ are open sets such that (F, U) and (F, V) are both inessential, then $(F, U \cup V)$ is inessential as well.*
To prove this claim note that by assumption we find functions $h_1, \ldots, h_m$ and $k_1, \ldots k_n$ such that $|h_i(x)| = 1 = |k_j(x)|$ for all i, j and $x \in X$, and for all nontrivial elements $t \in F$
$$\frac{1}{m}\sum_{i=1}^{m} h_i(x)\overline{h_i(t^{-1} \cdot x)} = 0, \qquad x \in U,$$
$$\frac{1}{n}\sum_{j=1}^{n} k_j(x)\overline{k_j(t^{-1} \cdot x)} = 0, \qquad x \in V.$$
If we consider the functions $h_i k_j$ for all i, j then $|h_i k_j(x)| = 1$ and
$$\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} h_i k_j(x)\overline{h_i k_j(t^{-1} \cdot x)}$$
$$= \left(\frac{1}{m}\sum_{i=1}^{m} h_i(x)\overline{h_i(t^{-1} \cdot x)}\right)\left(\frac{1}{n}\sum_{j=1}^{n} k_j(x)\overline{k_j(t^{-1} \cdot x)}\right)$$
vanishes for all $x \in U \cup V$ and all nontrivial elements $t \in F$.

(3) *If $E, F \subseteq G$ are finite and $U \subseteq X$ is open such that (E, U) and (F, U) are both inessential, then $(E \cup F, U)$ is inessential as well.*
By assumption we find $h_1, \ldots, h_m$ and $k_1, \ldots k_n$ such that $|h_i(x)| = 1 = |k_j(x)|$ for all i, j and $x \in X$, and for all $x \in U$
$$\frac{1}{m}\sum_{i=1}^{m} h_i(x)\overline{h_i(t^{-1} \cdot x)} = 0, \qquad t \in E,$$
$$\frac{1}{n}\sum_{j=1}^{n} k_j(x)\overline{k_j(t^{-1} \cdot x)} = 0, \qquad t \in F.$$

The same calculation as in step (2) then shows

$$\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} h_i k_j(x)\overline{h_i k_j(t^{-1}\cdot x)} = 0$$

for all $x \in U$ and $t \in E \cup F$.

Now we are ready to finish the proof. Let $t \in F \backslash \{e\}$ be a nontrivial group element. Using compactness of X and step (1) we find $n \in \mathbb{N}$ and open sets $U_1, \ldots, U_n$ such that $(\{t\}, U_j)$ is inessential for all $1 \leq j \leq n$ and $\bigcup_{j=1}^{n} U_j = X$. Then $n-1$ applications of step (2) show that $(\{t\}, X)$ is inessential. Since F is finite we may then use step (3) to conclude that (F, X) is inessential. This completes the proof. □

Lemma 10.5 is the key ingredient in the following proposition.

Proposition 10.6. *Let (X, G, α) be a free classical dynamical system on a compact space X. Then for every $\varepsilon > 0$ and $y \in C(X) \rtimes_{\mathrm{r}} G$ there exists $n \in \mathbb{N}$ and elements $h_j \in C(X) \subseteq C(X) \rtimes_{\mathrm{r}} G$ for $1 \leq j \leq n$ such that $|h_j(x)| = 1$ for all $x \in X$ and $1 \leq j \leq n$ and*

$$\|E(y) - \frac{1}{n}\sum_{j=1}^{n} h_j y h_j^*\| < \varepsilon,$$

where E is the canonical conditional expectation.

Proof. Let $y \in C(X) \rtimes_{\mathrm{r}} G$ be arbitrary and choose $y' \in C_c(G, C(X), \alpha)$ such that $\|y - y'\| < \varepsilon/2$. Write $y' = \sum_{t \in F} f_t \delta_t$ for a finite set $F \subseteq G$, and assume without loss of generality $e \in F$. According to Lemma 10.5 we find elements $h_j \in C(X)$ for $1 \leq j \leq n$ such that $|h_j(x)| = 1$ for all $x \in X$ and

$$\sum_{j=1}^{n} h_j(x)\overline{h_j(t^{-1}\cdot x)} = 0$$

for all $x \in X$ and $t \in F \setminus \{e\}$. That is, $\sum_{j=1}^{n} h_j(t \cdot h_j^*) = 0$ for $t \in F \setminus \{e\}$.

The map $P : C(X) \rtimes_{\mathrm{r}} G \to C(X) \rtimes_{\mathrm{r}} G$ defined by $P(y) = \frac{1}{n}\sum_{j=1}^{n} h_j y h_j^*$ is contractive, and hence

$$\begin{aligned}\|E(y) - P(y)\| &= \|E(y) - E(y')\| + \|E(y') - P(y')\| + \|P(y') - P(y)\| \\ &\leq \|y - y'\| + \|E(y') - P(y')\| + \|y' - y\| \leq \|E(y') - P(y')\| + \varepsilon.\end{aligned}$$

It therefore suffices to show $E(y') = P(y')$. For $a\delta_t \in C_c(G, C(X), \alpha)$ we compute

$$P(a\delta_t) = \frac{1}{n}\sum_{j=1}^{n} h_j a(t \cdot h_j^*)\delta_t = \frac{1}{n} a \sum_{j=1}^{n} h_j(t \cdot h_j^*)\delta_t.$$

According to our above considerations, this expression vanishes if $t \neq e$. For $t = e$ we obtain $P(a\delta_e) = \frac{1}{n} a \sum_{j=1}^{n} h_j h_j^* \delta_e = a\delta_e$. Using linearity we conclude $E(y') = P(y')$ as required. □

We are now ready to prove the following result.

Theorem 10.7. *Let (X, G, α) be a classical dynamical system such that the action of $\alpha : G \to \operatorname{Homeo}(X)$ on the compact space X is free. Then tracial states on the associated reduced crossed product $C(X) \rtimes_{\mathrm{r},\alpha} G$ correspond bijectively to G-invariant probability measures on X.*

Proof. We have already seen that every G-invariant probability measure μ on X defines a tracial state φ^μ_E on $C(X) \rtimes_{\mathrm{r},\alpha} G$. Moreover, the restriction of φ^μ_E to $C(X) \subseteq C(X) \rtimes_{\mathrm{r},\alpha} G$ agrees with φ^μ, so that the resulting map from G-invariant probability measures on X to tracial states on $C(X) \rtimes_{\mathrm{r},\alpha} G$ is injective.

Assume that $\tau : C(X) \rtimes_{\mathrm{r},\alpha} G \to \mathbb{C}$ is a tracial state. Then the restriction of τ to $C(X)$ determines a probability measure μ on X such that $\tau(f) = \varphi^\mu(f)$ is given by integration against μ for all $f \in C(X)$. Since τ is a trace we get

$$\int_X (t \cdot f)(x) d\mu(x) = \tau(t \cdot f) = \tau(\delta_t f \delta_{t^{-1}}) = \tau(f) = \int_X f(x) d\mu(x)$$

for all $t \in G$, which implies that μ is G-invariant.

We claim that τ agrees with the tracial state φ^μ_E. According to Proposition 10.6, for every $y \in C(X) \rtimes_{\mathrm{r},\alpha} G$ and every $\varepsilon > 0$ we find $n \in \mathbb{N}$ and unitary elements $h_j \in C(X)$ for $1 \leq j \leq n$ such that

$$|\tau(E(y) - y)| = |\tau(E(y) - \frac{1}{n} \sum_{j=1}^n h_j^* h_j y)| = |\tau(E(y) - \frac{1}{n} \sum_{j=1}^n h_j y h_j^*)| \leq \|E(y) - \frac{1}{n} \sum_{j=1}^n h_j y h_j^*\| < \varepsilon,$$

which means $\tau(E(y)) = \tau(y)$. We therefore get

$$\tau(y) = \tau(E(y)) = \int_X E(y)(x) d\mu(x) = \varphi^\mu_E(y).$$

In other words, every trace on $C(X) \rtimes_{\mathrm{r},\alpha} G$ is induced from a G-invariant probability measure on X. □

Theorem 10.7 provides a neat description of tracial states on $C(X) \rtimes_{\mathrm{r},\alpha} G$ in terms of the underlying dynamical system, under the assumption that the action of G on X is free. Without freeness, the assertion of 10.7 fails, that is, not all traces on the crossed product come from invariant measures in general.

However, let us note the following result, which is a byproduct of Power's proof of simplicity of $C^*_{\mathrm{r}}(\mathbb{F}_2)$ mentioned in the previous lecture, see [39].

Theorem 10.8. *The reduced group C^*-algebra $C^*_{\mathrm{r}}(\mathbb{F}_2)$ of the free group on two generators has a unique tracial state.*

Of course, the unique tracial state in Theorem 10.8 is given by the canonical trace, compare the discussion before Proposition 9.15.

10.3. **Minimal actions and simplicity.** Simplicity of the crossed product of a classical dynamical system turns out to be related to the concept of minimality. If a group G acts on a topological space X then we say that a subset $U \subseteq X$ is G-invariant if $G \cdot U \subseteq U$. That is, U is G-invariant iff $t \cdot x \in U$ for all $t \in G$ and $x \in U$. In this case the action on X restricts to an action of G on U.

Definition 10.9. A classical dynamical system (X, G, α) is called *minimal* if the only closed G-invariant subsets of X are the empty set and X.

In other words, minimality is a topological counterpart to the notion of ergodicity in measurable dynamics. We also say that the action $\alpha : G \to \mathrm{Homeo}(X)$ is minimal if (X, G, α) is minimal.

Let us consider the case of actions of $\mathbb{Z}$ in more detail.

Proposition 10.10. *For a classical dynamical system $(X, \mathbb{Z}, \alpha)$ over a compact space X the following conditions are equivalent.*

(a) *The system $(X, \mathbb{Z}, \alpha)$ is minimal.*
(b) *If $T \subseteq X$ is a closed subset such that $\alpha_1(T) = T$ then $T = \emptyset$ or $T = X$.*
(c) *For every $x \in X$ the orbit $\mathbb{Z} \cdot x = \{\alpha_n(x) \mid n \in \mathbb{Z}\}$ is dense in X.*

Proof. $(a) \Rightarrow (b)$ This is obvious.
$(b) \Rightarrow (c)$ Let $x \in X$ be arbitrary and let T be the closure of $\mathbb{Z} \cdot x$. We clearly have $\alpha_1(\mathbb{Z} \cdot x) = \mathbb{Z} \cdot x$, which implies $\alpha_1(T) = T$. Since T contains x our hypothesis yields $T = X$. That is, $\mathbb{Z} \cdot x$ is dense in X.
$(c) \Rightarrow (a)$ Assume that $T \subseteq X$ is a nonempty closed G-invariant set. Choose a point $x \in T$ and note that $\mathbb{Z} \cdot x \subseteq T$ by invariance. Since $\mathbb{Z} \cdot x$ is dense in X by assumption we conclude that T is equal to X since it is closed. $\square$

We note that the property of the homeomorphism α_1 in part (b) of Proposition 10.10 is usually taken as the definition of minimality for a single homeomorphism. That is, a homeomorphism $\sigma \in \mathrm{Homeo}(X)$ is called minimal if for every closed subset $T \subseteq X$ with $\sigma(T) = T$ we have $T = \emptyset$ or $T = X$.

Example 10.11. Let $\vartheta \in \mathbb{R}$ be irrational. Then the action of $\mathbb{Z}$ on $\mathbb{T}$ by rotation by ϑ is minimal: Due to Proposition 10.10 it suffices to show that the set $\Theta_x = \{e^{2\pi i n\vartheta} x \mid n \in \mathbb{Z}\}$ is dense in $\mathbb{T}$ for every $x \in \mathbb{T}$. This can be verified by using that the action is free: freeness means that Θ_x is an infinite subset of a compact space, so has an accumulation point. In particular, considering $x = 1$ it follows that for every $\varepsilon > 0$ there are $n, m \in \mathbb{N}$ such that $|e^{2\pi i m\vartheta} - e^{2\pi i n\vartheta}| < \varepsilon$. Since Θ_1 is a subgroup of $\mathbb{T}$ and $\Theta_x = \Theta_1 x$ this yields the claim.

Minimality of $\alpha : G \to \mathrm{Homeo}(X)$ is a necessary condition for simplicity of the crossed products $C(X) \rtimes_{\mathrm{f},\alpha} G$ and $C(X) \rtimes_{\mathrm{r},\alpha} G$. Indeed, if $T \subseteq X$ is a nontrivial G-invariant subset then the canonical projection homomorphism $C(X) \to C(T)$ induces a surjective $*$-homomorphism $C_c(G, C(X)) \to C_c(G, C(T))$ between the convolution algebras with nontrivial kernel. Since both the full and reduced crossed products are functorial for $*$-homomorphisms by Theorem 9.13, we see that the induced $*$-homomorphisms $C(X) \rtimes_{\mathrm{f}} G \to C(T) \rtimes_{\mathrm{f}} G$ and $C(X) \rtimes_{\mathrm{r}} G \to C(T) \rtimes_{\mathrm{r}} G$ have nontrivial kernels as well. In particular, the crossed products are not simple.

It is also not difficult to see that minimality is not a sufficient condition for the full or reduced crossed products to be simple. Take, for instance, a finite group G acting trivially on a point. Such an action is clearly minimal and, as we have seen in Exercise 9.6, the corresponding crossed product is $C^*_{\mathrm{f}}(G) = C^*_{\mathrm{r}}(G)$, a finite direct sum of matrix algebras. Moreover this C^*-algebra always admits a character, which means that it fails to be simple as soon as G has more than one element.

In order to obtain a general simplicity result, valid for all groups, we shall restrict ourselves to free actions, compare Definition 10.3.

Theorem 10.12. *Let (X, G, α) be a free classical dynamical system on a compact space X. Then the crossed product $C(X) \rtimes_{\mathrm{r},\alpha} G$ is simple if and only if (X, G, α) is minimal.*

Proof. We have already observed that minimality is a necessary condition for the crossed product $C(X) \rtimes_{\mathrm{r}} G = C(X) \rtimes_{\mathrm{r},\alpha} G$ to be simple. Let us show that, under the assumption of freeness, this condition is also sufficient.

Let $I \subseteq C(X) \rtimes_{\mathrm{r}} G$ be a closed ideal. We claim that if I contains a nonzero positive element of $C(X)$ then $I = C(X) \rtimes_{\mathrm{r}} G$. To this end let $f \in I \cap C(X)$ be nonzero and positive, and let $U = \{x \in X \mid f(x) > 0\}$. Then U is a nonempty open set. If $U = X$ then f is invertible, which implies that I equals $C(X) \rtimes_{\mathrm{r}} G$. If U is not equal to X then $K = X \setminus U$ is a proper closed subset of X. Hence the intersection $\bigcap_{t \in G}(t \cdot K)$ is again a proper closed subset of X, which in addition is G-invariant. Since the action of G on X is minimal this intersection must be empty. Equivalently, the translates $t \cdot U$ for $t \in G$ define an open covering of X. By compactness of X there exists a finite set $t_1, \ldots, t_k$ of elements in G such that $\bigcup_{j=1}^k t_j \cdot U = X$. Therefore

$$g = \sum_{j=1}^{k} t_j \cdot f$$

is a strictly positive element of $C(X)$, and hence invertible in $C(X)$. Since g is also contained in I we conclude that I equals $C(X) \rtimes_{\mathrm{r}} G$.

Next we claim that any element of the form $E(x)$ for $x \in I$ is again contained in I, where $E : C(X) \rtimes_{\mathrm{r}} G \to C(X)$ is the canonical conditional expectation. Indeed, for every $\varepsilon > 0$ we find elements h_j such that

$$\|E(x) - \frac{1}{n} \sum_{j=1}^{n} h_j x h_j^*\| < \varepsilon$$

due to Proposition 10.6. Since $h_j x h_j^* \in I$ for all j and I is closed we conclude $E(x) \in I$ as desired.

Finally, assume that $x \in I$ is an arbitrary nonzero element. Then x^*x is positive and nonzero, so that Proposition 10.2 implies that $E(x^*x) \in C(X)$ is a nonzero positive element contained in I. Hence the first part of our proof shows $I = C(X) \rtimes_{\mathrm{r}} G$. In other words, the only ideals in $C(X) \rtimes_{\mathrm{r}} G$ are the zero ideal and the entire algebra. □

We note that the conditions in Theorem 10.12 are not necessary for a reduced crossed product to be simple. The trivial action of the free group $\mathbb{F}_2$ on a point is highly non-free, and yet the corresponding reduced crossed product, namely the reduced group C^*-algebra $C^*_{\mathrm{r}}(\mathbb{F}_2)$, is simple.

In the special case $G = \mathbb{Z}$ we obtain the following useful variant of Theorem 10.12.

Theorem 10.13. *Let $(X, \mathbb{Z}, \alpha)$ be a classical dynamical system on an infinite compact space X. Then the crossed product $C(X) \rtimes_{\mathrm{r},\alpha} \mathbb{Z}$ is simple if and only if $(X, \mathbb{Z}, \alpha)$ is minimal.*

Proof. According to Theorem 10.12 it suffices to observe that a minimal action of $\mathbb{Z}$ on an infinite compact space is automatically free. To this end note that if the stabilizer $\mathrm{Stab}(x)$ of some point $x \in X$ is nontrivial, then $\mathrm{Stab}(x) = m\mathbb{Z} \subseteq \mathbb{Z}$ for some $m > 0$, and the orbit $\mathbb{Z} \cdot x = \{k \cdot x \mid 0 \leq k < m\}$ is finite. Since the system is minimal this contradicts our assumption that X is infinite, see Proposition 10.10. Hence all stabilizers are trivial, or equivalently, the action is free. □

Remark 10.14. Let $\vartheta \in \mathbb{R}$ be irrational. As a special case of Theorem 10.13 we obtain the simplicity of the *reduced* crossed product $C(\mathbb{T}) \rtimes_{\mathrm{r},\alpha} \mathbb{Z}$ of the action by irrational rotations by ϑ on $\mathbb{T}$, since this action is minimal as explained in Example 10.11. Note that the simplicity of the *full* crossed product $A_\vartheta \cong C(\mathbb{T}) \rtimes_{\mathrm{f},\alpha} \mathbb{Z}$ was already obtained in Theorem 7.11, compare the discussion in Example 9.11. However, there is no difference between full and reduced crossed products since $\mathbb{Z}$ is amenable, see Remark 9.17.

10.4. **Odometers.** In order to obtain a larger supply of examples of dynamical systems we shall now discuss certain homeomorphisms of the Cantor set studied in ergodic theory. See [13] for further information.

Let $(n_i)_{i=1}^\infty$ be a sequence of integers such that $n_i > 1$ for all i. Let $X_i = \{0, 1, \dots, n_i - 1\}$ and form the direct product

$$X = \prod_{i=1}^\infty X_i.$$

If we equip X_i with the discrete topology for all $i \in \mathbb{N}$ then X, equipped with the product topology, is a Cantor set. That is, X is a compact totally disconnected metrizable space with no isolated points.

Consider the uniform probability measure μ_i on X_i, so that μ_i assigns the mass n_i^{-1} to each point of X_i. Let μ be the product measure on X constructed from the measures μ_i. On a cylinder set $E = \prod_{i=1}^{\infty} E_i$ such that $E_i = X_i$ except for a finite set of numbers $i_1, \dots, i_n$, this measure is given by

$$\mu(E) = \prod_{j=1}^{n} \mu_{i_j}(E_{i_j}).$$

We think of an element $x = (x_i)_{i=1}^{\infty}$ as a formal sum

$$x = \sum_{i=1}^{\infty} x_i N_{i-1},$$

where $N_0 = 1$ and $N_i = N_{i-1} n_i$ for all $i \in \mathbb{N}$. One can then define a map $X \times X \to X$ by addition with carryover, that is, $(x_i) + (y_i) = (z_i)$ where z_i is uniquely determined by

$$\sum_{i=1}^{n} (x_i + y_i) N_{i-1} \equiv \sum_{i=1}^{n} z_i N_{i-1} \bmod N_n$$

for $n \geq 1$. This turns X into a compact abelian group.

We obtain a homeomorphism σ of X by defining $\sigma(x) = x + 1$, where $1 = (1, 0, 0, \dots) \in X$. The resulting classical dynamical system is called *odometer*, because of its similarity to the odometer in a car. Of course, we are allowing some additional flexibility: an actual odometer would be modeled by taking $n_i = 10$ for all $i \in \mathbb{N}$.

Lemma 10.15. *The measure μ is the unique invariant probability measure on X.*

Proof. On each set $E = \prod_{i=0}^{\infty} E_i$, with E_j a singleton for some j and $E_i = X_i$ for $j \neq i$, we see that any invariant measure ν must be given by $\nu(E) = \mu(E_j) = n_j^{-1}$. Since these sets generate the Borel σ-algebra of X this yields the claim. $\square$

Theorem 10.16. *The crossed product $A = C(X) \rtimes_{\mathrm{r}} \mathbb{Z}$ of an odometer action is simple and has a unique trace.*

Proof. The action of $\mathbb{Z}$ on X is minimal since the orbit $x + \mathbb{Z}$ is dense in X for every $x \in X$. Hence the claim follows from Theorem 10.13 and Theorem 10.7, keeping in mind Lemma 10.15 and the fact that minimal actions of $\mathbb{Z}$ on infinite compact spaces are automatically free. $\square$

In view of Theorem 10.16 it is natural to ask to which extent the C^*-algebra $C(X) \rtimes_{\mathrm{r},\alpha} \mathbb{Z}$ remembers the odometer action from which it is constructed. Note that any nonzero $*$-homomorphism between simple C^*-algebras is necessarily an isomorphism. We will address this question in the next lecture.

10.5. **Exercises.**

Exercise 10.1. Verify in detail that the bounded linear map $E : A \rtimes_{\mathrm{r},\alpha} G \to A$ constructed in Proposition 10.1 is a conditional expectation.

Exercise 10.2. Show that the action of $\mathbb{Z}$ by rotations by $\vartheta \in \mathbb{R}$ on $X = \mathbb{T}$ is free if and only if ϑ is irrational.

Exercise 10.3. Show that an action $\alpha : G \to \mathrm{Homeo}(X)$ is minimal if and only if the orbits $G \cdot x = \{t \cdot x \mid t \in G\}$ are dense in X for all $x \in X$.

Exercise 10.4. Verify the details of the assertions about the Cantor space X in the construction of the odometer.

(a) Check that X has no isolated points. That is, show that there is no $x \in X$ such that $\{x\}$ is open.
(b) Check that X is compact.
(d) Check that X is metrizable.
(c) Check that X is totally disconnected. More precisely, show that X has a basis of the topology consisting of clopen subsets.

It can be shown that any two topological spaces with these properties are homeomorphic. This is known as *Brouwer's Theorem*.

Exercise 10.5. Verify that the odometer X becomes a compact topological group by formal addition with carryover.

Exercise 10.6. Check that odometer actions are minimal.

11. Crossed product C^*-algebras of odometers – a case study

Abstract. In this lecture we study crossed products attached to odometers, which were introduced in the previous lecture, and investigate how much information the crossed product $C(X) \rtimes_{\sigma,\mathrm{r}} \mathbb{Z}$ remembers of the original odometer (X, σ). In other words, given two odometers (X, σ) and (Y, τ), our goal is to find out when $C(X) \rtimes_{\sigma,\mathrm{r}} \mathbb{Z}$ and $C(Y) \rtimes_{\tau,\mathrm{r}} \mathbb{Z}$ are isomorphic as C^*-algebras. In the next lecture, we discuss the same question for general minimal homeomorphisms of the Cantor set. Here and in the sequel, the notation (X, σ) stands for a single homeomorphism σ of a topological space X, which is nothing else but a classical dynamical system $(X, \mathbb{Z}, \alpha)$ with acting group $\mathbb{Z}$ (where $\sigma = \alpha_1$ is the generator and $\alpha_n = \sigma^n$).

11.1. **Inductive limit decompositions.** Our first goal is to write crossed products attached to odometers as inductive limits of simpler building blocks. We use the same notation as in § 10.4, i.e., if $\boldsymbol{n} = (n_i)$ is a sequence of natural numbers $n_i > 1$, then we set $X_i := \{0, \dots, n_i - 1\}$ and $X := \prod_{i=1}^{\infty} X_i$. Moreover, we set $N_0 := 1$, $N_1 := n_1$ and $N_{i+1} := N_i n_{i+1}$ for all $i \in \mathbb{N}$. The odometer $\sigma : X \to X$ attached to the sequence $\boldsymbol{n}$ is given by $\sigma(\boldsymbol{x}) = \boldsymbol{z}$, where $\boldsymbol{x} = (x_i)$, $\boldsymbol{z} = (z_i)$, and z_i is determined by

$$\sum_{h=1}^{i} z_h N_{h-1} \equiv \Big(\sum_{h=1}^{i} x_h N_{h-1} \Big) + 1 \mod N_i,$$

for all $i \in \mathbb{N}$. Alternatively, making use of the group structure on X as explained in § 10.4, we have $\sigma(\boldsymbol{x}) = \boldsymbol{x} + 1$, where 1 denotes the element $(1, 0, 0, \dots)$ of X.

For $\boldsymbol{x} \in \prod_{i=1}^{j} X_i$, we define the cylinder set $C(\boldsymbol{x}) := \{(y_i) \in X : y_i = x_i \ \forall\, 1 \leq i \leq j\}$. The collection $\{C(\boldsymbol{x}) : \boldsymbol{x} \in \prod_{i=1}^{j} X_i,\ j \in \mathbb{N}\}$ forms a basis of clopen subsets of X. We denote by $\mathbf{1}_{C(\boldsymbol{x})}$ the characteristic function of $C(\boldsymbol{x})$.

Recall that (X, σ) induces a C^*-dynamical system $(C(X), \mathbb{Z}, \sigma)$ as in Example 9.3 (c) and the crossed product $C(X) \rtimes_{\sigma,\mathrm{r}} \mathbb{Z}$ attached to $(C(X), \mathbb{Z}, \sigma)$ is constructed as a completion of $C_c(\mathbb{Z}, C(X), \sigma)$ (see Definition 9.7). Following the notation introduced before Definition 9.6, we set $u := \delta_1 \in C_c(\mathbb{Z}, C(X), \sigma) \subseteq C(X) \rtimes_{\sigma,\mathrm{r}} \mathbb{Z}$. Here δ_1 is the function $\mathbb{Z} \to C(X)$ taking the value $1 \in C(X)$ at the canonical generator $1 \in \mathbb{Z}$ and the value $0 \in C(X)$ everywhere else. u is the unitary corresponding to the canonical generator of $\mathbb{Z}$, and we have the twisted commutation relation

$$u f u^* = f \circ \sigma^{-1}$$

for all $f \in C(X)$.

Now let us fix $j \in \mathbb{N}$ and consider the sub-C^*-algebra $C^*(u, \{\mathbf{1}_{C(\boldsymbol{x})} : \boldsymbol{x} \in \prod_{i=1}^{j} X_i\})$ of $C(X) \rtimes_{\sigma,\mathrm{r}} \mathbb{Z}$ generated by u and $\{\mathbf{1}_{C(\boldsymbol{x})} : \boldsymbol{x} \in \prod_{i=1}^{j} X_i\}$. Write $N := N_j$.

The twisted commutation relation implies

$$u^p \mathbf{1}_{C(\boldsymbol{x})} u^{-p} = \mathbf{1}_{C(\boldsymbol{x})+p}, \tag{11.1}$$

for $\boldsymbol{x} = (x_i) \in \prod_{i=1}^{j} X_i$, where $C(\boldsymbol{x}) + p$ is the cylinder set $C(\boldsymbol{z})$, where $\boldsymbol{z} = (z_i) \in \prod_{i=1}^{j} X_i$ is determined by

$$\sum_{h=1}^{i} z_h N_{h-1} \equiv \Big(\sum_{h=1}^{i} x_h N_{h-1}\Big) + p \mod N_i,$$

for all $1 \leq i \leq j$. Hence we have for all $\boldsymbol{x} = (x_i) \in \prod_{i=1}^{j} X_i$

$$(C(\boldsymbol{x}) + p) \cap (C(\boldsymbol{x}) + q) = \emptyset \quad \forall\, p, q \in \{0, \ldots, N-1\},\, p \neq q, \tag{11.2}$$

$$(C(\boldsymbol{x}) + p) = (C(\boldsymbol{x}) + q) \quad \text{if } p \equiv q \mod N. \tag{11.3}$$

In particular, we have

$$u^N \mathbf{1}_{C(\boldsymbol{x})} u^{-N} = \mathbf{1}_{C(\boldsymbol{x})} \tag{11.4}$$

for all $\boldsymbol{x} = (x_i) \in \prod_{i=1}^{j} X_i$.

Lemma 11.1. *We have*

$$C^*(u,\, \{\mathbf{1}_{C(\boldsymbol{x})} :\, \boldsymbol{x} \in \prod_{i=1}^{j} X_i\}) \cong M_N(C(\mathbb{T})).$$

Remark 11.2. Here and in the sequel, given $N \in \mathbb{N}$ and a C^*-algebra C, $M_N(C)$ denotes the C^*-algebra of $N \times N$-matrices over C. The algebra structure of $M_N(C)$ is given by entry-wise addition and scalar multiplication, while multiplication is given by matrix multiplication. The involution is given by $(c^*)_{ij} := c_{ji}^*$ for $c = (c_{ij})$. To define a C*-norm on $M_N(C)$, we apply Theorem 5.19 to obtain a faithful representation $\pi : C \to B(H)$. This representation induces a representation $M_N(C) \to B(H^N)$ given by $(\pi(c_{ij})(\xi_j))_k = \sum_l \pi(c_{kl})\xi_l$ (the usual way the matrix $(\pi(c_{ij}))$ acts on a vector $(\xi_j) \in H^N$). The operator norm on $B(H^N)$ then induces a C*-norm on $M_N(C)$, i.e., we set $\|(c_{ij})\| := \|(\pi(c_{ij}))\|_{B(H^N)}$. This C*-norm is unique by Corollary 2.15.

We now come to the proof of Lemma 11.1, which will actually yield an explicit isomorphism of C^*-algebras.

Proof. Let $\mathbf{0} = (0, \ldots, 0) \in \prod_{i=1}^{j} X_i$. Define $\varepsilon_{p0} := u^p \mathbf{1}_{C(\mathbf{0})}$ for all $0 \leq p \leq N-1$, and set $\varepsilon_{pq} := \varepsilon_{p0}\varepsilon_{q0}^* = u^p \mathbf{1}_{C(\mathbf{0})} u^{-q}$. We claim that $\{\varepsilon_{pq} :\, 0 \leq p, q \leq N-1\}$ are matrix units, i.e., they satisfy $\varepsilon_{pq}\varepsilon_{rs} = \delta_{qr}\varepsilon_{ps}$ as in Proposition 6.11 (ii). Indeed, using (11.1), we get $\varepsilon_{pq} = u^p \mathbf{1}_{C(\mathbf{0})} u^{-q} = u^{p-q} \mathbf{1}_{C(\mathbf{0})+q}$ and $\varepsilon_{rs} = u^r \mathbf{1}_{C(\mathbf{0})} u^{-s} = \mathbf{1}_{C(\mathbf{0})+r} u^{r-s}$ and thus $\varepsilon_{pq}\varepsilon_{rs} = u^{p-q} \mathbf{1}_{C(\mathbf{0})+q} \mathbf{1}_{C(\mathbf{0})+r} u^{r-s}$. By (11.2), this term vanishes unless $q = r$, in which case we obtain

$$\varepsilon_{pq}\varepsilon_{qs} = u^{p-q} \mathbf{1}_{C(\mathbf{0})+q} u^{q-s} = u^{p-q} u^q \mathbf{1}_{C(\mathbf{0})} u^{-q} u^{q-s} = u^p \mathbf{1}_{C(\mathbf{0})} u^{-s} = \varepsilon_{ps}.$$

Note that by Corollary 6.12, the sub-C^*-algebra $C^*(\{\varepsilon_{pq} :\, 0 \leq p, q \leq N-1\})$ of $C^*(u,\, \{\mathbf{1}_{C(\boldsymbol{x})} :\, \boldsymbol{x} \in \prod_{i=1}^{j} X_i\})$ is isomorphic to $M_N(\mathbb{C})$.

Now let $A := C^*(u,\, \{\mathbf{1}_{C(\boldsymbol{x})} :\, \boldsymbol{x} \in \prod_{i=1}^{j} X_i\})$ and consider the map $\alpha :\, A \to M_N(\varepsilon_{00} A \varepsilon_{00})$, $a \mapsto (\varepsilon_{0p} a \varepsilon_{q0})_{pq}$. The entries of $\alpha(a)$ lie in the corner $\varepsilon_{00} A \varepsilon_{00}$ because

$\varepsilon_{0p} a \varepsilon_{q0} = \varepsilon_{00}(\varepsilon_{0p} a \varepsilon_{q0})\varepsilon_{00}$, and it is straightforward to check – using that $\{\varepsilon_{pq}\}$ forms a set of matrix units – that α is a *-homomorphism. To see that α is actually an isomorphism, we construct its inverse by defining $\beta : M_N(\varepsilon_{00} A \varepsilon_{00}) \to A$, $(a_{pq}) \mapsto \sum_{p,q} \varepsilon_{p0} a_{pq} \varepsilon_{0q}$. Again, it is straightforward to check that β is a *-homomorphism, and we have

$$\beta(\alpha(a)) = \beta((\varepsilon_{0p} a \varepsilon_{q0})_{pq}) = \sum_{p,q} \varepsilon_{pp} a \varepsilon_{qq} = a,$$

$$\alpha(\beta((a_{pq})))_{rs} = \alpha\Big(\sum_{p,q} \varepsilon_{p0} a_{pq} \varepsilon_{0q}\Big)_{rs} = \varepsilon_{0r}\Big(\sum_{p,q} \varepsilon_{p0} a_{pq} \varepsilon_{0q}\Big)\varepsilon_{s0} = a_{rs}.$$

This shows that $\beta \circ \alpha = \mathrm{id}_A$ and $\alpha \circ \beta = \mathrm{id}_{M_N(\varepsilon_{00} A \varepsilon_{00})}$, as desired.

It remains to identify $\varepsilon_{00} A \varepsilon_{00}$. First note that

$$A = \overline{\mathrm{span}}(\{\mathbf{1}_{C(\boldsymbol{x})} u^k : \boldsymbol{x} \in \prod_{i=1}^{j} X_i,\, k \in \mathbb{Z}\})$$

because $\{\mathbf{1}_{C(\boldsymbol{x})} u^k : \boldsymbol{x} \in \prod_{i=1}^{j} X_i,\, k \in \mathbb{Z}\}$ is *-invariant and multiplicatively closed. Indeed, $(\mathbf{1}_{C(\boldsymbol{x})} u^k)^* = u^{-k}\mathbf{1}_{C(\boldsymbol{x})} = \mathbf{1}_{C(\boldsymbol{x})-k} u^{-k}$, and

$$\mathbf{1}_{C(\boldsymbol{x})} u^k \mathbf{1}_{C(\boldsymbol{y})} u^l = \mathbf{1}_{C(\boldsymbol{x})} \mathbf{1}_{C(\boldsymbol{y})+k} u^{k+l} = \begin{cases} \mathbf{1}_{C(\boldsymbol{x})} u^{k+l} & \text{if } C(\boldsymbol{x}) = C(\boldsymbol{y}) + k, \\ 0 & \text{else.} \end{cases}$$

Hence we conclude that

$$\varepsilon_{00} A \varepsilon_{00} = \overline{\mathrm{span}}(\{\varepsilon_{00} \mathbf{1}_{C(\boldsymbol{x})} u^k \varepsilon_{00} : \boldsymbol{x} \in \prod_{i=1}^{j} X_i,\, k \in \mathbb{Z}\}).$$

Moreover, $\varepsilon_{00} \mathbf{1}_{C(\boldsymbol{x})} u^k \varepsilon_{00}$ vanishes unless $\boldsymbol{x} = \mathbf{0}$ and $k \in N\mathbb{Z}$, in which case we have, for $k = \kappa N$,

$$\varepsilon_{00} \mathbf{1}_{C(\boldsymbol{x})} u^k \varepsilon_{00} = \varepsilon_{00} u^{\kappa N} \varepsilon_{00} = (\varepsilon_{00} u^N \varepsilon_{00})^{\kappa}.$$

This shows that

$$\varepsilon_{00} A \varepsilon_{00} = \overline{\mathrm{span}}(\{(\varepsilon_{00} u^N \varepsilon_{00})^{\kappa} : \kappa \in \mathbb{Z}\}) = C^*(\varepsilon_{00} u^N \varepsilon_{00}).$$

With respect to the unit ε_{00} of $\varepsilon_{00} A \varepsilon_{00}$, $\varepsilon_{00} u^N \varepsilon_{00}$ is a unitary, and it turns out to have full spectrum, i.e., $\mathrm{sp}(\varepsilon_{00} u^N \varepsilon_{00}) = \mathbb{T}$ (see Exercise 11.4). Hence functional calculus (see Theorem 3.28) induces an isomorphism $\gamma : \varepsilon_{00} A \varepsilon_{00} \xrightarrow{\sim} C(\mathbb{T})$, $\varepsilon_{00} u^N \varepsilon_{00} \mapsto z$, where $z = \mathrm{id}_{\mathbb{T}} : \mathbb{T} \to \mathbb{T} \subseteq \mathbb{C}$. By applying it entry-wise to matrices, we obtain an isomorphism $M_N(\gamma) : M_N(\varepsilon_{00} A \varepsilon_{00}) \xrightarrow{\sim} M_N(C(\mathbb{T}))$. So, putting everything together, we obtain the desired isomorphism

$$\vartheta : A \overset{\alpha}{\xrightarrow{\sim}} M_N(\varepsilon_{00} A \varepsilon_{00}) \overset{M_N(\gamma)}{\xrightarrow{\sim}} M_N(C(\mathbb{T})).$$

□

Applying Lemma 11.1 to every $j \in \mathbb{N}$, we obtain isomorphisms $\vartheta_j : A_j \xrightarrow{\sim} M_{N_j}(C(\mathbb{T}))$ with $\varepsilon_{pq} \mapsto e_{pq}$ and $\varepsilon_{00} u^{N_j} \varepsilon_{00} \mapsto z e_{00}$. Here $A_j = C^*(u, \{\mathbf{1}_{C(\boldsymbol{x})} : \boldsymbol{x} \in \prod_{i=1}^{j} X_i\})$ and e_{pq} are the canonical matrix units for $M_{N_j} \subseteq M_{N_j}(C(\mathbb{T}))$. Note that $A_j \subseteq A_{j+1}$ because $C(\boldsymbol{x}) = \coprod_{x_{j+1}=0}^{n_{j+1}-1} C(\boldsymbol{x}, x_{j+1})$ implies that $\mathbf{1}_{C(\boldsymbol{x})} = \sum_{x_{j+1}=0}^{n_{j+1}-1} \mathbf{1}_{C(\boldsymbol{x}, x_{j+1})} \in A_{j+1}$ for all $\boldsymbol{x} \in \prod_{i=1}^{j} X_i$. Moreover, $C(X) \rtimes_{\sigma,\mathrm{r}} \mathbb{Z} = \overline{\bigcup_{j=1}^{\infty} A_j}$ because cylinder sets of the form $C(\boldsymbol{x})$ are a basis for the topology of X. Thus, if we denote by ι_j the canonical inclusion maps $A_j \hookrightarrow A_{j+1}$, then Remark 8.3 (a) tells us that $C(X) \rtimes_{\sigma,\mathrm{r}} \mathbb{Z}$ is isomorphic to the inductive limit of $A_1 \xrightarrow{\iota_1} A_2 \xrightarrow{\iota_2} A_3 \xrightarrow{\iota_3} \dots$, i.e.,

$$C(X) \rtimes_{\sigma,\mathrm{r}} \mathbb{Z} \cong \varinjlim_j \{A_j, \iota_j\}.$$

The isomorphisms ϑ_j fit into the following commutative diagram

$$\begin{tikzcd}
A_1 \arrow[r, "\iota_1"] \arrow[d, "\vartheta_1"] & A_2 \arrow[r, "\iota_2"] \arrow[d, "\vartheta_2"] & \dots \arrow[r, "\iota_{j-1}"] & A_j \arrow[r, "\iota_j"] \arrow[d, "\vartheta_j"] & A_{j+1} \arrow[r, "\iota_{j+1}"] \arrow[d, "\vartheta_{j+1}"] & \dots \\
M_{N_1}(C(\mathbb{T})) \arrow[r, "\varphi_1"] & M_{N_2}(C(\mathbb{T})) \arrow[r, "\varphi_2"] & \dots \arrow[r, "\varphi_{j-1}"] & M_{N_j}(C(\mathbb{T})) \arrow[r, "\varphi_j"] & M_{N_{j+1}}(C(\mathbb{T})) \arrow[r, "\varphi_{j+1}"] & \dots
\end{tikzcd}$$

Here we set $\varphi_j := \vartheta_{j+1} \circ \iota_j \circ \vartheta_j^{-1}$. Therefore, we obtain

$$C(X) \rtimes_{\sigma,\mathrm{r}} \mathbb{Z} \cong \varinjlim_j \{A_j, \iota_j\} \cong \varinjlim_j \{M_{N_j}(C(\mathbb{T})), \varphi_j\}.$$

We now set out to compute the maps φ_j. To do so, it is convenient to identify $M_{N_{j+1}}(C(\mathbb{T}))$ with $M_{N_j}(M_{n_{j+1}}(C(\mathbb{T})))$, using the bijection

$$\{0, \dots, N_{j+1} - 1\} \xrightarrow{\sim} \{0, \dots, n_{j+1} - 1\} \times \{0, \dots, N_j - 1\},\ rN_j + p \mapsto (r, p).$$

More precisely, a matrix $(a_{rN_j+p, sN_j+q}) \in M_{N_{j+1}}(C(\mathbb{T}))$ is identified with the element $(\mathring{a}_{p,q}) \in M_{N_j}(M_{n_{j+1}}(C(\mathbb{T})))$ whose (p,q)-entry is the $n_{j+1} \times n_{j+1}$-matrix $((\mathring{a}_{p,q})_{r,s}) = (a_{rN_j+p, sN_j+q})$.

Lemma 11.3. *With respect to the identification $M_{N_{j+1}}(C(\mathbb{T})) \cong M_{N_j}(M_{n_{j+1}}(C(\mathbb{T})))$ introduced above, the map $\varphi_j : M_{N_j}(C(\mathbb{T})) \to M_{N_{j+1}}(C(\mathbb{T}))$ is given by*

$$(\varphi_j(e_{pq}))_{p',q'} = \begin{cases} 1 & \text{if } (p',q') = (p,q), \\ 0 & \text{else}, \end{cases} \tag{11.5}$$

where 1 and 0 denote the unit and zero element in $M_{n_{j+1}}(C(\mathbb{T}))$, respectively, and

$$(\varphi_j(z e_{00}))_{p',q'} = \begin{cases} \begin{pmatrix} 0 & 0 & \dots & 0 & z \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} & \text{if } (p',q') = (0,0), \\ 0 & \text{else}, \end{cases} \tag{11.6}$$

where $z = \mathrm{id}_{\mathbb{T}}$ as before.

Note that this determines φ_j as e_{pq} and ze_{00} generate $M_{N_j}(C(\mathbb{T}))$. For instance, we must have

$$(\varphi_j(ze_{pq}))_{p',q'} = \begin{cases} \left(\begin{smallmatrix} 0 & 0 & \dots & 0 & z \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{smallmatrix}\right) & \text{if } (p',q') = (p,q), \\ 0 & \text{else.} \end{cases}$$

Proof. We write $\mathbf{0}_j := (0,\dots,0) \in \prod_{i=1}^j X_i$. We have $\vartheta_j(u^p \mathbf{1}_{C(\mathbf{0}_j)} u^{-q}) = e_{pq}$. Moreover,

$$\begin{aligned}\iota_j(u^p \mathbf{1}_{C(\mathbf{0}_j)} u^{-q}) = u^p \mathbf{1}_{C(\mathbf{0}_j)} u^{-q} = \sum_{r=0}^{n_{j+1}-1} u^p \mathbf{1}_{C(\mathbf{0}_j,r)} u^{-q} = \sum_{r=0}^{n_{j+1}-1} u^p u^{rN_j} \mathbf{1}_{C(\mathbf{0}_{j+1})} u^{-rN_j} u^{-q} \\ = \sum_{r=0}^{n_{j+1}-1} u^{rN_j+p} \mathbf{1}_{C(\mathbf{0}_{j+1})} u^{-(rN_j+q)},\end{aligned}$$

and thus

$$\vartheta_{j+1}(\iota_j(\vartheta_j^{-1}(e_{pq})))_{p',q'} = \begin{cases} \sum_{r=0}^{n_{j+1}-1} \tilde{e}_{rr} = 1 & \text{if } (p',q') = (p,q), \\ 0 & \text{else.} \end{cases}$$

Here $\tilde{e}_{rs}$ denotes the matrix units in $M_{n_{j+1}}(C(\mathbb{T}))$.

This, together with $\varphi_j = \vartheta_{j+1} \circ \iota_j \circ \vartheta_j^{-1}$, shows (11.5).

Similarly, we have $\vartheta_j(\mathbf{1}_{C(\mathbf{0}_j)} u^{N_j} \mathbf{1}_{C(\mathbf{0}_j)}) = ze_{00}$. Moreover,

$$\begin{aligned}&\iota_j(\mathbf{1}_{C(\mathbf{0}_j)} u^{N_j} \mathbf{1}_{C(\mathbf{0}_j)}) = \mathbf{1}_{C(\mathbf{0}_j)} u^{N_j} \mathbf{1}_{C(\mathbf{0}_j)} = \sum_{r,s=0}^{n_{j+1}-1} \mathbf{1}_{C(\mathbf{0}_j,r)} u^{N_j} \mathbf{1}_{C(\mathbf{0}_j,s)} \\ &= \left(\sum_{r=0}^{n_{j+1}-1} \sum_{s=0}^{n_{j+1}-2} \mathbf{1}_{C(\mathbf{0}_j,r)} \mathbf{1}_{C(\mathbf{0}_j,s+1)} u^{N_j} \right) + \left(\sum_{r=0}^{n_{j+1}-1} \mathbf{1}_{C(\mathbf{0}_j,r)} \mathbf{1}_{C(\mathbf{0}_j,0)} u^{N_j} \right) \\ &= \left(\sum_{s=0}^{n_{j+1}-2} \mathbf{1}_{C(\mathbf{0}_j,s+1)} u^{N_j} \right) + \mathbf{1}_{C(\mathbf{0}_{j+1})} u^{N_j} \\ &= \left(\sum_{s=0}^{n_{j+1}-2} u^{(s+1)N_j} \mathbf{1}_{C(\mathbf{0}_{j+1})} u^{-sN_j} \right) + (\mathbf{1}_{C(\mathbf{0}_{j+1})} u^{N_{j+1}} \mathbf{1}_{C(\mathbf{0}_{j+1})})(\mathbf{1}_{C(\mathbf{0}_{j+1})} u^{-(n_{j+1}-1)N_j}).\end{aligned}$$

Therefore

$$\vartheta_{j+1}(\iota_j(\vartheta_j^{-1}(ze_{00})))_{p',q'} = \begin{cases} \left(\sum_{s=0}^{n_{j+1}-2} \tilde{e}_{s+1,s} \right) + z\tilde{e}_{0,n_{j+1}-1} & \text{if } (p',q') = (0,0), \\ 0 & \text{else.} \end{cases}$$

Here $\tilde{e}_{rs}$ denotes the matrix units in $M_{n_{j+1}}(C(\mathbb{T}))$ as before.

This, together with $\varphi_j := \vartheta_{j+1} \circ \iota_j \circ \vartheta_j^{-1}$, shows (11.6). $\square$

It turns out that the inductive limit C^*-algebras $\varinjlim_j \{M_{N_j}(C(\mathbb{T})), \varphi_j\}$ attached to the inductive systems $M_{N_1}(C(\mathbb{T})) \xrightarrow{\varphi_1} M_{N_2}(C(\mathbb{T})) \xrightarrow{\varphi_2} M_{N_3}(C(\mathbb{T})) \xrightarrow{\varphi_3} \dots$ have been studied before by Bunce and Deddens in [9]. Hence they are called Bunce-Deddens algebras.

Definition 11.4. Let $\boldsymbol{n} = (n_i)$ be a sequence of natural numbers with $n_i > 1$ and $\boldsymbol{N} = (N_j)$ be given by $N_1 := n_1$, $N_{j+1} = N_j n_{j+1}$. The Bunce-Deddens algebra $\mathcal{BD}(\boldsymbol{N})$ is given by

$$\mathcal{BD}(\boldsymbol{N}) := \varinjlim_j \{M_{N_j}(C(\mathbb{T})), \varphi_j\},$$

where $\varphi_j : M_{N_j}(C(\mathbb{T})) \to M_{N_{j+1}}(C(\mathbb{T}))$ is determined by (11.5) and (11.6), with respect to the identification $M_{N_{j+1}}(C(\mathbb{T})) \cong M_{N_j}(M_{n_{j+1}}(C(\mathbb{T})))$ as in Lemma 11.3.

We can now summarize our findings as follows:

Corollary 11.5. *Let $\boldsymbol{n} = (n_i)$ be a sequence of natural numbers with $n_i > 1$ and $\boldsymbol{N} = (N_j)$ be given by $N_1 := n_1$, $N_{j+1} = N_j n_{j+1}$. Let (X, σ) be the odometer attached to the sequence $\boldsymbol{n}$. Then we have*

$$C(X) \rtimes_{\sigma,\mathrm{r}} \mathbb{Z} \cong \mathcal{BD}(\boldsymbol{N}).$$

Therefore, if we want to find out which odometers give rise to isomorphic crossed product C^*-algebras, we have to classify Bunce-Deddens algebras.

11.2. **Classification of Bunce-Deddens algebras.** Let us first formulate the classification result for Bunce-Deddens algebras. We need the notion of supernatural numbers. As before, let $\boldsymbol{n} = (n_i)$ be a sequence of natural numbers with $n_i > 1$ and $\boldsymbol{N} = (N_j)$ be given by $N_1 := n_1$, $N_{j+1} = N_j n_{j+1}$. For every prime p, define $v_p(\boldsymbol{N}) := \sup\{v \in \mathbb{N} : p^v \mid N_j \text{ for some } j \in \mathbb{N}\} \in \{0, 1, \dots\} \cup \{\infty\}$. We set $\mathcal{S}(\boldsymbol{N}) := \prod_p p^{v_p(\boldsymbol{N})}$. The product is taken over all primes p, and as such, it is just a formal product. If however $v_p(\boldsymbol{N}) \neq \infty$ for all primes p and $v_p(\boldsymbol{N}) = 0$ for all but finitely many primes p, then we just obtain the usual prime factorization of natural numbers. Given another sequence $\boldsymbol{m} = (m_i)$ of natural numbers $m_i > 1$ and $\boldsymbol{M} = (M_j)$ given by $M_1 := m_1$, $M_{j+1} = M_j m_{j+1}$, we write $\mathcal{S}(\boldsymbol{M}) \mid \mathcal{S}(\boldsymbol{N})$ if $v_p(\boldsymbol{M}) \leq v_p(\boldsymbol{N})$ for all primes p, and $\mathcal{S}(\boldsymbol{M}) = \mathcal{S}(\boldsymbol{N})$ if $v_p(\boldsymbol{M}) = v_p(\boldsymbol{N})$ for all primes p. Note that $\mathcal{S}(\boldsymbol{M}) = \mathcal{S}(\boldsymbol{N})$ if and only if $\mathcal{S}(\boldsymbol{M}) \mid \mathcal{S}(\boldsymbol{N})$ and $\mathcal{S}(\boldsymbol{N}) \mid \mathcal{S}(\boldsymbol{M})$.

Our goal is to prove the following classification result:

Theorem 11.6. *Let $\boldsymbol{m}$, $\boldsymbol{M}$, $\boldsymbol{n}$ and $\boldsymbol{N}$ be as before. We have $\mathcal{BD}(\boldsymbol{M}) \cong \mathcal{BD}(\boldsymbol{N})$ if and only if $\mathcal{S}(\boldsymbol{M}) = \mathcal{S}(\boldsymbol{N})$.*

For the proof, we need two technical lemmas.

Given a C^*-algebra B, an element $e \in B$ and a sub-C^*-algebra D of B, we write $\mathrm{dist}(e, D) := \inf\{\|e - d\| : d \in D\}$.

Lemma 11.7. *Given $\varepsilon > 0$ and $N \in \mathbb{N}$, there exists $\delta > 0$ such that whenever $e_1, \dots, e_N$ are pairwise orthogonal projections in a unital C^*-algebra B and D is a* *sub-C^*-algebra of B containing the unit of B, with* $\operatorname{dist}(e_p, D) < \delta$ *for all* $1 \le p \le N$, *then there exist pairwise orthogonal projections* $f_p \in D$, $1 \le p \le N$, *such that* $\|e_p - f_p\| < \varepsilon$ *for all* $1 \le p \le N$.

If we have in addition $\sum_{p=1}^N e_p = 1$, *then we can also arrange* $\sum_{p=1}^N f_p = 1$.

Here and in the sequel, we call two projections orthogonal if their product vanishes.

Proof. We proceed inductively on N. First consider the case $N = 1$. Set $\delta := \min(\frac{1}{3}, \frac{\varepsilon}{2})$. Suppose $e \in B$ is a projection with $\operatorname{dist}(e, D) < \delta$. Find $x \in D$ with $\|e - x\| < \delta$. By replacing x by $\frac{1}{2}(x + x^*)$, we may assume that x is self-adjoint. We claim that $\operatorname{sp}(x) \subseteq [-\delta, \delta] \cup [1-\delta, 1+\delta]$. Indeed, given $\lambda \in \mathbb{C}$ with $|\lambda|, |1-\lambda| > \delta$, we have that $e - \lambda 1$ is invertible as $\operatorname{sp}(e) = \{0, 1\}$, and $\|(e - \lambda 1)^{-1}\| = \max(|\lambda|^{-1}, |1 - \lambda|^{-1}) < \delta^{-1}$, so that $\|(e - \lambda 1)^{-1}\|^{-1} > \delta$. Hence

$$\|(e - \lambda 1) - (x - \lambda 1)\| = \|e - x\| < \delta < \|(e - \lambda 1)^{-1}\|^{-1}.$$

Thus Lemma 2.6 (b) implies that $\lambda \notin \operatorname{sp}(x)$. This shows $\operatorname{sp}(x) \subseteq [-\delta, \delta] \cup [1-\delta, 1+\delta]$, as desired.

As $\delta \le \frac{1}{3}$, we see that $\operatorname{sp}(x)$ is a disjoint union of two intervals, so that the characteristic function $\mathbf{1}_{[1-\delta, 1+\delta]}$ is continuous on $\operatorname{sp}(x)$. Thus we can apply functional calculus to define $f := \mathbf{1}_{[1-\delta,1+\delta]}(x) \in D$. Then f is self-adjoint, and $\operatorname{sp}(f) \subseteq \{0, 1\}$ implies that f is a projection. Moreover, we have

$$\|e - f\| \le \|e - x\| + \|x - f\| = \|e - x\| + \|(\operatorname{id} - \mathbf{1}_{[1-\delta,1+\delta]})|_{\operatorname{sp}(x)}\|_\infty < \delta + \delta \le \varepsilon.$$

Now suppose $N > 1$, and let $1 > \delta > 0$ (to be specified later). As we proceed inductively, we may assume that we have already constructed pairwise orthogonal projections $f_p \in D$, for $1 \le p \le N-1$, such that $\|e_p - f_p\| < \frac{1}{3N}\delta$. Set $e := \sum_{p=1}^{N-1} e_p$ and $f := \sum_{p=1}^{N-1} f_p$. As before, find a self-adjoint element $x \in D$ with $\|e_N - x\| < \delta$. Let us now estimate

$$\begin{aligned}
&\|e_N - (1-f)x(1-f)\| = \|(1-e)e_N(1-e) - (1-f)x(1-f)\| \\
&\le \|(1-e)e_N(1-e) - (1-f)e_N(1-e)\| \\
&\quad + \|(1-f)e_N(1-e) - (1-f)x(1-e)\| \\
&\quad + \|(1-f)x(1-e) - (1-f)x(1-f)\| \\
&= \|(f-e)e_N(1-e)\| + \|(1-f)(e_N - x)(1-e)\| + \|(1-f)x(f-e)\| \\
&\le \|f - e\| + \|e_N - x\| + \|x\|\|f - e\| \\
&\le (2+\delta)\|f - e\| + \delta \le (2+\delta)\tfrac{N-1}{3N}\delta + \delta < 2\delta.
\end{aligned}$$

Thus, if $2\delta \le \frac{1}{3} \Leftrightarrow \delta \le \frac{1}{6}$, then we can use functional calculus as in the first part of the proof (the case $N = 1$) to produce a projection $f_N \in (1-f)D(1-f)$ such that $\|(1-f)x(1-f) - f_N\| < 2\delta$, so that $\|e_N - f_N\| < 4\delta$. Hence if we set $\delta := \min(\frac{1}{6}, \frac{\varepsilon}{4})$, then we obtain $\|e_N - f_N\| < \varepsilon$, as desired.

Now assume that $\sum_{p=1}^N e_p = 1$. Then

$$\|1 - \sum_{p=1}^N f_p\| = \|\sum_{p=1}^N (e_p - f_p)\| \le \sum_{p=1}^N \|e_p - f_p\| \le \tfrac{N-1}{3N}\delta + 4\delta < 5\delta < 1.$$

But $\sum_{p=1}^N f_p$ is a projection, so that $1 - \sum_{p=1}^N f_p$ is a projection as well, and the only projection with norm strictly less than 1 is zero. Hence we conclude that $1 - \sum_{p=1}^N f_p = 0$, i.e., $\sum_{p=1}^N f_p = 1$, as desired. $\square$

Lemma 11.8. *Given $\varepsilon > 0$ and $N \in \mathbb{N}$, there exists $\delta > 0$ such that whenever C, D are sub-C^*-algebras of a unital C^*-algebra B containing the unit of B, with $\dim C < N$ and $\operatorname{dist}(e_{pq}^{(k)}, D) < \delta$ for all k, p, q, where $e_{pq}^{(k)}$ are matrix units for C as in § 8.2, there exists a unitary v in $C^*(C, D) \subseteq B$ with $\|v - 1\| < \varepsilon$ such that $vCv^* \subseteq D$.*

Proof. Let $\delta > 0$ (to be specified later). Take $0 < \varepsilon' < 1$ (also to be specified later) and set $\eta := \frac{1}{3(N+1)}\varepsilon'$. Apply Lemma 11.7 to $\{e_{pp}^{(k)}\} \subseteq C$ to obtain δ' such that as soon as $\delta \le \delta'$, we can find pairwise orthogonal projections $\{f_{pp}^{(k)}\} \subseteq D$ with $\sum_{k,p} f_{pp}^{(k)} = 1$ and $\|e_{pp}^{(k)} - f_{pp}^{(k)}\| < \eta$ for all k, p. Set $x := \sum_{k,p} f_{pp}^{(k)} e_{pp}^{(k)}$. As $\|e_{pp}^{(k)} f_{pp}^{(k)} e_{pp}^{(k)} - e_{pp}^{(k)}\| \le \|e_{pp}^{(k)} - f_{pp}^{(k)}\| < \eta$, we have

$$x^*x = \sum_{k,p} e_{pp}^{(k)} f_{pp}^{(k)} e_{pp}^{(k)} \ge (1-\eta) \sum_{k,p} e_{pp}^{(k)} = (1-\eta)1,$$

and similarly $xx^* \ge (1-\eta)1$. Hence it follows that x is invertible. Moreover, we have $f_{pp}^{(k)} x = x e_{pp}^{(k)}$ for all k, p by construction. Considering polar decomposition of x, we obtain a unitary $w \in C^*(x) \subseteq C^*(C, D)$ with $x = w|x|$. In other words, we set $w := x|x|^{-1}$. $e_{pp}^{(k)}$ commutes with x^*x, hence with $|x|$, so that

$$we_{pp}^{(k)} = x|x|^{-1} e_{pp}^{(k)} = x e_{pp}^{(k)} |x|^{-1} = f_{pp}^{(k)} x|x|^{-1} = f_{pp}^{(k)} w$$

for all k, p. Moreover, we have the estimate (for sufficiently small ε')

$$\begin{aligned} \|w - 1\| &\le \|w - x\| + \|x - 1\| = \|x|x|^{-1} - x\| + \|\sum_{k,p} (f_{pp}^{(k)} - e_{pp}^{(k)}) e_{pp}^{(k)}\| \\ &\le \|x\| \||x|^{-1} - 1\| + \sum_{k,p} \|f_{pp}^{(k)} - e_{pp}^{(k)}\| < ((1-\eta)^{-\frac{1}{2}} - 1) + N\eta \le (N+1)\eta = \tfrac{\varepsilon'}{3}. \end{aligned}$$

Here we used $\|x\| \le 1$ since $x^*x \le \sum_{k,p} e_{pp}^{(k)} = 1$, and that for sufficiently small η, we have $(1-\eta)^{-\frac{1}{2}} - 1 \le \eta$.

Let $\tilde{e}_{pq}^{(k)} := we_{pq}^{(k)} w^*$ be the matrix units for wCw^*. If $\delta \le \frac{\varepsilon'}{3}$, then we have

$$\operatorname{dist}(\tilde{e}_{pq}^{(k)}, D) < \delta + \|e_{pq}^{(k)} - \tilde{e}_{pq}^{(k)}\| \le \delta + 2\|w - 1\| \le 3\tfrac{\varepsilon'}{3} = \varepsilon'$$

Hence we can find $y_{0q}^{(k)} \in D$ with $\|\tilde{e}_{0q}^{(k)} - y_{0q}^{(k)}\| < \varepsilon'$, for all k, q. By multiplying with $\tilde{e}_{00}^{(k)}$ from the left and $\tilde{e}_{qq}^{(k)}$ from the right if necessary, we can arrange that $y_{0q}^{(k)} = \tilde{e}_{00}^{(k)} y_{0q}^{(k)} \tilde{e}_{qq}^{(k)}$. By normalizing $y_{0q}^{(k)}$ and choosing ε' sufficiently small, we obtain $x_{0q}^{(k)} \in C^*(C, D)$ with $x_{0q}^{(k)} = \tilde{e}_{00}^{(k)} x_{0q}^{(k)} \tilde{e}_{qq}^{(k)}$ and $\|x_{0q}^{(k)}\| = 1$ such that $\|\tilde{e}_{0q}^{(k)} - x_{0q}^{(k)}\| < \frac{\varepsilon}{3}$ and

$$\|\tilde{e}_{qq}^{(k)} - (x_{0q}^{(k)})^* x_{0q}^{(k)}\| = \|(\tilde{e}_{0q}^{(k)})^* \tilde{e}_{0q}^{(k)} - (x_{0q}^{(k)})^* x_{0q}^{(k)}\| < \tfrac{\varepsilon}{3},$$
$$\|\tilde{e}_{00}^{(k)} - x_{0q}^{(k)} (x_{0q}^{(k)})^*\| = \|\tilde{e}_{0q}^{(k)} (\tilde{e}_{0q}^{(k)})^* - x_{0q}^{(k)} (x_{0q}^{(k)})^*\| < \tfrac{\varepsilon}{3},$$

for all k, q.

A similar argument as before shows that if ε is sufficiently small, then $(x_{0q}^{(k)})^* x_{0q}^{(k)}$ is invertible in $\tilde{e}_{qq}^{(k)} D \tilde{e}_{qq}^{(k)}$ and $x_{0q}^{(k)} (x_{0q}^{(k)})^*$ is invertible in $\tilde{e}_{00}^{(k)} D \tilde{e}_{00}^{(k)}$. Thus polar decomposition of $x_{0q}^{(k)}$ yields partial isometries $f_{0q}^{(k)} \in C^*(C, D)$. In other words, we set $f_{0q}^{(k)} := x_{0q}^{(k)} |x_{0q}^{(k)}|^{-1}$, where the inverse of $|x_{0q}^{(k)}|$ is taken in $\tilde{e}_{qq}^{(k)} D \tilde{e}_{qq}^{(k)}$. We have $(f_{0q}^{(k)})^* f_{0q}^{(k)} = e_{qq}^{(k)}$, $f_{0q}^{(k)} (f_{0q}^{(k)})^* = e_{00}^{(k)}$, and – by a similar argument as before – $\|x_{0q}^{(k)} - f_{0q}^{(k)}\| < (1 - \frac{\varepsilon}{3})^{-\frac{1}{2}} - 1 < \frac{\varepsilon}{3}$. Define $f_{pq}^{(k)} := (f_{0p}^{(k)})^* f_{0q}^{(k)}$. Moreover, we construct the unitary $\tilde{w} := \sum_{k,q} (f_{0q}^{(k)})^* \tilde{e}_{0q}^{(k)}$. By construction, we have $f_{pq}^{(k)} \tilde{w} = \tilde{w} \tilde{e}_{pq}^{(k)}$. Furthermore, we can estimate

$$\|\tilde{w} - 1\| = \max_{k,q} \|(f_{0q}^{(k)})^* \tilde{e}_{0q}^{(k)} - (\tilde{e}_{0q}^{(k)})^* \tilde{e}_{0q}^{(k)}\| = \max_{k,q} \|f_{0q}^{(k)} - \tilde{e}_{0q}^{(k)}\| < \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} = \tfrac{2}{3}\varepsilon.$$

Now let us set $v := \tilde{w} w$ and check that this is a unitary with the desired properties if, in addition to the requirements on ε' we have collected so far, we choose ε' so small that $\varepsilon' < \varepsilon$. We have $\|v - 1\| \leq \|\tilde{w} w - w\| + \|w - 1\| < \frac{2}{3}\varepsilon + \frac{1}{3}\varepsilon' < \varepsilon$, and $v e_{pq}^{(k)} v^* = \tilde{w} w e_{pq}^{(k)} w^* \tilde{w}^* = \tilde{w} \tilde{e}_{pq}^{(k)} \tilde{w}^* = f_{pq}^{(k)} \in D$ for all k, p, q. □

After these preparations, we now set out to classify Bunce-Deddens algebras.

Proof of Theorem 11.6. Let $\boldsymbol{m}$, $\boldsymbol{M}$, $\boldsymbol{n}$ and $\boldsymbol{N}$ be as in Theorem 11.6. Set $A := \mathcal{BD}(\boldsymbol{M})$ and $B := \mathcal{BD}(\boldsymbol{N})$. By definition of Bunce-Deddens algebras, we have $A = \varinjlim_j \{A_j, \varphi_j\}$ and $B = \varinjlim_j \{B_j, \psi_j\}$, where $A_j = M_{M_j}(C(\mathbb{T}))$, $B_j = M_{N_j}(C(\mathbb{T}))$, and the connecting maps φ_j and ψ_j are given as in Definition 11.4.

First, let us assume that $\mathcal{S}(\boldsymbol{M}) = \mathcal{S}(\boldsymbol{N})$ and prove $A \cong B$. Without loss of generality, we may assume $M_j \mid N_j$ and $N_j \mid M_{j+1}$ (pass to subsequences to arrange this). We construct for all $j \in \mathbb{N}$ a commutative diagram

(11.7)
$$\begin{tikzcd} A_j \arrow[r, "\varphi_j"] \arrow[d, "\pi_j"'] & A_{j+1} \arrow[d, "\pi_{j+1}"] \\ B_j \arrow[r, "\psi_j"'] \arrow[ur, "\rho_j"] & B_{j+1} \end{tikzcd}$$

as follows: First define $\tilde{\pi}_j \colon A_j \to B_j$ by setting

$$
\begin{aligned}
(\tilde{\pi}_j(e_{pq}))_{p',q'} &= \begin{cases} 1 & \text{if } (p',q') = (p,q), \\ 0 & \text{else,} \end{cases} \\
(\tilde{\pi}_j(ze_{00}))_{p',q'} &= \begin{cases} \left(\begin{smallmatrix} 0 & 0 & \dots & 0 & z \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{smallmatrix}\right) & \text{if } (p',q') = (0,0), \\ 0 & \text{else,} \end{cases}
\end{aligned}
$$

where we use similar notation as in (11.5) and (11.6) as well as the identification $M_{N_j}(C(\mathbb{T})) \cong M_{M_j}(M_{N_j/M_j}(C(\mathbb{T})))$ as in Lemma 11.3. Similarly, define $\tilde{\rho}_j \colon B_j \to A_{j+1}$ by setting

$$
\begin{aligned}
(\tilde{\rho}_j(e_{pq}))_{p',q'} &= \begin{cases} 1 & \text{if } (p',q') = (p,q), \\ 0 & \text{else,} \end{cases} \\
(\tilde{\rho}_j(ze_{00}))_{p',q'} &= \begin{cases} \left(\begin{smallmatrix} 0 & 0 & \dots & 0 & z \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{smallmatrix}\right) & \text{if } (p',q') = (0,0), \\ 0 & \text{else.} \end{cases}
\end{aligned}
$$

It is now straightforward to check that the diagram

$$
\begin{tikzcd}
A_j \arrow[r, "\varphi_j"] \arrow[d, "\tilde{\pi}_j"'] & A_{j+1} \arrow[d, "\tilde{\pi}_{j+1}"] \\
B_j \arrow[r, "\psi_j"] \arrow[ur, "\tilde{\rho}_j"] & B_{j+1}
\end{tikzcd}
$$

commutes up to conjugation by suitable permutation matrices. Hence we can modify $\tilde{\pi}_j$ and $\tilde{\rho}_j$ by suitable inner automorphisms to obtain maps π_j and ρ_j with the desired properties.

Since diagram (11.7) commutes, the maps π_j induce a map $\pi \colon A \to B$, and the maps ρ_j induce a map $\rho \colon B \to A$. Using again commutativity of (11.7), we see that $\rho \circ \pi = \mathrm{id}_A$ and $\pi \circ \rho = \mathrm{id}_B$. Hence $A \cong B$, as desired.

Now we turn to the converse, i.e., let us assume that $A \cong B$, and our goal is to show that $\mathcal{S}(\boldsymbol{M}) = \mathcal{S}(\boldsymbol{N})$. Consider the unital embedding $M_{M_j}(\mathbb{C}) \hookrightarrow M_{M_j}(C(\mathbb{T})) = A_j \hookrightarrow A$. Composing with $A \cong B$, we obtain a unital embedding $\iota \colon M_{M_j}(\mathbb{C}) \hookrightarrow B$. Let $\{e_{pq}\}$ be matrix units for $\iota(M_{M_j}(\mathbb{C}))$. Let δ be as in Lemma 11.8, for some $\varepsilon > 0$ (it does not matter which). As $B = \overline{\bigcup_{k=1}^\infty B_k}$ there exists a (sufficiently big) k such that $\operatorname{dist}(e_{pq}, B_k) < \delta$ for all p, q. Lemma 11.8 then yields a unitary $v \in B$ such that $v\iota(M_{M_j}(\mathbb{C}))v^* \subseteq B_k$. This means that we obtain a unital embedding $M_{M_j}(\mathbb{C}) \hookrightarrow B_k$, $x \mapsto v\iota(x)v^*$. Moreover, evaluation at $1 \in \mathbb{T}$ (or any other point) gives a unital homomorphism $B_k = M_{N_k}(C(\mathbb{T})) \to M_{N_k}(\mathbb{C})$. The composition is a unital homomorphism $M_{M_j}(\mathbb{C}) \to M_{N_k}(\mathbb{C})$. As $M_{M_j}(\mathbb{C})$ is simple, this homomorphism must be injective. Using for instance an argument involving traces, it is straightforward to conclude that we must have $M_j \mid N_k$. Hence we obtain that for all $j \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that $M_j \mid N_k$. This implies that $\mathcal{S}(\boldsymbol{M}) \mid \mathcal{S}(\boldsymbol{N})$. By symmetry, we also get $\mathcal{S}(\boldsymbol{N}) \mid \mathcal{S}(\boldsymbol{M})$. Therefore, $\mathcal{S}(\boldsymbol{M}) = \mathcal{S}(\boldsymbol{N})$, as desired. $\square$

Combining Corollary 11.5 and Theorem 11.6, we obtain

Corollary 11.9. *Let $\boldsymbol{m}$, $\boldsymbol{M}$, $\boldsymbol{n}$ and $\boldsymbol{N}$ be as Theorem 11.6, and let (X, σ) and (Y, τ) be the odometers attached to the sequences $\boldsymbol{m}$ and $\boldsymbol{n}$, respectively. We have $C(X) \rtimes_{\sigma,\mathrm{r}} \mathbb{Z} \cong C(Y) \rtimes_{\tau,\mathrm{r}} \mathbb{Z}$ if and only if $\mathcal{S}(\boldsymbol{M}) = \mathcal{S}(\boldsymbol{N})$.*

11.3. **Exercises.**

Exercise 11.1. In this exercise we develop an alternative approach to odometers. As before, let $\boldsymbol{n} = (n_i)$ be a sequence of natural numbers with $n_i > 1$ and $\boldsymbol{N} = (N_j)$ be given by $N_1 := n_1$, $N_{j+1} = N_j n_{j+1}$. By construction, we have $N_{j+1}\mathbb{Z} \subseteq N_j\mathbb{Z}$. Thus we obtain canonical projections $\pi_{j+1} : \mathbb{Z}/N_{j+1}\mathbb{Z} \twoheadrightarrow \mathbb{Z}/N_j\mathbb{Z}$. Form the projective limit $Y := \varprojlim_j \{\mathbb{Z}/N_j\mathbb{Z}, \pi_j\}$. With the discrete topology and the usual addition, $\mathbb{Z}/N_j\mathbb{Z}$ becomes a compact group. Thus Y inherits the structure of a compact group. It is abelian, so let us use additive notation. The canonical projections $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/N_j\mathbb{Z}$ induce a group homomorphism $\mathbb{Z} \to Y$. Check that this map is injective. We obtain a homeomorphism $\tau : Y \xrightarrow{\sim} Y$ given by $\tau(y) := y + 1$, where 1 denotes the image of the canonical generator $1 \in \mathbb{Z}$ in Y under the map $\mathbb{Z} \to Y$ we just constructed.

Now let (X, σ) be the odometer attached to the sequence $\boldsymbol{n}$, as constructed in § 10.4. Show that there exists a homeomorphism $\varphi : Y \xrightarrow{\sim} X$ such that $\varphi \circ \tau \circ \varphi^{-1} = \sigma$. In other words, (X, σ) and (Y, τ) are conjugate, and thus we may view them as the "same" dynamical system.

Exercise 11.2. It turns out that the construction of odometers can be generalized as follows: Let G be a (discrete) group and G_j a family of finite index subgroups of G such that $G_{j+1} \subseteq G_j$ for all j. Denote by $\pi_{j+1} : G/G_{j+1} \twoheadrightarrow G/G_j$ the canonical projection. As we did in the previous exercise, we may form the projective limit $X := \varprojlim_j \{G/G_j, \pi_j\}$, where the discrete topology on G/G_j induces the topology on X. The group G acts on each coset G/G_j by left translations, i.e., $g \cdot xG_j = (gx)G_j$. This induces an action of G on X, i.e., a map $\alpha : G \to \mathrm{Homeo}(X)$. Show that (X, G, α) is minimal, and that there is a unique G-invariant probability measure on X. Is (X, G, α) free?

Exercise 11.3. Show that $M_N(C(\mathbb{T}))$ can be described as a universal C^*-algebra as follows:

$$\begin{aligned} &M_N(C(\mathbb{T})) \\ &\cong C^*\left(\{e_{pq} :\ 0 \le p, q \le N-1\}, u \,\middle|\, \begin{matrix} e_{pq}^* = e_{qp},\ e_{pq}e_{rs} = \delta_{q,r}e_{ps}\ \forall\ p,q,r,s \\ u^*u = 1 = uu^*, \\ e_{pq}u = ue_{pq}\ \forall\ p,q \end{matrix}\right) \end{aligned}$$

Use this description to construct a homomorphism $M_N(C(\mathbb{T})) \to A$ (A as in the proof of Lemma 11.1) which is the inverse of the map ϑ constructed in the proof of Lemma 11.1.

Exercise 11.4. Here is one possible route to showing $\mathrm{sp}(\varepsilon_{00}u^N\varepsilon_{00}) = \mathbb{T}$ in the proof of Lemma 11.3:

We use the same notation as in Lemma 11.3 and its proof.

First show that given an arbitrary C*-dynamical system $(A, \mathbb{Z}, \alpha)$ (where A is unital), if $u \in A \rtimes_{\alpha,\mathrm{r}} \mathbb{Z}$ is the unitary constructed as at the beginning of § 11.1, then we always have $\mathrm{sp}(u) = \mathbb{T}$.

Next suppose that $\boldsymbol{n} = (n_i)$ is a sequence of natural numbers with $n_i > 1$ and let $\boldsymbol{N} = (N_j)$ be given by $N_1 := n_1$, $N_{j+1} = N_j n_{j+1}$. Let (X, σ) be the odometer attached to $\boldsymbol{n}$. Fix $j \in \mathbb{N}$ and let (Y, τ) be the odometer attached to the shifted sequence $(n_{j+1}, n_{j+2}, \dots)$. Let $v \in C(Y) \rtimes_{\tau,\mathrm{r}} \mathbb{Z}$ be the unitary constructed as at the beginning of § 11.1. Show that we have an isomorphism

$$C(Y) \rtimes_{\tau,\mathrm{r}} \mathbb{Z} \xrightarrow{\sim} C^*(\mathbf{1}_{C(\mathbf{0})}C(X)\mathbf{1}_{C(\mathbf{0})}, \mathbf{1}_{C(\mathbf{0})}u^N\mathbf{1}_{C(\mathbf{0})}) \subseteq C(X) \rtimes_{\sigma,\mathrm{r}} \mathbb{Z}$$

sending v to $\mathbf{1}_{C(\mathbf{0})}u^N\mathbf{1}_{C(\mathbf{0})}$.

Finally, conclude that $\mathrm{sp}(\varepsilon_{00}u^N\varepsilon_{00}) = \mathbb{T}$.

Exercise 11.5. Using Corollary 11.9, show that there are continuum many pairwise non-isomorphic C^*-algebras of the form $C(X) \rtimes_{\sigma,\mathrm{r}} \mathbb{Z}$, where (X, σ) are odometers.

11.4. **Comments.**

Remark 11.10. In this lecture, we have mostly followed the exposition in [14, Sections VIII.4 and V.3]

Remark 11.11. Originally, Bunce-Deddens algebras were introduced as C^*-algebras of weighted shifts (see [9]). By a weighted shift, we mean a bounded linear operator $T :\ \ell^2\mathbb{N} \to \ell^2\mathbb{N}$ given by $T(e_k) = a_k e_{k+1}$ on the canonical orthonormal basis $\{e_k\}$, where (a_k) is a sequence of complex numbers. Such a weighted shift operator is called periodic with period N if $a_{k+N} = a_k$ for all k. Now let $\boldsymbol{n} = (n_i)$ be a sequence of natural numbers with $n_i > 1$ and $\boldsymbol{N} = (N_j)$ be given by $N_1 := n_1$, $N_{j+1} = N_j n_{j+1}$. Let $\mathcal{W}$ be the C^*-algebra generated by all weighted shift operators which are periodic of period N_j for some $j \in \mathbb{N}$. It turns out that $\mathcal{W}$ contains the algebra of compact operators $\mathcal{K} := \mathcal{K}(\ell^2\mathbb{N})$. Then the Bunce-Deddens algebra attached to (N_j) was originally defined as $\mathcal{BD}(N_j) := \mathcal{W}/\mathcal{K}$.

It turned out that Bunce-Deddens algebras provide interesting examples of C^*-algebras. For instance, it can be shown that they are not AF. Even more, it is possible to construct a $\mathbb{Z}/2\mathbb{Z}$-action on a Bunce-Deddens algebra $\mathcal{BD}$ such that the crossed product $\mathcal{BD} \rtimes_{\mathrm{r}} \mathbb{Z}/2\mathbb{Z}$ is AF. By taking dual actions, we obtain an AF algebra together with a $\mathbb{Z}/2\mathbb{Z}$-action such that its crossed product is not AF. Thinking about it, the existence of such an example is quite surprising!

For more about this example, we refer the interested reader to [14, Section VIII.9] and to the original papers [4] and [33].

Remark 11.12. There is a conceptual explanation why the supernatural numbers $\mathcal{S}(\boldsymbol{N})$ are isomorphism invariants of $\mathcal{BD}(\boldsymbol{N})$, and it is provided by K-theory. It turns out that $\mathcal{S}(\boldsymbol{N})$ can be read off from the K-theory of $\mathcal{BD}(\boldsymbol{N})$, and it is clear that K-theory is an isomorphism invariant. K-theory can also be used to show that Bunce-Deddens algebras are not AF (because they have $K_1 \cong \mathbb{Z}$, while AF algebras have vanishing K_1).

Remark 11.13. Lemma 11.8 is a perturbation result for finite-dimensional C^*-algebras. It plays a key role in the classification of AF algebras.

12. Crossed product C^*-algebras of Cantor minimal systems

Abstract. In this lecture we take a closer look at crossed product C^*-algebras $C(X) \rtimes_{\sigma,\mathrm{r}} \mathbb{Z}$ attached to Cantor minimal systems (X, σ) (i.e., X is the Cantor set and σ is a homeomorphism of X which is minimal in the sense of Proposition 10.10 (b)). Our main goal is to decompose $C(X) \rtimes_{\sigma,\mathrm{r}} \mathbb{Z}$ into simpler building blocks, just as we did in the previous lecture in the case of odometers. We then apply classification results for a class of C^*-algebras (so-called $A\mathbb{T}$-algebras) to determine when two Cantor minimal systems give rise to isomorphic crossed product C^*-algebras.

12.1. **Kakutani-Rokhlin partitions.** A key feature of odometers (X, σ) is that we have partitions $\{C(\boldsymbol{x}) : \boldsymbol{x} \in \prod_{i=1}^{j} X_i\}$ of X which are left invariant under σ. This was crucial in our analysis of crossed product C^*-algebras of odometers in the previous lecture. A general Cantor minimal system might not admit such nice partitions. But there is a substitute: Kakutani-Rokhlin partitions, which we now introduce.

In the following, let (X, σ) be a general Cantor minimal system, i.e., X is the Cantor set and σ is a homeomorphism of X which is minimal in the sense of Proposition 10.10 (b). By a partition of X we mean a finite family of non-empty clopen subsets of X which are pairwise disjoint and whose union is X. Given two partitions $\mathcal{P}$ and $\mathcal{Q}$ of X, we say that $\mathcal{Q}$ refines $\mathcal{P}$, or that $\mathcal{Q}$ is finer than $\mathcal{P}$, if for every $P \in \mathcal{P}$, there exists a subset $Q_P \subseteq \mathcal{Q}$ such that $P = \coprod_{Q \in Q_P} Q$. Given a partition $\mathcal{P}$ of X, we introduce the notation $C(\mathcal{P}) := \mathrm{span}(\{\mathbf{1}_P : P \in \mathcal{P}\}) \subseteq C(X)$.

Definition 12.1. Let $Y \subseteq X$ be a non-empty clopen subset. A Kakutani-Rokhlin partition based on Y is a partition $\mathcal{Y}$ of X such that there exist $K \in \mathbb{N}$ and $J_k \in \mathbb{N}$ for $k \in K$ such that $\mathcal{Y} = \{\mathcal{Y}(k, j) : 1 \leq k \leq K, 1 \leq j \leq J_k\}$, and we have

$$\sigma(\mathcal{Y}(k, j)) = \mathcal{Y}(k, j+1) \quad \forall\ 1 \leq k \leq K,\ 1 \leq j \leq J_k - 1, \tag{12.1}$$

$$Y = \bigcup_{k=1}^{K} \mathcal{Y}(k, J_k), \tag{12.2}$$

$$\sigma(Y) = \bigcup_{k=1}^{K} \mathcal{Y}(k, 1). \tag{12.3}$$

For fixed k, $\{\mathcal{Y}(k, j) : 1 \leq j \leq J_k\}$ is called the kth tower. And since the partition $\mathcal{Y}$ is a disjoint union of towers, it is also called a castle. It might be illuminating to draw a picture of the situation, to see that this terminology makes sense.

Our goal now is to show the existence of Kakutani-Rokhlin partitions. We need a preparation first.

Lemma 12.2. *Let $Y \subseteq X$ be a non-empty clopen subset. The map $\lambda_Y : Y \to \mathbb{N}$ given by $\lambda_Y(y) := \min\{n \in \mathbb{N} : \sigma^n(y) \in Y\}$ is well-defined and continuous.*

Note that $\lambda_Y(y)$ is the first time y returns to Y under iterates of σ.

Proof. First of all, λ_Y is well-defined because minimality implies that $\{\sigma^n(y) : n \in \mathbb{N}\}$ is dense in X, and since Y is open, we must have $\{\sigma^n(y) : n \in \mathbb{N}\} \cap Y \neq \emptyset$ for all $y \in Y$.

To prove that λ_Y is continuous, we have to show that for all $y_0 \in Y$, the set $\{y \in Y : \lambda_Y(y) = \lambda_Y(y_0)\}$ is open, as this implies that the preimage of every $n \in \mathbb{N}$ under λ_Y is open. Suppose that $\lambda_Y(y_0) = n$. As σ^n is continuous and Y is open, there exists an open neighbourhood U_n of y_0 such that $\sigma^n(y) \in Y$ for all $y \in U_n$. Moreover, for all $1 \leq m < n$, we have $\sigma^m(y_0) \in X \setminus Y$. Since σ^m is continuous and $X \setminus Y$ is open, there exists an open neighbourhood U_m of y_0 such that $\sigma^m(y) \in X \setminus Y$ for all $y \in U_m$. Then $\bigcap_{m=1}^n U_m$ is an open neighbourhood of y_0, and we have $\lambda_Y(y) = n$ for all $y \in \bigcap_{m=1}^n U_m$ by construction. $\square$

Lemma 12.3. *Given any non-empty clopen subset $Y \subseteq X$ and any partition $\mathcal{P}$ of X, there exists a Kakutani-Rokhlin partition based on Y which refines $\mathcal{P}$.*

Proof. Let us first ignore $\mathcal{P}$ and construct a Kakutani-Rokhlin partition based on Y. Let λ_Y be as in Lemma 12.2. Since Y is compact and λ_Y is continuous, λ_Y has finite image, and we can write $\lambda_Y(Y) = \{J_1, \ldots, J_K\}$. Now set $\mathcal{Y}(k,j) := \sigma^j(\lambda_Y^{-1}(J_k))$ for all $1 \leq k \leq K$ and $1 \leq j \leq J_k$, and $\mathcal{Y} := \{\mathcal{Y}(k,j) : 1 \leq k \leq K, 1 \leq j \leq J_k\}$. By construction, the sets $\mathcal{Y}(k,j)$ are non-empty and clopen. They are also pairwise disjoint: Suppose we are given $y \in \lambda_Y^{-1}(J_k)$ and $y' \in \lambda_Y^{-1}(J_{k'})$ with $\sigma^j(y) = \sigma^{j'}(y')$. Assume without loss of generality that $j' \geq j$. Then $\sigma^{j'-j}(y') = y$, which implies $j' = j$ because otherwise, we would have $1 \leq j' - j < J_{k'}$ and thus $\sigma^{j'-j}(y') \notin Y$ by definition of $\lambda_Y(y')$. Hence we conclude $j' = j$, which implies $y = y'$ and $k = k'$.

Condition (12.1) holds by construction.

We have $Y = \bigcup_k \lambda_Y^{-1}(J_k)$ and thus $\sigma(Y) = \bigcup_k \sigma^1(\lambda_Y^{-1}(J_k)) = \bigcup_k \mathcal{Y}(k,1)$. This shows (12.3).

By construction, we have $\bigcup_k \mathcal{Y}(k, J_k) \subseteq Y$, and so we see that $\bigcup_{k,j} \mathcal{Y}(k,j)$ is σ-invariant. Hence minimality implies that $X = \bigcup_{k,j} \mathcal{Y}(k,j)$. This shows that $\mathcal{Y}$ is a partition of X.

Finally, $Y \cap \mathcal{Y}(k,j) = \emptyset$ for all $1 \leq k \leq K$ and $1 \leq j \leq J_k - 1$. This, together with $X = \bigcup_{k,j} \mathcal{Y}(k,j)$, implies that $Y \subseteq \bigcup_k \mathcal{Y}(k, J_k)$. As we already observed, we also have $\bigcup_k \mathcal{Y}(k, J_k) \subseteq Y$. Hence (12.2) holds.

Now consider a partition $\mathcal{P}$ of X. If $\mathcal{Y}$ does not refine $\mathcal{P}$, then there exists $P \in \mathcal{P}$ and $\mathcal{Y}(k, j_0) \in \mathcal{Y}$ such that $Z(k, j_0) := P \cap \mathcal{Y}(k, j_0)$ is a proper non-empty subset of $\mathcal{Y}(k, j_0)$. Define $Z(k,j) := \sigma^{j-j_0}(Z(k, j_0))$ for all $1 \leq j \leq J_k$. Produce a finer partition by replacing $\mathcal{Y}(k,j)$ by $Z(k,j)$ and $\mathcal{Y}(k,j) \setminus Z(k,j)$ for all $1 \leq j \leq J_k$. Now continue this process for all proper intersections of the form $P \cap \mathcal{Y}(k, j_0)$. This yields the desired Kakutani-Rokhlin partition based on Y refining $\mathcal{P}$. $\square$

Our goal is to present $C(X) \rtimes_{\sigma,\mathrm{r}} \mathbb{Z}$ as an inductive limit of building blocks of the form $M_N(C(\mathbb{T})) \oplus F$, where F is a finite-dimensional C^*-algebra. The first step is to observe that Kakutani-Rokhlin partitions give rise to finite-dimensional sub-C^*-algebras of the crossed product. In the following, let $u \in C(X) \rtimes_{\sigma,\mathrm{r}} \mathbb{Z}$ be the unitary constructed as at the beginning of § 11.1.

Lemma 12.4. *Let $\mathcal{Y} = \{\mathcal{Y}(k,j)\}$ be a Kakutani-Rokhlin partition based on $Y \subseteq X$. Define $F := C^*(C(\mathcal{Y}), u\mathbf{1}_{X\setminus Y}) \subseteq C(X) \rtimes_{\sigma,\mathrm{r}} \mathbb{Z}$. Then*

$$F \cong \bigoplus_{k=1}^{K} M_{J_k}(\mathbb{C}).$$

In particular, F is finite-dimensional.

Proof. Set $e_{ij}^{(k)} := u^{i-j}\mathbf{1}_{\mathcal{Y}(k,j)}$ for $1 \leq k \leq K$, $1 \leq i,j \leq J_k$. Using the identity

$$e_{ij}^{(k)} = u^{i-j}\mathbf{1}_{\mathcal{Y}(k,j)}u^{j-i}u^{i-j} = \mathbf{1}_{\mathcal{Y}(k,i)}u^{i-j},$$

we deduce that $e_{ij}^{(k)}e_{lm}^{(n)} = 0$ unless $k = n$ and $j = l$, in which case we get $e_{ij}^{(k)}e_{jm}^{(k)} = u^{i-j}\mathbf{1}_{\mathcal{Y}(k,j)}\mathbf{1}_{\mathcal{Y}(k,j)}u^{j-m} = u^{i-m}\mathbf{1}_{\mathcal{Y}(k,m)} = e_{im}^{(k)}$. This shows that $\{e_{ij}^{(k)} : 1 \leq k \leq K, 1 \leq i,j \leq J_k\}$ forms a set of matrix units. It remains to show that $F = C^*(\{e_{ij}^{(k)} : 1 \leq k \leq K, 1 \leq i,j \leq J_k\})$. On the one hand, we have $e_{ij}^{(k)} = (u\mathbf{1}_{X\setminus Y})^{i-j}\mathbf{1}_{\mathcal{Y}(k,j)}$ if $1 \leq j < i \leq J_k$. This shows that $e_{ij}^{(k)} \in F$ for all k,i,j. On the other hand, $\mathbf{1}_{\mathcal{Y}(k,j)} = e_{jj}^{(k)}$ and

$$u\mathbf{1}_{X\setminus Y} = u\sum_{k=1}^{K}\sum_{j=1}^{J_k-1}\mathbf{1}_{\mathcal{Y}(k,j)} = \sum_{k=1}^{K}\sum_{j=1}^{J_k-1}u^{(j+1)-j}\mathbf{1}_{\mathcal{Y}(k,j)} = \sum_{k=1}^{K}\sum_{j=1}^{J_k-1}e_{j+1,j}^{(k)}$$

show that $F \subseteq C^*(\{e_{ij}^{(k)} : 1 \leq k \leq K, 1 \leq i,j \leq J_k\})$.

We conclude that $F = C^*(\{e_{ij}^{(k)} : 1 \leq k \leq K, 1 \leq i,j \leq J_k\}) \cong \bigoplus_{k=1}^{K} M_{J_k}(\mathbb{C})$, as desired. □

12.2. **Inductive limit decompositions of crossed product C^*-algebras attached to Cantor minimal systems.** A Kakutani-Rokhlin partition $\mathcal{Y} = \{\mathcal{Y}(k,j)\}$ as in Lemma 12.3 produces a model for the Cantor minimal system (X,σ) in the sense that the partition $\mathcal{P}$ models X and we know exactly what σ does on $\mathcal{Y}(k,j)$ for all $1 \leq k \leq K$ and $1 \leq j \leq J_k - 1$. What the Kakutani-Rokhlin partition $\mathcal{Y}$ does not tell us is where points in $Y = \bigcup_k \mathcal{Y}(k, J_k)$ are mapped to under σ. In view of this interpretation, we can improve our model by replacing $\mathcal{P}$ by finer and finer partitions of X and by shrinking Y. This motivates the following: Let $y \in X$ be an arbitrary point and choose a decreasing sequence of clopen subsets $Y_n \subseteq X$ with $\bigcap_n Y_n = \{y\}$. Moreover, choose a sequence $\mathcal{P}_n$ of finer and finer partitions of X such that $\bigcup_n \mathcal{P}_n$ generates the topology of X, in the sense that any clopen subset of X is a finite disjoint union of some $P_n \in \mathcal{P}_n$, for some n. Let $\mathcal{Y}_n = \{\mathcal{Y}(n,k,j) : 1 \leq k \leq K_n, 1 \leq j \leq J_{n,k}\}$ be Kakutani-Rokhlin partitions based on Y_n and refining $\mathcal{P}_n$. Such $\mathcal{Y}_n$ exist by Lemma 12.3. Replacing (Y_n) by a subsequence if necessary, we may arrange that $\sigma^j(Y_n)$ are pairwise disjoint for $0 \le j \le 2^n + 1$, and that for each $0 \le j \le 2^n + 1$, $\sigma^j(Y_n)$ is contained in a single element of the partition $\mathcal{Y}_{n-1}$.

In the following, we fix $n \in \mathbb{N}$ and drop the index n whenever convenient, i.e., we write $Y = Y_n$, $\mathcal{Y} = \mathcal{Y}_n$, $\mathcal{Y}(k,j) = \mathcal{Y}(n,k,j)$, $K = K_n$, $J_k = J_{n,k}$ for brevity. As in Lemma 12.4, set $F_n := C^*(C(\mathcal{Y}), u\mathbf{1}_{X\setminus Y}) \subseteq C(X) \rtimes_{\sigma,\mathrm{r}} \mathbb{Z}$. Recall that matrix units for F_n are given by $e_{ij}^{(k)} = u^{i-j}\mathbf{1}_{\mathcal{Y}(k,j)}$ for $1 \le k \le K$, $1 \le i,j \le J_k$. We now set out to approximate the unitary $u \in C(X) \rtimes_{\sigma,\mathrm{r}} \mathbb{Z}$.

First, we define a unitary v_n in F_n by

$$v_n := u\mathbf{1}_{X\setminus Y} + \sum_{k=1}^{K} u^{1-J_k}\mathbf{1}_{\mathcal{Y}(k,J_k)} = \sum_{k=1}^{K}\Big(e_{1,J_k}^{(k)} + \sum_{j=1}^{J_k-1} e_{j+1,j}^{(k)}\Big).$$

The following lemma makes precise in what sense v_n models u.

Lemma 12.5. (i) *We have* $v_n\mathbf{1}_{X\setminus Y} = u\mathbf{1}_{X\setminus Y}$.

(ii) *If* $f \in C(\mathcal{Y}) = C(\mathcal{Y}_n)$ *is constant on* $Y = Y_n$, *then* $v_n f v_n^* = f \circ \sigma^{-1}$. *In particular,* $v_n\mathbf{1}_Y v_n^* = \mathbf{1}_{\sigma(Y)}$.

(iii) *We have* $v_n f v_n^* = f\circ\sigma^{-1}$ *for all* $f \in C(\mathcal{Y}_{n-1})$.

Proof. (i) follows by construction. For (ii), take $f \in C(\mathcal{Y})$, and assume without loss of generality that $f \cdot \mathbf{1}_Y = 0$, i.e., $f|_Y \equiv 0$. Then we obtain using (i) that

$$v_n f v_n^* = (v_n\mathbf{1}_{X\setminus Y})f(v_n\mathbf{1}_{X\setminus Y})^* = (u\mathbf{1}_{X\setminus Y})f(u\mathbf{1}_{X\setminus Y})^* = ufu^* = f\circ\sigma^{-1}.$$

Moreover, because of the way we constructed v_n, we have

$$v_n\mathbf{1}_Y v_n^* = v_n\Big(\sum_{k=1}^{K}\mathbf{1}_{\mathcal{Y}(k,J_k)}\Big)v_n^* = \sum_{k=1}^{K}\mathbf{1}_{\mathcal{Y}(k,1)} = \mathbf{1}_{\sigma(Y)}.$$

Thus we obtain (ii). (iii) follows because, by assumption, Y is contained in a single element of $\mathcal{Y}_{n-1}$, so that every $f \in C(\mathcal{Y}_{n-1})$ is constant on Y. ☐

Next, we construct a unitary $w_n \in F_{n+1}$ commuting with $C(\mathcal{Y}_{n-1})$ such that $w_n v_{n+1} w_n^*$ approximates v_n. First observe that $v_{n+1}\mathbf{1}_{X\setminus Y} = u\mathbf{1}_{X\setminus Y} = v_n\mathbf{1}_{X\setminus Y}$ and $u\mathbf{1}_{X\setminus Y} = \mathbf{1}_{X\setminus\sigma(Y)}u$ imply that $\mathbf{1}_{X\setminus\sigma(Y)}v_{n+1} = \mathbf{1}_{X\setminus\sigma(Y)}v_n$. Therefore, we deduce $\mathbf{1}_{X\setminus\sigma(Y)}v_n v_{n+1}^* = \mathbf{1}_{X\setminus\sigma(Y)} = v_n v_{n+1}^*\mathbf{1}_{X\setminus\sigma(Y)}$ and hence $\mathbf{1}_{\sigma(Y)}v_n v_{n+1}^* = v_n v_{n+1}^*\mathbf{1}_{\sigma(Y)}$. Secondly, as $\dim F_{n+1} < \infty$, we know that $\mathrm{sp}(v_n v_{n+1}^*)$ is finite. Thus the function h on $\mathbb{T}$ defined by $h(\mathrm{e}^{\mathrm{i}\vartheta}) = \mathrm{e}^{\mathrm{i}2^{-n}\vartheta}$ for $-\pi < \vartheta \le \pi$ restricts to a continuous function on $\mathrm{sp}(v_n v_{n+1}^*)$, and we may apply functional calculus to obtain $z := h(v_n v_{n+1}^*)$. By construction, we have $z^{2^n} = v_n v_{n+1}^*$. Moreover, we have

$$\|z-1\| = \|(h-1)|_{\mathrm{sp}(v_n v_{n+1}^*)}\|_\infty < 2^{-n}\pi. \tag{12.4}$$

As $v_n v_{n+1}^*$ commutes with $\mathbf{1}_{\sigma(Y)}$, z also commutes with $\mathbf{1}_{\sigma(Y)}$ and $\mathbf{1}_{X\setminus\sigma(Y)}$. It follows that $u^{j-1}zu^{1-j}$ commutes with $\mathbf{1}_{\sigma^j(Y)}$. Moreover, by assumption, we have that $\sigma^j(Y)$

are pairwise disjoint for $1 \leq j \leq 2^n$. Now set $Z := X \setminus \bigcup_{j=1}^{2^n} \sigma^j(Y)$ and

$$w_n := \mathbf{1}_Z + \sum_{j=1}^{2^n} u^{j-1} z^{2^n+1-j} u^{1-j} \mathbf{1}_{\sigma^j(Y)}.$$

The idea here is that w_n acts as the identity on $\mathbf{1}_Z$ and as z^j on $\mathbf{1}_{\sigma^{2^n+1-j}(Y)}$, thus we are shifting from 1 to $v_n v_{n+1}^*$ as we pass through $Z, \sigma^{2^n}(Y), \sigma^{2^n-1}(Y), \ldots, \sigma^1(Y)$.

The following lemma tells us that w_n has all the desired properties.

Lemma 12.6. *The unitary w_n lies in F_{n+1} and commutes with $C(\mathcal{Y}_{n-1})$. Moreover, we have $\|w_n v_{n+1} w_n^* - v_n\| < 2^{-n}\pi$.*

Proof. By construction, we have $z \in F_{n+1}$. In addition, $\mathbf{1}_{\sigma^j(Y)}$ and hence $\mathbf{1}_Z$ lie in $C(\mathcal{Y}) \subseteq F_n \subseteq F_{n+1}$. Moreover, $u^{j-1} z^{2^n+1-j} u^{1-j} \mathbf{1}_{\sigma^j(Y)} = \mathbf{1}_{\sigma^j(Y)} u^{j-1} z^{2^n+1-j} u^{1-j} \mathbf{1}_{\sigma^j(Y)}$ and $\mathbf{1}_{\sigma^j(Y)} u^{j-1} = u^{j-1} \mathbf{1}_{\sigma(Y)} = v_n^{j-1} \mathbf{1}_{\sigma(Y)}$ for all $1 \leq j \leq 2^n$ (by Lemma 12.5 (i)). Putting these observations together, we obtain $w_n \in F_{n+1}$.

By construction, $w_n \mathbf{1}_Z = \mathbf{1}_Z$ and w_n commutes with $\mathbf{1}_{\sigma^j(Y)}$ for all $1 \leq j \leq 2^n$. As every $f \in C(\mathcal{Y}_{n-1})$ is constant on $\sigma^j(Y)$, it must therefore commute with w_n.

Using Lemma 12.5 for v_n and for v_{n+1} in place of v_n, we compute

$$\begin{aligned}
(v_n w_n - w_n v_{n+1}) \mathbf{1}_{\sigma^j(Y)} &= v_n \mathbf{1}_{\sigma^j(Y)} u^{j-1} z^{2^n+1-j} u^{1-j} - w_n \mathbf{1}_{\sigma^{j+1}(Y)} u \\
&= \mathbf{1}_{\sigma^{j+1}(Y)} (u u^{j-1} z^{2^n+1-j} u^{1-j} - u^j z^{2^n-j} u^{-j} u) \mathbf{1}_{\sigma^j(Y)} \\
&= \mathbf{1}_{\sigma^{j+1}(Y)} u^j z^{2^n-j} (z-1) u^{1-j} \mathbf{1}_{\sigma^j(Y)}
\end{aligned}$$

Hence $\|(v_n w_n - w_n v_{n+1}) \mathbf{1}_{\sigma^j(Y)}\| \leq \|z - 1\| < 2^{-n}\pi$ by (12.4).

Moreover, we have

$$(v_n w_n - w_n v_{n+1}) \mathbf{1}_Y = v_n \mathbf{1}_Y - w_n \mathbf{1}_{\sigma(Y)} v_{n+1} = (v_n - z^{2^n} v_{n+1}) \mathbf{1}_Y = 0.$$

And using $\sigma(Z \setminus Y) = (X \setminus \bigcup_{j=1}^{2^n} \sigma^{j+1}(Y)) \setminus \sigma(Y) = Z \setminus \sigma^{2^n+1}(Y)$, we obtain

$$\begin{aligned}
(v_n w_n - w_n v_{n+1}) \mathbf{1}_{Z \setminus Y} &= v_n \mathbf{1}_{Z \setminus Y} - w_n \mathbf{1}_{\sigma(Z \setminus Y)} u \mathbf{1}_{Z \setminus Y} \\
&= u \mathbf{1}_{Z \setminus Y} - w_n \mathbf{1}_{Z \setminus \sigma^{2^n+1}(Y)} u \mathbf{1}_{Z \setminus Y} \\
&= u \mathbf{1}_{Z \setminus Y} - u \mathbf{1}_{Z \setminus Y} = 0.
\end{aligned}$$

Hence it follows that $(v_n w_n - w_n v_{n+1}) \mathbf{1}_Z = 0$.

Since $\{\mathbf{1}_{\sigma^{j+1}(Y)} : 1 \leq j \leq 2^n\}$ and $\{\mathbf{1}_{\sigma^j(Y)} : 1 \leq j \leq 2^n\}$ are pairwise orthogonal, we can estimate

$$\begin{aligned}
\|v_n w_n - w_n v_{n+1}\| &= \|\sum_{j=1}^{2^n} \mathbf{1}_{\sigma^{j+1}(Y)} (v_n w_n - w_n v_{n+1}) \mathbf{1}_{\sigma^j(Y)}\| \\
&= \max_{1 \leq j \leq 2^n} \|(v_n w_n - w_n v_{n+1}) \mathbf{1}_{\sigma^j(Y)}\| < 2^{-n}\pi.
\end{aligned}$$

□

We can now find building blocks of the desired form which will induce an inductive limit decomposition of $C(X) \rtimes_{\sigma,\mathrm{r}} \mathbb{Z}$.

Proposition 12.7. *Let $A_n := C^*(C(\mathcal{Y}_{n-1}), w_n^* v_n w_n, v_{n+1}^* u) \subseteq C(X) \rtimes_{\sigma,\mathrm{r}} \mathbb{Z}$. Then there exists an isomorphism $A_n \xrightarrow{\sim} M_{p(n,1)}(C(\mathbb{T})) \oplus M_{p(n,2)}(\mathbb{C}) \oplus \ldots \oplus M_{p(n,\kappa_n)}(\mathbb{C})$ sending $v_{n+1}^* u$ to the element $(zE_{11}^{(1)}, 1_{M_{p(n,2)}(\mathbb{C})}, \ldots, 1_{M_{p(n,\kappa_n)}(\mathbb{C})})$, where $z = \mathrm{id}_{\mathbb{T}}$, and $E_{ij}^{(k)}$ are matrix units for $M_{p(n,1)}(C(\mathbb{T})) \oplus M_{p(n,2)}(\mathbb{C}) \oplus \ldots \oplus M_{p(n,\kappa_n)}(\mathbb{C})$.*

In addition, there exists a unitary $u_n \in A_n$ with $\|u_n - u\| < 2^{-n}\pi$.

Proof. Set $B_n := C^*(C(\mathcal{Y}_{n-1}), w_n^* v_n w_n) \subseteq C(X) \rtimes_{\sigma,\mathrm{r}} \mathbb{Z}$. As w_n commutes with $C(\mathcal{Y}_{n-1}$, we have $B_n = w_n^* C^*(C(\mathcal{Y}_{n-1}), v_n) w_n \subseteq w_n^* F_n w_n$. As F_n is finite-dimensional, it follows that B_n is finite-dimensional as well, so that we obtain an isomorphism $B_n \cong M_{p(n,1)}(\mathbb{C}) \oplus \ldots \oplus M_{p(n,\kappa_n)}(\mathbb{C})$, producing matrix units $e_{ij}^{(k)}$ for B_n with the additional property that the projections $\{e_{ii}^{(k)}\}$ in B_n correspond to a partition of X between $\mathcal{Y}_{n-1}$ and $\mathcal{Y}_n$ (i.e., it is finer than $\mathcal{Y}_{n-1}$ and it is refined by $\mathcal{Y}_n$).

Moreover, we have $v_{n+1}^* u \mathbf{1}_{X \setminus Y_{n+1}} = \mathbf{1}_{X \setminus Y_{n+1}}$. By construction Y_{n+1} is contained in a single element, say $Y(n)$, of the partition $\mathcal{Y}_n$. As $\mathcal{Y}_n$ is finer than the partition corresponding to $\{e_{ii}^{(k)}\}$, there exists a projection in $\{e_{ii}^{(k)}\}$, say $e_{11}^{(1)}$, such that $\mathbf{1}_{Y_{n+1}} \leq e_{11}^{(1)}$. It follows that $v_{n+1}^* u = (e_{11}^{(1)} v_{n+1}^* u e_{11}^{(1)}) + (1 - e_{11}^{(1)})$. If we now set $z_n := e_{11}^{(1)} v_{n+1}^* u e_{11}^{(1)}$, then z_n is a partial isometry with source and range projection given by $e_{11}^{(1)}$ (in other words, z_n is a unitary in $e_{11}^{(1)} A_n e_{11}^{(1)}$). Furthermore, we have $v_{n+1}^* u = z_n e_{11}^{(1)} + (1 - e_{11}^{(1)})$.

Putting all this together, we obtain $A_n \cong M_{p(n,1)}(C^*(z_n)) \oplus M_{p(n,2)}(\mathbb{C}) \oplus \ldots \oplus M_{p(n,\kappa_n)}(\mathbb{C})$. Now we claim that $\mathrm{sp}(z_n) = \mathbb{T}$. To see this, Let σ_n be the homeomorphism of Y_{n+1} given by $\sigma_n(y) = \sigma^{\lambda_{Y_{n+1}}(y)}(y)$, where $\lambda_{Y_{n+1}}$ is the function introduced in Lemma 12.2. We leave it to the reader to check that (Y_{n+1}, σ_n) is a Cantor minimal system and that we have $z_n f z_n^* = f \circ \sigma_n^{-1}$ for all $f \in C(Y_{n+1})$. As in Exercise 11.4, we obtain an isomorphism $C(Y_{n+1}) \rtimes_{\sigma_n,r} \mathbb{Z} \xrightarrow{\sim} C^*(C(Y_{n+1}), z_n)$ sending the canonical unitary in $C(Y_{n+1}) \rtimes_{\sigma_n,r} \mathbb{Z}$ corresponding to the canonical generator of $\mathbb{Z}$ to z_n, which then implies that $\mathrm{sp}(z_n) = \mathbb{T}$. Thus we arrive at $A_n \cong M_{p(n,1)}(C(\mathbb{T})) \oplus M_{p(n,2)}(\mathbb{C}) \oplus \ldots \oplus M_{p(n,\kappa_n)}(\mathbb{C})$, and the image of $v_{n+1}^* u$ under this isomorphism is of the desired form by construction.

Finally, we set $u_n := (w_n^* v_n w_n)(v_{n+1}^* u) \in A_n$. We estimate

$$\|u_n - u\| = \|(w_n^* v_n w_n v_{n+1}^* - 1)u\| = \|w_n^* v_n w_n v_{n+1}^* - 1\| = \|v_n - w_n v_{n+1} w_n^*\| < 2^{-n}\pi$$

by Lemma 12.6. □

We are now ready to construct the desired inductive limit decomposition of $C(X) \rtimes_{\sigma,\mathrm{r}} \mathbb{Z}$.

Theorem 12.8. *By passing to a subsequence of A_n if necessary, we can find *-homomorphisms $\varphi_n \colon A_n \to A_{n+1}$ such that $C(X) \rtimes_{\sigma,\mathrm{r}} \mathbb{Z} \cong \varinjlim_n \{A_n, \varphi_n\}$.*

Proof. First let $m \in \mathbb{N}$ be arbitrary, and let $B_m \subseteq A_m$ be as in the proof of Proposition 12.7. Given $\varepsilon > 0$, let $\delta > 0$ be as in Lemma 11.8 (for $\frac{\varepsilon}{6}$ in place of ε in Lemma 11.8 and $N = \dim B_m + 1$), and assume without loss of generality that

$\delta \leq \frac{\varepsilon}{6}$. Let $e_{ij}^{(k)}$ be matrix units for $B_m \subseteq A_m$ as in the proof of Proposition 12.7. By Proposition 12.7, there exists a (sufficiently big) $n \in \mathbb{N}$ such that $\operatorname{dist}(e_{ij}^{(k)}, A_n) < \delta$ and $\operatorname{dist}(v_{m+1}^* u, A_n) < \delta$. Indeed, every element of $C(X) \rtimes_{\sigma,\mathrm{r}} \mathbb{Z}$ can be approximated by a finite sum of the form $\sum_{l=-L}^{L} f_l u^l$ for some $f_l \in C(X)$, and by choosing n sufficiently big, we can approximate f_l by elements of $C(\mathcal{Y}_{n-1})$ and u by u_n because of Proposition 12.7. This allows us to approximate elements of the form $\sum_{l=-L}^{L} f_l u^l$ and thus arbitrary elements of $C(X) \rtimes_{\sigma,\mathrm{r}} \mathbb{Z}$ by elements in A_n.

Lemma 11.8 produces a unitary $W_n \in C(X) \rtimes_{\sigma,\mathrm{r}} \mathbb{Z}$ with $\|W_n - 1\| < \frac{\varepsilon}{6}$ such that $e_{ij}^{(k)} \in W_n A_n W_n^*$ for all k, i, j.

As in the proof of Proposition 12.7, write $v_{m+1}^* u = z_m e_{11}^{(1)} + (1 - e_{11}^{(1)})$. Since $\operatorname{dist}(v_{m+1}^* u, A_n) < \delta$, we can find $x \in A_n$ such that $\|v_{m+1}^* u - x\| < \delta$. Now set $z_m' := e_{11}^{(1)} W_n x W_n^* e_{11}^{(1)} \in W_n A_n W_n^*$. We have

$$\begin{aligned}
\|z_m - z_m'\| &= \|e_{11}^{(1)}(v_{m+1}^* u - W_n x W_n^*) e_{11}^{(1)}\| \leq \|v_{m+1}^* u - W_n x W_n^*\| \\
&\leq \|v_{m+1}^* u - W_n v_{m+1}^* u\| + \|W_n v_{m+1}^* u - W_n x\| + \|W_n x - W_n x W_n^*\| \\
&\leq \|1 - W_n\| + \|v_{m+1}^* u - x\| + \|1 - W_n\| < \tfrac{\varepsilon}{6} + \delta + \tfrac{\varepsilon}{6} \leq \tfrac{\varepsilon}{2}.
\end{aligned}$$

As in Lemma 11.8, polar decomposition yields a unitary $\tilde{z}_m \in e_{11}^{(1)} W_n A_n W_n^* e_{11}^{(1)}$ with $\|z_m - \tilde{z}_m\| < \varepsilon$.

Now define $\varphi_m : A_m \to W_n A_n W_n^*$ by $\varphi_m(e_{ij}^{(k)}) := e_{ij}^{(k)}$ for all k, i, j and $\varphi_m(z_m e_{11}^{(1)}) := \tilde{z}_m$. We then have $\varphi_m(e_{ij}^{(k)}) - e_{ij}^{(k)} = 0$ and $\|\varphi_m(z_m e_{11}^{(1)}) - z_m e_{11}^{(1)}\| < \varepsilon$. Since $\{e_{ij}^{(k)}\}_{k,i,j}$ and $z_m e_{11}^{(1)}$ generate A_n, it follows that given any finite set of elements $\mathcal{F}_m \subseteq A_m$, we can arrange that, for all $x \in \mathcal{F}_m$, $\|\varphi_m(x) - x\|$ is as small as we want.

Now proceed inductively to obtain – after passing to a subsequence of A_n if necessary – unitaries $W_n \in C(X) \rtimes_{\sigma,\mathrm{r}} \mathbb{Z}$, *-homomorphisms $\varphi_n : W_n A_n W_n^* \to W_{n+1} A_{n+1} W_{n+1}^*$ and finite subsets $\mathcal{F}_n \subseteq W_n A_n W_n^*$ with $\varphi_n(\mathcal{F}_n) \subseteq \mathcal{F}_{n+1}$ such that $\bigcup_n \overline{\varphi_n}(\mathcal{F}_n)$ is dense in $\varinjlim_n \{W_n A_n W_n^*, \varphi_n\}$ and $\|\varphi_n(x) - x\| < 2^{-n}$ for all $x \in \mathcal{F}_n$. Here $\overline{\varphi_n} : W_n A_n W_n^* \to \varinjlim_n \{W_n A_n W_n^*, \varphi_n\}$ are the maps to the inductive limit as in Proposition 8.2.

Let us set $\varphi_{N,n} := \varphi_{N-1} \circ \ldots \circ \varphi_n : W_n A_n W_n^* \to W_N A_N W_N^*$. We now leave it to the reader to check that

$$\Theta : \varinjlim_n \{W_n A_n W_n^*, \varphi_n\} \to C(X) \rtimes_{\sigma,\mathrm{r}} \mathbb{Z},\ \Theta(\overline{\varphi_n}(x)) := \lim_{N\to\infty} \varphi_{N,n}(x)$$

is well-defined, isometric and has dense image. It then follows that Θ is an isomorphism, and we conclude that

$$C(X) \rtimes_{\sigma,\mathrm{r}} \mathbb{Z} \cong \varinjlim_n M_{p(n,1)}(C(\mathbb{T})) \oplus M_{p(n,2)}(\mathbb{C}) \oplus \ldots \oplus M_{p(n,\kappa_n)}(\mathbb{C}),$$

as desired. $\square$

Inductive limits of the form $\varinjlim_n M_{p(n,1)}(C(\mathbb{T})) \oplus M_{p(n,2)}(\mathbb{C}) \oplus \ldots \oplus M_{p(n,\kappa_n)}(\mathbb{C})$ are examples of A$\mathbb{T}$-algebras. The terminology is in analogy to the case of AF-algebras, which are inductive limits of finite-dimensional C^*-algebras. To construct A$\mathbb{T}$-algebras, finite-dimensional building blocks are replaced by more complicated ones involving matrix algebras over continuous functions on the circle. Hence F (for finite-dimensional) in AF is replaced by $\mathbb{T}$.

12.3. **C*-classification of Cantor minimal systems.** The importance of Theorem 12.8 stems from the possibility of applying classification results for simple A$\mathbb{T}$-algebras to decide when two Cantor minimal systems give rise to isomorphic crossed product C^*-algebras. It turns out that there is a complete isomorphism invariant, which we set out to describe now.

We need some terminology.

Definition 12.9. A ordered abelian group is a pair (G, G^+) consisting of an abelian group $(G, +)$ and a subset $G^+ \subseteq G$ such that

- $0 \in G^+$, $G^+ + G^+ \subseteq G^+$ (i.e., G^+ is a submonoid of G),
- $G^+ \cap (-G^+) = \{0\}$,
- $G^+ - G^+ = G$.

For $x, y \in G$, we write $x \leq y$ if $y - x \in G^+$.

G^+ is called the positive cone.

Definition 12.10. Let (G, G^+) be an ordered abelian group. An order unit for (G, G^+) is an element $u \in G^+$ such that every $x \in G$ satisfies $-ku \leq x \leq ku$ for some $k \in \mathbb{N}$.

The triple (G, G^+, u) is called a scaled ordered abelian group, where (G, G^+) is an ordered abelian group and u an order unit for (G, G^+).

Definition 12.11. Two scaled ordered abelian group (G_1, G_1^+, u_1) and (G_2, G_2^+, u_2) are isomorphic if there exists a group isomorphism $\varphi : G_1 \xrightarrow{\sim} G_2$ with $\varphi(G_1^+) = G_2^+$ and $\varphi(u_1) = u_2$.

Now let (X, σ) be a Cantor minimal system. Let $C(X, \mathbb{Z})$ denote the set of continuous functions $X \to \mathbb{Z}$, where $\mathbb{Z}$ is equipped with the discrete topology. We construct a scaled ordered abelian group as follows:

Definition 12.12. We set

$$K^0(X, \sigma) := C(X, \mathbb{Z}) / \{f - f \circ \sigma^{-1} : \ f \in C(X, \mathbb{Z})\},$$

which is an abelian group under pointwise addition of functions, and define the submonoid

$$K^0(X, \sigma)^+ := \{[f] : \ f \in C(X, \mathbb{Z}), \ f \geq 0\}.$$

Here $[\cdot]$ denote equivalence classes of functions in $K^0(X, \sigma)$.

It turns out that for every Cantor minimal system (X, σ), the triple

$$(K^0(X, \sigma), K^0(X, \sigma)^+, [1_X])$$

is a scaled ordered abelian group.

We can now state without proof the classification result for crossed product C^*-algebras of Cantor minimal systems.

Theorem 12.13. *Let (X_1, σ_1) and (X_2, σ_2) be Cantor minimal systems. We have $C(X_1) \rtimes_{\sigma_1,\mathrm{r}} \mathbb{Z} \cong C(X_2) \rtimes_{\sigma_2,\mathrm{r}} \mathbb{Z}$ if and only if*

$$(K^0(X_1, \sigma_1), K^0(X_1, \sigma_1)^+, [1_{X_1}]) \cong (K^0(X_2, \sigma_2), K^0(X_2, \sigma_2)^+, [1_{X_2}])$$

as scaled ordered abelian groups.

12.4. **Exercises.**

Exercise 12.1. It is instructive to go through what we did in this lecture in the particular case of odometers and see that we recover our findings from the previous lecture. More precisely, let (X, σ) be the odometer attached to a sequence (n_i) of natural numbers $n_i > 1$.

- Check that $\{C(\boldsymbol{x}) : \boldsymbol{x} \in \prod_{i=1}^{j} X_i\}$ give rise to Kakutani-Rokhlin partitions.
- Using this special Kakutani-Rokhlin partitions, convince yourself that Theorem 12.8 boils down to Corollary 11.5.
- Show that in the case of odometers, the classification result Theorem 12.13 boils down to Corollary 11.9.

Exercise 12.2. Using the same notation as in the proof of Proposition 12.7, show that (Y_{n+1}, σ_n) is a Cantor minimal system and that we have $z_n f z_n^* = f \circ \sigma_n^{-1}$ for all $f \in C(Y_{n+1})$.

Exercise 12.3. Using the same notation as in the proof of Theorem 12.8, show that the map Θ is indeed well-defined, isometric, and that its image is dense. Thus conclude that Θ is an isomorphism.

12.5. **Comments.**

Remark 12.14. In this lecture, we mainly followed the exposition in [14, Sections VIII.6 and VIII.7] (see also [48, Chapter 1 and Section 3.2] for a discussion of the classification results mentioned in § 12.3).

Remark 12.15. Theorem 12.8 is originally due to Putnam (see [41, 42]).

Remark 12.16. From the definitions in § 12.3, it is not even clear that

$$(K^0(X, \sigma), K^0(X, \sigma)^+, [1_X])$$

is an isomorphism invariant of $C(X) \rtimes_{\sigma,\mathrm{r}} \mathbb{Z}$. The conceptual explanation is provided – once again – by K-theory: It turns out that there is a canonical isomorphism $K_0(C(X) \rtimes_{\sigma,\mathrm{r}} \mathbb{Z}) \cong K^0(X, \sigma)$ (and hence the notation). Morever, there is in general the notion of positive cone for K_0-groups of C^*-algebras, and the isomorphism $K_0(C(X) \rtimes_{\sigma,\mathrm{r}} \mathbb{Z}) \cong K^0(X, \sigma)$ identifies this positive cone with $K^0(X, \sigma)^+$. Moreover, the isomorphism sends the K_0-class of the unit of $C(X) \rtimes_{\sigma,\mathrm{r}} \mathbb{Z}$ to $[1_X] \in K^0(X, \sigma)$. The interested reader may consult [48, Chapter 1 and Section 3.2] for more information.

Remark 12.17. Talking about the classifying invariant, the following natural question comes to mind: Which scaled ordered abelian groups arise from Cantor minimal systems, i.e., are of the form

$$(K^0(X,\sigma), K^0(X,\sigma)^+, [1_X])?$$

It turns out that there is an abstract characterization of such groups. To explain it, we need some further terminology.

Definition 12.18. An ordered abelian group (G, G^+) is called simple if every non-zero element of G^+ is an order unit for (G, G^+).

Definition 12.19. An ordered abelian group (G, G^+) is called unperforated if for all $g \in G$ and all positive integers n, $ng \in G^+$ implies that $g \in G^+$.

An ordered abelian group (G, G^+) is said to have the Riesz interpolation property if for all $g_1, g_2, h_1, h_2 \in G$ with $g_i \leq h_j$ for all $i, j = 1, 2$, there exists $z \in G$ with $g_i \leq z \leq h_j$ for all $i, j = 1, 2$.

An ordered abelian group is called a dimension group if it is countable, unperforated and has the Riesz interpolation property.

Effros, Handelman and Shen proved that an ordered abelian group is a dimension group if and only if it arises as an inductive limit of an inductive system of ordered abelian groups of the form

$$(\mathbb{Z}^{r_1}, \mathbb{Z}_+^{r_1}) \to (\mathbb{Z}^{r_2}, \mathbb{Z}_+^{r_2}) \to (\mathbb{Z}^{r_3}, \mathbb{Z}_+^{r_3}) \to \dots,$$

where $\mathbb{Z}_+$ denotes the positive cone of non-negative integers.

Now it turns out that for every Cantor minimal system (X, σ),

$$(K^0(X,\sigma), K^0(X,\sigma)^+)$$

is a simple dimension group, and that conversely every simple dimension group (G, G^+, u) with distinguished order unit u is of the form

$$(K^0(X,\sigma), K^0(X,\sigma)^+, [1_X])$$

for some Cantor minimal system (X, σ).

This result is due to Herman, Putnam and Skau (see [24]).

Remark 12.20. Lemma 11.8 combined with the idea behind the construction of Θ in the proof of Theorem 12.8 actually leads to a proof of Proposition 8.15. For an analogue of Proposition 8.15 for A$\mathbb{T}$-algebras, the reader may consult [48, Proposition 3.2.3].

Remark 12.21. Theorem 12.13 follows from Elliott's classification result for A$\mathbb{T}$-algebras of real rank zero (see [18]). While a proof of this result is beyond the scope of these lectures, note that we have already seen a first instance of the so-called intertwining argument in the construction of Θ at the end of the proof of Theorem 12.8. This argument and related ideas form one of the key ingredients in proofs of classification results such as Theorem 12.13.

The classification programme for C^*-algebras has seen tremendous progress recently, leading to classification results which are in a certain sense optimal. We refer the interested reader to [48, 58] and the references therein for more about this fascinating topic.

In this context, it is worth mentioning one of the big remaining open questions in C^*-algebra classification: Given a classical dynamical system (X, G, α) on a finite-dimensional space X (for which we may take the Cantor set) which is free and minimal, with G amenable, does the crossed product C^*-algebra $C(X) \rtimes_{\alpha,\mathrm{r}} G$ fall within the scope of the C^*-algebra classification programme (i.e., are such crossed products classified by K-theoretic invariants)?

The interested reader may find more about this open question and partial progress in this direction in [31, 32].

13. Bratteli-Vershik models and classification up to orbit equivalence for Cantor minimal systems

Abstract. In the previous lecture, we have seen how to classify crossed product C^*-algebras attached to Cantor minimal systems up to isomorphism. While this answers a very natural question in a nice way, it is not clear how this C*-classification result helps in our understanding of the underlying Cantor minimal systems. And so in this last lecture, we would like to make the point that several of the key ideas which came up in our study of crossed product C^*-algebras also lead to advances in the study of Cantor minimal systems. This shows that there is a fruitful interplay between the research areas of C^*-algebras and topological dynamical systems.

13.1. **Bratteli-Vershik models for Cantor minimal systems.** We start by using Kakutani-Rokhlin partitions (as they appeared in § 12.1) and Bratteli diagrams (as in § 8.3) to construct combinatorial models for all Cantor minimal systems. More specifically, to every Cantor minimal system and a given point of the Cantor set, we will construct a Bratteli diagram (together with a partial order). We then present a recipe how to re-construct the initial Cantor minimal system out of the Bratteli diagram. All in all, this will establish a one-to-one correspondence between equivalence classes of pointed Cantor minimal systems and equivalence classes of ordered Bratteli diagrams (with particular properties).

Let us first introduce some terminology.

Definition 13.1. A pointed Cantor minimal system (X, σ, y) consists of a Cantor minimal system (X, σ) together with a point $y \in X$.

Given two pointed Cantor minimal systems (X_1, σ_1, y_1) and (X_2, σ_2, y_2), a pointed topological conjugacy between (X_1, σ_1, y_1) and (X_2, σ_2, y_2) is a homeomorphism $\varphi : X_1 \xrightarrow{\sim} X_2$ satisfying $\varphi \circ \sigma_1 \circ \varphi^{-1} = \sigma_2$ and $\varphi(y_1) = y_2$.

We write $(X_1, \sigma_1, y_1) \sim_{\mathrm{conj}} (X_2, \sigma_2, y_2)$ if there is a pointed topological conjugacy between (X_1, σ_1, y_1) and (X_2, σ_2, y_2).

Now we turn to Bratteli diagrams, which we already encountered in § 8.3. Let us introduce some notations which will be convenient.

Definition 13.2. A Bratteli diagram (V, E) is given by a set of vertices V, a set of edges E and source and range maps $s, r : E \to V$ such that $V = \coprod_{n=0}^{\infty} V_n$ for some finite non-empty sets of vertices $V_0, V_1, V_2, \dots$ and $E = \coprod_{n=1}^{\infty} E_n$ for some finite non-empty sets of edges $E_1, E_2, \dots$, and the source and range maps restrict to maps $s : E_n \to V_{n-1}$ and $r : E_n \to V_n$, for all $n = 1, 2, \dots$.

In addition, we require that $V_0 = \{v_0\}$ is a singleton, and that $s^{-1}\{v\} \neq \emptyset$ for all $v \in \coprod_{n=0}^{\infty} V_n$ and $r^{-1}\{v\} \neq \emptyset$ for all $v \in \coprod_{n=1}^{\infty} V_n$.

Given such a Bratteli diagram (V, E), set

$$E_{m,n} := \{(e_{m+1}, \dots, e_n) : e_i \in E_i \; \forall \, m{+}1 \leq i \leq n, \; r(e_i) = s(e_{i+1}) \; \forall \, m{+}1 \leq i \leq n{-}1\},$$

and for $(e_{m+1}, \dots, e_n) \in E_{m,n}$, we define $s(e_{m+1}, \dots, e_n) = s(e_{m+1})$, $r(e_{m+1}, \dots, e_n) = r(e_n)$. We also introduce the set of infinite paths as

$$E_{0,\infty} := \{(e_1, e_2, \dots) : \ e_i \in E_i \ \forall \ i \in \mathbb{N}, \ r(e_i) = s(e_{i+1}) \ \forall \ i \in \mathbb{N}\}.$$

Next, we introduce an equivalence relation on Bratteli diagrams.

Definition 13.3. Two Bratteli diagrams (V, E) and (V', E') are called isomorphic if there exist bijections $V_n \xrightarrow{\sim} V'_n$ for all $n = 0, 1, 2, \dots$ and $E_n \xrightarrow{\sim} E'_n$ for all $n = 1, 2, \dots$ such that the diagrams

$$\begin{tikzcd} E_n \arrow[r, "\cong"] \arrow[d, "s"'] & E'_n \arrow[d, "s"] \\ V_{n-1} \arrow[r, "\cong"] & V'_{n-1} \end{tikzcd}$$

and

$$\begin{tikzcd} E_n \arrow[r, "\cong"] \arrow[d, "r"'] & E'_n \arrow[d, "r"] \\ V_n \arrow[r, "\cong"] & V'_n \end{tikzcd}$$

commute for all $n \in \mathbb{N}$.

In this case, we write $(V, E) \cong (V', E')$.

Definition 13.4. Suppose that (V, E) is a Bratteli diagram. Let $(n_m)_{m=0}^{\infty}$ be a strictly increasing sequence, starting with $n_0 = 0$. For $m = 0, 1, 2, \dots$, set $V'_m := V_{n_m}$ and $E'_m := E_{n_{m-1}, n_m}$. The source and range maps on E_{n_{m-1}, n_m}, as introduced above, give rise to source and range maps on E'_m. In this way, we obtain a new Bratteli diagram (V', E').

We say that (V', E') is obtained from (V, E) by telescoping (or contracting).

Definition 13.5. Let $\sim$ be the equivalence relation on Bratteli diagrams generated by isomorphism and the operation of telescoping (i.e., we require that $(V, E) \sim (V', E')$ if (V', E') is obtained from (V, E) by telescoping).

We leave it to the reader to find an explicit description of the equivalence relation $\sim$.

Now let us introduce partial orders on Bratteli diagrams.

Definition 13.6. An ordered Bratteli diagram $(V, E, \geq)$ is a Bratteli diagram (V, E) together with a partial order $\geq$ on E such that two edges e_1 and e_2 are comparable with respect to $\geq$ if and only if $r(e_1) = r(e_2)$, i.e., $\geq$ is given by total orders on $r^{-1}\{v\}$ for all $v \in \coprod_{n=1}^{\infty} V_n$.

There is an obvious notion of isomorphism for ordered Bratteli diagrams.

Definition 13.7. Two ordered Bratteli diagrams $(V, E, \geq)$ and $(V', E', \geq')$ are isomorphic if (V, E) and (V', E') are isomorphic via bijections $E_n \xrightarrow{\sim} E'_n$ compatible with $\geq$ and $\geq'$.

Moreover, given an ordered Bratteli diagram $(V, E, \geq)$, the partial order $\geq$ induces partial orders on the set of finite paths $E_{m,n}$ as follows: Given two paths $(e_{m+1}, \dots, e_n), (f_{m+1}, \dots, f_n) \in E_{m,n}$, we say that $(f_{m+1}, \dots, f_n) > (e_{m+1}, \dots, e_n)$ if there exists $m + 1 \leq i \leq n$ such that $f_i > e_i$ and $f_j = e_j$ for all $i < j \leq n$, and $(f_{m+1}, \dots, f_n) \geq (e_{m+1}, \dots, e_n)$ if and only if $(f_{m+1}, \dots, f_n) > (e_{m+1}, \dots, e_n)$ or $(f_{m+1}, \dots, f_n) = (e_{m+1}, \dots, e_n)$.

Definition 13.8. Given two ordered Bratteli diagrams $(V, E, \geq)$ and $(V', E', \geq')$, we say that $(V', E', \geq')$ is obtained from $(V, E, \geq)$ by telescoping (or contracting) if (V', E') is obtained from (V, E) by telescoping (or contracting) in the sense of Definition 13.4, and, for each $m \in \mathbb{N}$, the partial order $\geq'$ on $E'_m = E_{n_{m-1},n_m}$ is the one induced by $\geq$ as we just explained.

It is now clear how to carry over the equivalence relation to ordered Bratteli diagrams.

Definition 13.9. Let $\sim$ be the equivalence relation on ordered Bratteli diagrams generated by isomorphism and the operation of telescoping (i.e., we require that $(V, E, \geq) \sim (V', E', \geq')$ if $(V', E', \geq')$ is obtained from $(V, E, \geq)$ by telescoping).

Now let us explain how to construct ordered Bratteli diagrams from pointed Cantor minimal systems. Actually, this construction motivates the concepts we have just introduced, in particular the notion of partial orders.

Let (X, σ, y) be a pointed Cantor minimal system. As in the previous lecture, let Y_n be a sequence of clopen subsets of X with $Y_{n+1} \subseteq Y_n$ and $\cap_n Y_n = \{y\}$. Let $\mathcal{Y}_n = \{\mathcal{Y}(n, k, j) : 1 \leq k \leq K_n, 1 \leq j \leq J_{n,k}\}$ be Kakutani-Rokhlin partitions based on Y_n. We require that, for each n, $\mathcal{Y}_{n+1}$ refines $\mathcal{Y}_n$, and that $\bigcup_n \mathcal{Y}_n$ generates the topology of X, in the sense that any clopen subset of X is a finite disjoint union of some $\mathcal{Y}(n) \in \mathcal{Y}_n$ for some n. We also require that $\lim_{n\to\infty} \min\{J_{n,k} : 1 \leq k \leq K_n\} = \infty$. It is convenient to set $K_0 := 1$, $J_1 := 1$, $\mathcal{Y}(0, 1, 1) := X$ and $\mathcal{Y}_0 := \{\mathcal{Y}(0, 1, 1)\} = \{X\}$.

Now we construct an ordered Bratteli diagram as follows: For $n = 0, 1, \dots$, set $V_n := \{(n, 1), \dots, (n, K_n)\}$, i.e, we introduce one vertex for each tower in the nth Kakutani-Rokhlin partition. Moreover, we set

$$E_n := \left\{ (n, k, k', j') : \begin{array}{c} 1 \leq k \leq K_{n-1},\, 1 \leq k' \leq K_n,\, 1 \leq j' \leq J_{n,k'}, \\ \mathcal{Y}(n, k', j') \subseteq \mathcal{Y}(n-1, k, 1) \end{array} \right\}.$$

Note that the condition $\mathcal{Y}(n, k', j') \subseteq \mathcal{Y}(n-1, k, 1)$ implies that $j' + J_{n-1,k} - 1 \leq J_{n,k'}$ and that $\mathcal{Y}(n, k', j' + j - 1) \subseteq \mathcal{Y}(n - 1, k, j)$ for all $1 \leq j \leq J_{n-1,k}$. In other words, the k'th tower of $\mathcal{Y}_n$ passes through the kth tower of $\mathcal{Y}_{n-1}$.

To introduce source and range maps, we set $s(n, k, k', j') := (n - 1, k) \in V_{n-1}$ and $r(n, k, k', j') := (n, k') \in V_n$.

To define a partial order $\geq$, the idea is that since each tower is ordered in a natural way, we obtain induced partial orders as follows: We write $(n_2, k_2, k_2', j_2') \geq (n_1, k_1, k_1', j_1')$ if (and only if) $n_2 = n_1$, $k_2' = k_1'$ and $j_2' \geq j_1'$.

This completes our construction. We now obtain an assignment

$$(X, \sigma, y) \mapsto (V, E, \geq)$$

of an ordered Bratteli diagram to a pointed Cantor minimal system.

Before we proceed, let us observe that ordered Bratteli diagrams arising in this way have particular properties which will be important later on.

First of all, the condition $\lim_{n \to \infty} \min\{J_{n,k} : 1 \leq k \leq K_n\} = \infty$ implies that

$$\#E_n > \#V_{n-1} \text{ for infinitely many } n. \tag{13.1}$$

Moreover, consider the following property.

Definition 13.10. A Bratteli diagram (V, E) is called simple if for all $m \in \mathbb{N}$ and $v \in V_m$ there exists $n > m$ such that for all $w \in V_n$ there exists $e \in E_{m,n}$ with $s(e) = v$ and $r(e) = w$.

We leave it to the reader to check that minimality of (X, σ) implies that the Bratteli diagram (V, E) constructed from (X, σ, y) is simple.

Let us introduce another property.

Definition 13.11. An ordered Bratteli diagram $(V, E, \geq)$ is called properly ordered if

- there exists a unique infinite path $e_{\min} = (e_n) \in E_{0,\infty}$ which is minimal in the sense that, for each $n \in \mathbb{N}$, e_n is the minimal element in $r^{-1}\{r(e_n)\}$,
- there exists a unique infinite path $e_{\max} = (e_n) \in E_{0,\infty}$ which is maximal in the sense that, for each $n \in \mathbb{N}$, e_n is the maximal element in $r^{-1}\{r(e_n)\}$.

Remark 13.12. It is easy to see that we always have unique minimal and maximal finite paths with prescribed range, i.e., given $v \in V_n$, there exists a unique finite path $e = (e_1, \ldots, e_n) \in E_{0,n}$ such that $s(e) = v_0$, $r(e) = v_n$ and e_i is the minimal element in $r^{-1}\{r(e_i)\}$ for all $1 \leq i \leq n$, and similarly for "maximal" in place of "minimal".

Lemma 13.13. *The ordered Bratteli diagram $(V, E, \geq)$ constructed from a pointed Cantor minimal system (X, σ, y) is properly ordered.*

Proof. We treat the case of $e_{\min}$, the case of $e_{\max}$ is similar and is left to the reader. Suppose that we are given an infinite path $(e_n) \in E_{0,\infty}$. Write $e_n = (n, k_n, k_n', j_n')$. First of all, observe that $r(e_n) = s(e_{n+1})$ implies that $k_n' = k_{n+1}$, for all $n \in \mathbb{N}$. Next use the facts $\bigcup_{k \in K_{n-1}} \mathcal{Y}(n-1, k, 1) = \sigma(Y_{n-1})$ and $\bigcup_{k' \in K_n} \mathcal{Y}(n, k', 1) = \sigma(Y_n) \subseteq \sigma(Y_{n-1})$ to deduce that we must have $j_n' = 1$ for all $n \in \mathbb{N}$ if e_n is the minimal element in $r^{-1}\{r(e_n)\}$ for all $n \in \mathbb{N}$.

Now conclude that $\emptyset \neq \bigcap_n \mathcal{Y}(n, k_n', 1) \subseteq \bigcap_n \sigma(Y_n) = \{\sigma(y)\}$. Thus k_n' is uniquely determined by the condition $\sigma(y) \in \mathcal{Y}(n, k_n', 1)$ for all $n \in \mathbb{N}$. This shows uniqueness (and also existence) of $e_{\min}$. $\square$

Now let us describe how to construct a pointed Cantor minimal system (X', σ', y') from an ordered Bratteli diagram with the particular properties we just discussed. Let $(V, E, \geq)$ be an ordered Bratteli diagram satisfying (13.1), which is simple and properly ordered. First we construct the topological space X' as the projective limit

$$X' := \varprojlim_n E_{0,n},$$

with respect to the canonical projection maps

$$E_{0,n+1} \twoheadrightarrow E_{0,n},\ (e_1, \dots, e_n, e_{n+1}) \mapsto (e_1, \dots, e_n).$$

In other words, as a set we have $X' = E_{0,\infty}$, and a basis of compact open subsets is given by

$$C(e_1, \dots, e_N) := \{(f_n) \in E_{0,\infty} :\ f_n = e_n \ \forall\ 1 \leq n \leq N\},$$

where N runs through all natural numbers and $(e_1, \dots, e_N)$ runs through all finite paths in $E_{0,N}$.

Observe that condition (13.1) and simplicity of (V, E) imply that X' is homeomorphic to the Cantor set.

Now, using that $(V, E, \geq)$ is properly ordered, we set $y' := e_{\max}$ and $\sigma'(y') := e_{\min}$. Moreover, given $x' \in X'$ with $x' \neq y'$, we have $x' = (e_n)$ and not all e_n are maximal in $r^{-1}\{r(e_n)\}$. So let $m \in \mathbb{N}$ be minimal such that e_m is not maximal in $r^{-1}\{r(e_m)\}$. Let f_m be the successor of e_m with respect to $\geq$. As explained in Remark 13.12, there is a unique minimal finite path $(f_1, \dots, f_{m-1})$ with source v_0 and range $s(f_m)$. We define

$$\sigma'(x') := (f_1, \dots, f_{m-1}, f_m, e_{m+1}, e_{m+2}, \dots).$$

We leave it to the reader to check that this defines a continuous map $\sigma' : X' \to X'$. To see that σ' is a homeomorphism, just apply our construction to the ordered Bratteli diagram obtained from $(V, E, \geq)$ by reversing the partial order. We also leave it to the reader to check that (X', σ') is minimal.

In this way, we obtain an assignment

$$(V, E, \geq) \mapsto (X', \sigma', y')$$

of a pointed Cantor minimal system to an ordered Bratteli diagram.

It turns out that these two assignments are inverse to each other.

Theorem 13.14. *The two assignments*

$$(X, \sigma, y) \mapsto (V, E, \geq) \text{ and } (V, E, \geq) \mapsto (X', \sigma', y')$$

establish a one-to-one correspondence

$$\left\{ \begin{matrix} \textit{pointed Cantor} \\ \textit{minimal systems} \end{matrix} \right\} \Big/ \sim_{\text{conj}} \ \overset{\textit{1-1}}{\longleftrightarrow} \ \left\{ \begin{matrix} \textit{ordered Bratteli diagrams} \\ \textit{satisfying (13.1), which are} \\ \textit{simple and properly ordered} \end{matrix} \right\} \Big/ \sim$$

Proof. First, we argue that in the assignment $(X, \sigma, y) \mapsto (V, E, \geq)$, the equivalence class of $(V, E, \geq)$ does not depend on the choices of Kakutani-Rokhlin partitions. Given Kakutani-Rokhlin partitions $\mathcal{Y}_n$ as above, let n_m be a subsequence. Suppose $(V', E', \geq')$ is obtained from $(V, E, \geq)$ by telescoping with respect to (n_m). Let $(V'', E'', \geq'')$ be the ordered Bratteli diagram constructed from (X, σ, y) using the Kakutani-Rokhlin partitions $(\mathcal{Y}_{n_m})$. Then it is straightforward to check that $(V', E', \geq')$ and $(V'', E'', \geq'')$ are isomorphic, so that we obtain $(V, E, \geq) \sim (V'', E'', \geq'')$.

Now suppose that we are given two sequences of Kakutani-Rokhlin partitions $(\mathcal{Y}_n)$ and $(\tilde{\mathcal{Y}}_n)$ for (X, σ, y), with $\mathcal{Y}_n$ based on Y_n and $(\tilde{\mathcal{Y}}_n)$ based on $\tilde{Y}_n$. By the above argument, we may pass to subsequences, and hence we can arrange that $Y_{n+2} \subseteq \tilde{Y}_{n+1} \subseteq Y_n$ and that $\mathcal{Y}_{n+2}$ refines $\tilde{\mathcal{Y}}_{n+1}$, which in turn refines $\mathcal{Y}_n$. Now define a new sequence of Kakutani-Rokhlin partitions by

$$\bar{\mathcal{Y}}_n := \begin{cases} \mathcal{Y}_n & \text{if } n \text{ is even,} \\ \tilde{\mathcal{Y}}_n & \text{if } n \text{ is odd.} \end{cases}$$

Suppose that $(V, E, \geq)$, $(\tilde{V}, \tilde{E}, \tilde{\geq})$ and $(\bar{V}, \bar{E}, \bar{\geq})$ are the ordered Bratteli diagrams constructed from $(\mathcal{Y}_n)$, $(\tilde{\mathcal{Y}}_n)$ and $(\bar{\mathcal{Y}}_n)$, respectively. Then the argument above shows that telescoping $(\bar{V}, \bar{E}, \bar{\geq})$ with respect to even indices gives us – up to isomorphism – $(V, E, \geq)$, while telescoping $(\bar{V}, \bar{E}, \bar{\geq})$ with respect to odd indices gives us – up to isomorphism – $(\tilde{V}, \tilde{E}, \tilde{\geq})$. Thus we conclude that

$$(V, E, \geq) \sim (\bar{V}, \bar{E}, \bar{\geq}) \sim (\tilde{V}, \tilde{E}, \tilde{\geq}),$$

as desired.

Next, we argue that in the assignment $(V, E, \geq) \mapsto (X', \sigma', y')$, the pointed Cantor minimal system (X', σ', y') – up to pointed conjugacy – only depends on the equivalence class of $(V, E, \geq)$. Indeed, suppose that $(V', E', \geq')$ is obtained from $(V, E, \geq)$ by telescoping with respect to a subsequence (n_m). Let (X'', σ'', y'') be the Cantor minimal system constructed from $(V', E', \geq')$ using our recipe above. Then it is straightforward to verify that $\varphi : X' \to X''$ given by $\varphi(e_1, e_2, e_3, \dots) := (f_1, f_2, \dots)$, where $f_1 = (e_1, \dots, e_{n_1})$, $f_2 = (e_{n_1+1}, \dots, e_{n_2})$, ..., is a pointed topological conjugacy between (X', σ', y') and (X'', σ'', y''). So we deduce that

$$(X', \sigma', y') \sim_{\text{conj}} (X'', \sigma'', y''),$$

as desired.

Now we content ourselves with outlining how to prove that if we start with a pointed Cantor minimal system (X, σ, y) and our assignments yield

$$(X, \sigma, y) \mapsto (V, E, \geq) \mapsto (X', \sigma', y'),$$

then we must have $(X, \sigma, y) \sim_{\text{conj}} (X', \sigma', y')$. The other half of the one-to-one correspondence is left to the reader (compare also [24]).

A point $x' \in X'$ is given by an infinite path $x' = (n, k_n, k'_n, j'_n)_n$. As we have seen before, that $(n, k_n, k'_n, j'_n)_n$ is an infinite path means that $k_{n+1} = k'_n$ for all n.

Define $j_1 := j'_1$ and $j_n := j_{n-1} + j'_n - 1$. By our construction of (V, E), we must have $\mathcal{Y}(n+1, k_{n+1}, j_{n+1}) \subseteq \mathcal{Y}(n, k_n, j_n)$. Hence there is a uniquely determined point $x \in X$ with $\bigcap_n \mathcal{Y}(n, k_n, j_n) = \{x\}$, and we define

$$\varphi(x') := x.$$

Now check that φ is a homeomorphism $X' \xrightarrow{\sim} X$ which produces a pointed topological conjugacy between (X, σ, y) and (X', σ', y'). $\square$

13.2. **Classification of Cantor minimal systems up to orbit equivalence.** In the previous lecture, we have seen how to classify crossed product C^*-algebras attached to Cantor minimal systems up to isomorphism. Let us now explain how to classify Cantor minimal systems themselves. Several ideas which originated from the C^*-algebra context will play a crucial role. Bratteli-Vershik models as developed in § 13.1 well be a key ingredient, too. Note that, however, that we will not classify Cantor minimal systems up to conjugacy (that would be too strong), but rather up to a weaker notion called orbit equivalence. So, as a first step, we need to introduce that notion. This requires a discussion of equivalence relations on Cantor sets.

Definition 13.15. Let (X, σ) be a Cantor minimal system. Define the equivalence relation

$$R_\sigma := \{(x, \sigma^n(x)) : \, x \in X, \, n \in \mathbb{Z}\} \subseteq X \times X.$$

We introduce a topology on R_σ by requiring that the bijection

$$\mathbb{Z} \times X \xrightarrow{\sim} R_\sigma, \, (n, x) \mapsto (x, \sigma^n(x))$$

is a homeomorphism, where we take the discrete topology on $\mathbb{Z}$, the given topology on X and the product topology on $\mathbb{Z} \times X$.

Later on, we will also need to consider equivalence relations constructed from Bratteli diagrams.

Definition 13.16. Let (V, E) be a Bratteli diagram. As we did before, equip $X := E_{0,\infty}$ with the projective limit topology, i.e.,

$$X \cong \varprojlim_n E_{0,n}.$$

Given $N \in \mathbb{N}$, define the equivalence relation

$$R_N := \{(e, f) \in X \times X : \, e_n = f_n \; \forall \, n > N\} \subseteq X \times X.$$

Equip R_N with the subspace topology coming from $X \times X$. Now construct the equivalence relation

$$R_E := \bigcup_{N=1}^{\infty} R_N \subseteq X \times X$$

and equip R_E with the inductive limit topology, i.e., $U \subseteq R_E$ is open if and only if $U \cap R_N$ is an open subset of R_N for all $N \in \mathbb{N}$.

Here are two ways to compare equivalence relations.

Definition 13.17. Let X_1 and X_2 be topological spaces and R_1, R_2 equivalence relations on X_1, X_2, each of them equipped with a topology.

We say that (X_1, R_1) and (X_2, R_2) are orbit equivalent (written $(X_1, R_1) \sim_{\mathrm{OE}} (X_2, R_2)$) if there exists a homeomorphism $\varphi: X_1 \xrightarrow{\sim} X_2$ with $(\varphi \times \varphi)(R_1) = R_2$. Such φ is called an orbit equivalence.

We say that (X_1, R_1) and (X_2, R_2) are isomorphic (written $(X_1, R_1) \cong (X_2, R_2)$) if there exists an orbit equivalence $\varphi: X_1 \xrightarrow{\sim} X_2$ such that $\varphi \times \varphi$ restricts to a homeomorphism $\varphi \times \varphi: R_1 \xrightarrow{\sim} R_2$.

We can now introduce orbit equivalence for Cantor minimal systems.

Definition 13.18. Two Cantor minimal systems (X_1, σ_1) and (X_2, σ_2) are orbit equivalent (written $(X_1, \sigma_1) \sim_{\mathrm{OE}} (X_2, \sigma_2)$) if $(X_1, R_{\sigma_1}) \sim_{\mathrm{OE}} (X_2, R_{\sigma_2})$.

If two Cantor minimal systems are orbit equivalent, then they have the same orbit structure. As orbit structures contain a lot of important dynamical information, orbit equivalence is a very natural way to compare dynamical systems. We could also define isomorphism for Cantor minimal systems, but that turns out to be very close to conjugacy, and hence too strong for our purposes.

Our goal now is to present a complete classification result for Cantor minimal systems up to orbit equivalence. First we need to introduce the classifying invariant. As for C*-classification, the invariant will be a scaled ordered abelian group. Let (X, σ) be a Cantor minimal system. As before, let $C(X, \mathbb{Z})$ denote the set of continuous functions $X \to \mathbb{Z}$, where $\mathbb{Z}$ is equipped with the discrete topology. Denote by $M(X, \sigma)$ the space of σ-invariant Borel probability measures on X. Define

$$\begin{aligned} B_m(X, \sigma) &:= \{f \in C(X, \mathbb{Z}) : \int_X f d\mu = 0 \ \forall \ \mu \in M(X, \sigma)\}, \\ D_m(X, \sigma) &:= C(X, \mathbb{Z})/B_m(X, \sigma); \\ D_m(X, \sigma)^+ &:= \{[f] \in D_m(X, \sigma) : \ f \in C(X, \mathbb{Z}), \ f \geq 0\}. \end{aligned}$$

It turns out that $(D_m(X, \sigma), D_m(X, \sigma)^+, [1_X])$ is a scaled ordered abelian group, in the sense of Definition 12.10.

Remark 13.19. We can construct $(D_m(X, \sigma), D_m(X, \sigma)^+, [1_X])$ from the invariant $(K^0(X, \sigma), K^0(X, \sigma)^+, [1_X])$ introduced in Definition 12.12 as follows: Given an ordered abelian group (G, G^+), we construct the subgroup of infinitesimal elements by

$$\mathrm{Inf}(G, G^+) := \{g \in G : \ h \geq kg \text{ for all order units } h \text{ and all } k \in \mathbb{Z}\}.$$

Now it turns out that the canonical projection $C(X, \mathbb{Z}) \twoheadrightarrow D_m(X, \sigma)$ induces an isomorphism

$$K^0(X, \sigma)/\mathrm{Inf}(K^0(X, \sigma), K^0(X, \sigma)^+) \xrightarrow{\sim} D_m(X, \sigma).$$

$(D_m(X, \sigma), D_m(X, \sigma)^+, [1_X])$ is the complete invariant we were looking for.

Theorem 13.20. *Given two Cantor minimal systems (X_1, σ_1) and (X_2, σ_2), we have $(X_1, \sigma_1) \sim_{\mathrm{OE}} (X_2, \sigma_2)$ if and only if*

$$(D_m(X_1, \sigma_1), D_m(X_1, \sigma_1)^+, [1_{X_1}]) \cong (D_m(X_2, \sigma_2), D_m(X_2, \sigma_2)^+, [1_{X_2}])$$

as scaled ordered abelian groups.

The proof of this theorem is beyond the scope of these lecture. But at least, let us try to outline some of the key steps.

First of all, for one direction, we have to prove that $(D_m(X, \sigma), D_m(X, \sigma)^+, [1_X])$ is an orbit equivalence invariant of (X, σ). Essentially, this boils down to showing that orbit equivalences preserve invariant measures (see [43, Theorem 2.8] for details).

Now let us turn to the more substantial direction. We have to show that if $(D_m(X_1, \sigma_1), D_m(X_1, \sigma_1)^+, [1_{X_1}]) \cong (D_m(X_2, \sigma_2), D_m(X_2, \sigma_2)^+, [1_{X_2}])$ as scaled ordered abelian groups, then we must have $(X_1, \sigma_1) \sim_{\mathrm{OE}} (X_2, \sigma_2)$. The proof proceeds in three steps.

I.) First, given a Cantor minimal system (X, σ), let $y \in X$ be some point, and construct the Bratteli diagram (V, E) as in § 13.1. Then consider the equivalence relation (X, R_E) as in Definition 13.16. The first step is to show that $(X, R_\sigma) \sim_{\mathrm{OE}} (X, R_E)$. The main ingredient is the absorption theorem (see for instance [43, § 2.8]), the most technical part of the proof of Theorem 13.20. It is also used for the second step.

To explain the second step, we need to introduce an invariant for Bratteli diagrams. First of all, positive homomorphisms between ordered abelian groups are group homomorphisms preserving the positive cones. We need the notion of inductive limits for ordered abelian groups.

Definition 13.21. Let $(G_1, G_1^+) \xrightarrow{h_1} (G_2, G_2^+) \xrightarrow{h_2} \dots$ be an inductive system of ordered abelian groups, where h_n are positive homomorphisms. The inductive limit $\varinjlim_n \{(G_n, G_n^+), h_n\}$ of this system is an ordered abelian group (G, G^+) together with positive homomorphisms $\eta_n : (G_n, G_n^+) \to (G, G^+)$ such that $\eta_{n+1} \circ h_n = \eta_n$, satisfying the following universal property: Given any ordered abelian group (H, H^+) and positive homomorphisms $\omega_n : (G_n, G_n^+) \to (H, H^+)$, $n \in \mathbb{N}$ with $\omega_{n+1} \circ h_n = \omega_n$, there is a unique positive homomorphism $\omega : (G, G^+) \to (H, H^+)$ such that the following diagram commutes:

$$\begin{tikzcd}
 & & & & (G, G^+) \arrow[dd, "\omega"] \\
\dots \arrow[r] & (G_n, G_n^+) \arrow[r, "h_n"] \arrow[urrr, "\eta_n"] \arrow[drrr, "\omega_n"'] & (G_{n+1}, G_{n+1}^+) \arrow[r, "\cdots"] \arrow[urr, "\eta_{n+1}"'] \arrow[drr, "\omega_{n+1}"] & {} & \\
 & & & & (H, H^+)
\end{tikzcd}$$

The proof that $\varinjlim_n\{(G_n, G_n^+), h_n\}$ exists and of uniqueness is analogous to – actually a bit simpler than – the one in the C^*-algebra setting (see Proposition 8.2). We are going to need the following facts:

$$G = \bigcup_{n=1}^{\infty} \eta_n(G_n) \quad \text{and} \quad G^+ = \bigcup_{n=1}^{\infty} \eta_n(G_n^+). \tag{13.2}$$

If we set $h_{n,N} := h_{N-1} \circ \ldots \circ h_n : (G_n, G_n^+) \to (G_N, G_N^+)$, then

$$\ker(\eta_n) = \bigcup_{N=n}^{\infty} \ker(h_{n,N}). \tag{13.3}$$

We also need the following way to go from edges and vertices to positive group homomorphisms (and back): Let V and V' be sets. Let $\mathbb{Z}V$ be the free abelian group with basis V. Typical elements of $\mathbb{Z}V$ are finite sums of the form $\sum_v z_v v$ ($z_v = 0$ for all but finitely many v). Let $\mathbb{Z}V^+ := \{\sum_v z_v v : z_v \geq 0 \; \forall \; v\}$. Now sets of edges E from V to V' (given by source and range maps $s : E \to V$ and $r : E \to V'$) are in one-to-one correspondence with positive homomorphisms $h : (\mathbb{Z}V, \mathbb{Z}V^+) \to (\mathbb{Z}V', \mathbb{Z}V'^+)$ via $h(v) := \sum_{e \in E,\, s(e)=v} r(e)$. Moreover, under this correspondence, concatenation of edges corresponds to composition of homomorphisms: Let E be a set of edges between V and V', with source and range maps s, r, and let E' be a set of edges between V' and V'', with source and range maps s', r'. Suppose that $h : (\mathbb{Z}V, \mathbb{Z}V^+) \to (\mathbb{Z}V', \mathbb{Z}V'^+)$ and $h' : (\mathbb{Z}V', \mathbb{Z}V'^+) \to (\mathbb{Z}V'', \mathbb{Z}V''^+)$ are the positive homomorphisms corresponding to E and E'. Concatenation of E and E' yields $E'' := \{(e, e') \in E \times E' : r(e) = s'(e')\}$, with source and range maps $s''(e, e') := s(e)$ and $r''(e, e') := r(e')$. If now $h'' : (\mathbb{Z}V, \mathbb{Z}V^+) \to (\mathbb{Z}V'', \mathbb{Z}V''^+)$ corresponds to E'', then we have $h'' = h' \circ h$.

With these preparations, we can now define the following invariant for Bratteli diagrams: Given a Bratteli diagram (V, E), let $h_{n-1} : (\mathbb{Z}V_{n-1}, \mathbb{Z}V_{n-1}^+) \to (\mathbb{Z}V_n, \mathbb{Z}{V_n}^+)$ correspond to the set of edges E_n from V_{n-1} to V_n. Now define

$$(K^0(V, E), K^0(V, E)^+) := \varinjlim_n \{(\mathbb{Z}V_n, \mathbb{Z}{V_n}^+), h_n\}.$$

II.) The second step in the proof of Theorem 13.20 is to produce a Bratteli diagram $(\tilde{V}, \tilde{E})$, giving rise to the equivalence relation $(\tilde{X}, R_{\tilde{E}})$ as in Definition 13.16, such that

$$(X, R_E) \sim_{\mathrm{OE}} (\tilde{X}, R_{\tilde{E}}),$$

where (X, R_E) is the equivalence relation of the Bratteli diagram (V, E) obtained from (X, σ, y) in the first step, and

$$(K^0(\tilde{V}, \tilde{E}), K^0(\tilde{V}, \tilde{E})^+, \tilde{\eta}_0(\tilde{v}_0)) \cong (D_m(X, \sigma), D_m(X, \sigma)^+, [1_X])$$

as scaled ordered abelian groups.

Here $\tilde{\eta}_n$ are the positive homomorphisms $(\mathbb{Z}\tilde{V}_n, \mathbb{Z}{\tilde{V}_n}^+) \to (K^0(\tilde{V}, \tilde{E}), K^0(\tilde{V}, \tilde{E})^+)$ which are part of the inductive limit structure.

III.) The third step of the proof is to apply the following

Theorem 13.22. *Let (V, E) and (V', E') be two Bratteli diagrams. Then the following are equivalent:*

(i) $(V, E) \sim (V', E')$,
(ii) $(X, R_E) \cong (X', R_{E'})$,
(iii) $(K^0(V, E), K^0(V, E)^+, \eta_0(v_0)) \cong (K^0(V', E'), K^0(V', E')^+, \eta'_0(v'_0))$ *as scaled ordered abelian groups.*

It remains to put everything together in order to prove the remaining direction, i.e., if $(D_m(X_1, \sigma_1), D_m(X_1, \sigma_1)^+, [1_{X_1}]) \cong (D_m(X_2, \sigma_2), D_m(X_2, \sigma_2)^+, [1_{X_2}])$ as scaled ordered abelian groups, then we must have $(X_1, \sigma_1) \sim_{\mathrm{OE}} (X_2, \sigma_2)$. Indeed, first apply I.) to (X_1, σ_1) and (X_2, σ_2) to obtain Bratteli diagrams (V_1, E_1) and (V_2, E_2) such that $(X_1, R_{\sigma_1}) \sim_{\mathrm{OE}} (X_1, R_{E_1})$ and $(X_2, R_{\sigma_2}) \sim_{\mathrm{OE}} (X_2, R_{E_2})$. Now apply II.) twice to produce Bratteli diagrams $(\tilde{V}_1, \tilde{E}_1)$ and $(\tilde{V}_2, \tilde{E}_2)$ such that $(X_1, R_{E_1}) \sim_{\mathrm{OE}} (\tilde{X}_1, R_{\tilde{E}_1})$ and $(X_2, R_{E_2}) \sim_{\mathrm{OE}} (\tilde{X}_2, R_{\tilde{E}_2})$, and

$$(K^0(\tilde{V}_1, \tilde{E}_1), K^0(\tilde{V}_1, \tilde{E}_1)^+, (\tilde{\eta}_{1;0}(\tilde{v}_{1;0})) \cong (D_m(X_1, \sigma_1), D_m(X_1, \sigma_1)^+, [1_{X_1}])$$
$$(K^0(\tilde{V}_2, \tilde{E}_2), K^0(\tilde{V}_2, \tilde{E}_2)^+, (\tilde{\eta}_{2;0}(\tilde{v}_{2;0})) \cong (D_m(X_2, \sigma_2), D_m(X_2, \sigma_2)^+, [1_{X_2}])$$

as scaled ordered abelian groups. Finally, since we assume

$$(D_m(X_1, \sigma_1), D_m(X_1, \sigma_1)^+, [1_{X_1}]) \cong (D_m(X_2, \sigma_2), D_m(X_2, \sigma_2)^+, [1_{X_2}]),$$

we can apply Theorem 13.22 to deduce $(\tilde{X}_1, R_{\tilde{E}_1}) \cong (\tilde{X}_2, R_{\tilde{E}_2})$.

So all in all, we arrive at

$$(X_1, R_{\sigma_1}) \overset{\text{I.)}}{\sim}_{\mathrm{OE}} (X_1, R_{E_1}) \overset{\text{II.)}}{\sim}_{\mathrm{OE}} (\tilde{X}_1, R_{\tilde{E}_1}) \overset{\text{III.)}}{\cong} (\tilde{X}_2, R_{\tilde{E}_2}) \overset{\text{II.)}}{\sim}_{\mathrm{OE}} (X_2, R_{E_2}) \overset{\text{I.)}}{\sim}_{\mathrm{OE}} (X_2, R_{\sigma_2}),$$

as desired.

13.3. **Exercises.**

Exercise 13.1. Apply Theorem 13.20 to odometers and compare the result to Corollary 11.9.

In the proof of Theorem 13.22, the implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are rather straightforward, and the most substantial implication is (iii) $\Rightarrow$ (i). In the following exercise, we develop a proof for (iii) $\Rightarrow$ (i).

Exercise 13.2. Let (V, E) and (V', E') be two Bratteli diagrams. Assume that

$$(K^0(V, E), K^0(V, E)^+, \eta_0(v_0)) \cong (K^0(V', E'), K^0(V', E')^+, \eta'_0(v'_0))$$

via an isomorphism $\alpha : K^0(V, E) \xrightarrow{\sim} K^0(V', E')$ with $\alpha(K^0(V, E)^+) = K^0(V', E')^+$ and $\alpha(\eta_0(v_0)) = \eta'_0(v'_0)$.

a) Show that in order to prove $(V, E) \sim (V', E')$, it suffices to find a strictly increasing sequence (n_m) with $n_0 = 0$, $n_1 = 1$ and construct a Bratteli diagram (F, W) such that

$$W_m = \begin{cases} V'_{n_m} & \text{if } m \text{ is odd} \\ V_{n_m} & \text{if } m \text{ is even,} \end{cases}$$

$F_1 = E'_1$ (identifying $W_0 = V_0$ with V'_0), and such that we have, for all even m, bijections $F_{m-2,m} \xrightarrow{\sim} E_{n_{m-2},n_m}$ fitting into commutative diagrams

$$\begin{tikzcd} F_{m-2,m} \arrow[r, "\cong"] \arrow[d, "s"'] & E_{n_{m-2},n_m} \arrow[d, "s"] \\ W_{m-2} \arrow[r, "="] & V_{n_{m-2}} \end{tikzcd}$$

and

$$\begin{tikzcd} F_{m-2,m} \arrow[r, "\cong"] \arrow[d, "r"'] & E_{n_{m-2},n_m} \arrow[d, "r"] \\ W_{m} \arrow[r, "="] & V_{n_{m}} \end{tikzcd}$$

and, for all odd n, bijections $F_{m-2,m} \xrightarrow{\sim} E'_{n_{m-2},n_m}$ fitting into commutative diagrams

$$\begin{tikzcd} F_{m-2,m} \arrow[r, "\cong"] \arrow[d, "s"'] & E'_{n_{m-2},n_m} \arrow[d, "s"] \\ W_{m-2} \arrow[r, "="] & V'_{n_{m-2}} \end{tikzcd}$$

and

$$\begin{tikzcd} F_{m-2,m} \arrow[r, "\cong"] \arrow[d, "r"'] & E'_{n_{m-2},n_m} \arrow[d, "r"] \\ W_{m} \arrow[r, "="] & V'_{n_{m}} \end{tikzcd}$$

b) Using the one-to-one correspondence between sets of edges and positive homomorphisms, show that instead of the statement in a), we can equally well construct, for all odd m, positive homomorphisms

$$i_m : (\mathbb{Z}V_{n_{m-1}}, \mathbb{Z}V^+_{n_{m-1}}) \to (\mathbb{Z}V'_{n_m}, \mathbb{Z}{V'_{n_m}}^+)$$

such that $i_1 = h'_1$ and $i_m \circ i_{m-1} = h'_{n_{m-2},n_m}$, and, for all even m, positive homomorphisms

$$i_m : (\mathbb{Z}V'_{n_{m-1}}, \mathbb{Z}V'_{n_{m-1}}{}^+) \to (\mathbb{Z}V_{n_m}, \mathbb{Z}V_{n_m}{}^+)$$

such that $i_m \circ i_{m-1} = h_{n_{m-2},n_m}$.

Here $h'_\bullet$ and $h_\bullet$ denote the positive homomorphisms corresponding to $E'_\bullet$ and $E_\bullet$, respectively.

c) Using (13.2) and (13.3), proceed inductively on m to construct positive homomorphisms i_m as in b), with the additional property that $\eta'_{n_m} \circ i_m = \alpha \circ \eta_{n_{m-1}}$ for odd m and $\eta_{n_m} \circ i_m = \alpha^{-1} \circ \eta'_{n_{m-1}}$ for even m.

Here $\eta_\bullet : (\mathbb{Z}V_\bullet, \mathbb{Z}V_\bullet{}^+) \to (K^0(V,E), K^0(V,E)^+)$ are the positive homomorphisms which are part of the inductive limit structure, and similarly for $\eta'_\bullet$.

13.4. **Comments.**

Remark 13.23. In this lecture, we mainly followed the exposition in [24] and [44].

Remark 13.24. The idea of using orders on Bratteli diagrams to construct dynamical systems is due to Vershik, whose work was not in the topological, but in the measurable context. Bratteli-Vershik models as we introduced them in this lecture first appeared in [24].

Remark 13.25. Theorem 13.22 is due to Bratteli, Elliott and Krieger. Actually, because of the work of Bratteli and Elliott, we can add another item to the list of equivalent statements in Theorem 13.22:

(iv) The AF algebras with associated Bratteli diagrams (V,E) and (V',E') (in the sense of § 8.3) are isomorphic as C^*-algebras.

Remark 13.26. Theorem 13.20 is due to Giordano, Putnam and Skau (see [22]).

Remark 13.27. As a consequence of Theorem 13.20, we obtain that two Cantor minimal systems (X_1,σ_1) and (X_2,σ_2) are orbit equivalent if and only if there exists a homeomorphism $\varphi : X_1 \xrightarrow{\sim} X_2$ which induces an identification

$$M(X_1,\sigma_1) \xrightarrow{\sim} M(X_2,\sigma_2),\ \mu \mapsto \mu \circ \varphi^{-1}.$$

In the special case of two Cantor minimal systems (X_1,σ_1) and (X_2,σ_2) which are uniquely ergodic, i.e., each of them admits a unique invariant Borel probability measure (say μ_1 and μ_2), we have $(X_1,\sigma_1) \sim_{\mathrm{OE}} (X_2,\sigma_2)$ if and only if

$$\{\mu_1(C_1) : C_1 \subseteq X_1 \text{ clopen}\} = \{\mu_2(C_2) : C_2 \subseteq X_2 \text{ clopen}\}.$$

The reader may consult [22] for details.

Remark 13.28. Another consequence of Theorem 13.20 is that every Cantor minimal system is orbit equivalent to an odometer, as introduced in § 10.4, or to a Denjoy system, which are closely related to irrational rotations on the circle (see [44, § 0.1 in Appendix A] or [40] for more about Denjoy systems).

Again, the reader may consult [22] for details.

Remark 13.29. Giordano, Matui, Putnam and Skau generalized Theorem 13.20 to Cantor minimal $\mathbb{Z}^d$-systems (see [20, 21]). It is currently unknown whether it is possible to cover even more general acting groups.

Remark 13.30. Combining Theorem 12.13 with Theorem 13.20 and using Remark 13.19, we see that, given two Cantor minimal systems (X_1, σ_1) and (X_2, σ_2), isomorphism of C^*-algebras $C(X_1) \rtimes_{\sigma_1,\mathrm{r}} \mathbb{Z} \cong C(X_2) \rtimes_{\sigma_2,\mathrm{r}} \mathbb{Z}$ actually implies $(X_1, \sigma_1) \sim_{\mathrm{OE}} (X_2, \sigma_2)$. The proof, however, is not direct, as it is obtained by comparing classifying invariants. It is an intriguing question whether a direct proof exists, though it is likely that analogous statements are not true for more general acting groups.

Acknowledgements. M. Weber acknowledges a lecture given by J. Cuntz in Münster 2004/2005; to some extent, the first five lectures were based on the notes taken by M. Weber participating in this course.

M. Weber thanks everyone for sending corrections and posting alternative proofs, references and thoughts regarding the ISem24 lecture notes, in particular (but not only), the Darmstadt team, the Wuppertal team, Fabian Gabel, Manuel Schiffer, Sahiba Arora, Katharina Klioba, Sascha Trostorff, Mitja Roeder, Moritz Proell, Markus Haase, Alessio Ranallo, Hendrik Voigt – and many others (sorry for not listing everyone).

Moreover, M. Weber thanks Christian Budde and Balint Farkas for proofreading the notes for Lecture 1–8 in advance; and a very special thanks goes to Jürgen Voigt for his very careful proofreading of those lectures and his numerous valuable suggestions for improvements.

Finally, M. Weber thanks Michael Hartz for various discussions on some details of the proofs given in the lecture notes.

X. Li has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 817597). Chr. Voigt was supported by EPSRC grant EP/T03064X/1 "Quantum groups in action". M. Weber was supported by *SFB-TRR 195*.

References

[1] https://ncatlab.org/nlab/show/gelfand+duality.

[2] J. Bellissard, A. van Elst, and H. Schulz-Baldes. The noncommutative geometry of the quantum Hall effect. *J. Math. Phys.*, 35(10):5373–5451, 1994. Topology and physics.

[3] B. Blackadar. *Operator algebras*, volume 122 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2006. Theory of C^*-algebras and von Neumann algebras, Operator Algebras and Non-commutative Geometry, III.

[4] Bruce Blackadar. Symmetries of the CAR Algebra. *Annals of Mathematics*, 131(3):589–623, 1990.

[5] Ola Bratteli. Inductive limits of finite dimensional C^*-algebras. *Trans. Amer. Math. Soc.*, 171:195–234, 1972.

[6] Emmanuel Breuillard, Mehrdad Kalantar, Matthew Kennedy, and Narutaka Ozawa. C^*-simplicity and the unique trace property for discrete groups. *Publ. Math. Inst. Hautes Études Sci.*, 126:35–71, 2017.

[7] Lawrence G. Brown and George A. Elliott. Extensions of AF-algebras are determined by K_0. *C. R. Math. Rep. Acad. Sci. Canada*, 4(1):15–19, 1982.

[8] Nathanial P. Brown and Narutaka Ozawa. *C^*-algebras and finite-dimensional approximations*, volume 88 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.

[9] John W. Bunce and James A. Deddens. A family of simple C^*-algebras related to weighted shift operators. *J. Functional Analysis*, 19:13–24, 1975.

[10] Alain Connes. *Noncommutative geometry*. Academic Press, Inc., San Diego, CA, 1994.

[11] J. B. Conway. *A course in functional analysis*, volume 96 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1990.

[12] Joachim Cuntz. Simple C^*-algebras generated by isometries. *Comm. Math. Phys.*, 57(2):173–185, 1977.

[13] Kenneth R. Davidson. *C^*-algebras by example*, volume 6 of *Fields Institute Monographs*. American Mathematical Society, Providence, RI, 1996.

[14] Kenneth R. Davidson. *C^*-algebras by example*, volume 6 of *Fields Institute Monographs*. American Mathematical Society, Providence, RI, 1996.

[15] Pierre de la Harpe. On simplicity of reduced C^*-algebras of groups. *Bull. Lond. Math. Soc.*, 39(1):1–26, 2007.

[16] Jacques Dixmier. *Les C^*-algèbres et leurs représentations*. Les Grands Classiques Gauthier-Villars. [Gauthier-Villars Great Classics]. Éditions Jacques Gabay, Paris, 1996. Reprint of the second (1969) edition.

[17] George A. Elliott. Automorphisms determined by multipliers on ideals of a C^*-algebra. *J. Functional Analysis*, 23(1):1–10, 1976.

[18] George A. Elliott. On the classification of C^*-algebras of real rank zero. *J. Reine Angew. Math.*, 443:179–219, 1993.

[19] I. Gel'fand and M. Neumark. On the imbedding of normed rings into the ring of operators in Hilbert space. In *C^*-algebras: 1943–1993 (San Antonio, TX, 1993)*, volume 167 of *Contemp. Math.*, pages 2–19. Amer. Math. Soc., Providence, RI, 1994. Corrected reprint of the 1943 original.

[20] Thierry Giordano, Hiroki Matui, Ian F. Putnam, and Christian F. Skau. Orbit equivalence for Cantor minimal $\mathbb{Z}^2$-systems. *J. Amer. Math. Soc.*, 21(3):863–892, 2008.

[21] Thierry Giordano, Hiroki Matui, Ian F. Putnam, and Christian F. Skau. Orbit equivalence for Cantor minimal $\mathbb{Z}^d$-systems. *Invent. Math.*, 179(1):119–158, 2010.

[22] Thierry Giordano, Ian F. Putnam, and Christian F. Skau. Topological orbit equivalence and C^*-crossed products. *J. Reine Angew. Math.*, 469:51–111, 1995.

[23] José M. Gracia-Bondía, Joseph C. Várilly, and Héctor Figueroa. *Elements of noncommutative geometry*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Boston, Inc., Boston, MA, 2001.

[24] Richard H. Herman, Ian F. Putnam, and Christian F. Skau. Ordered Bratteli diagrams, dimension groups and topological dynamics. *Internat. J. Math.*, 3(6):827–864, 1992.

[25] R. V. Kadison. Notes on the Gel'fand-Neumark theorem. In *C^*-algebras: 1943–1993 (San Antonio, TX, 1993)*, volume 167 of *Contemp. Math.*, pages 20–53. Amer. Math. Soc., Providence, RI, 1994.

[26] Richard V. Kadison and John R. Ringrose. *Fundamentals of the theory of operator algebras. Vol. III*. Birkhäuser Boston, Inc., Boston, MA, 1991. Special topics, Elementary theory—an exercise approach.

[27] Richard V. Kadison and John R. Ringrose. *Fundamentals of the theory of operator algebras. Vol. IV*. Birkhäuser Boston, Inc., Boston, MA, 1992. Special topics, Advanced theory—an exercise approach.

[28] Richard V. Kadison and John R. Ringrose. *Fundamentals of the theory of operator algebras. Vol. I*, volume 15 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1997. Elementary theory, Reprint of the 1983 original.

[29] Richard V. Kadison and John R. Ringrose. *Fundamentals of the theory of operator algebras. Vol. II*, volume 16 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1997. Advanced theory, Corrected reprint of the 1986 original.

[30] Mehrdad Kalantar and Matthew Kennedy. Boundaries of reduced C^*-algebras of discrete groups. *J. Reine Angew. Math.*, 727:247–267, 2017.

[31] David Kerr. Dimension, comparison, and almost finiteness. *J. Eur. Math. Soc. (JEMS)*, 22(11):3697–3745, 2020.

[32] David Kerr and Gábor Szabó. Almost finiteness and the small boundary property. *Comm. Math. Phys.*, 374(1):1–31, 2020.

[33] A. Kumjian. An involutive automorphism of the Bunce-Deddens algebra. *C. R. Math. Rep. Acad. Sci. Canada*, 10(5):217–218, 1988.

[34] Gerard J. Murphy. *C^*-algebras and operator theory*. Academic Press, Inc., Boston, MA, 1990.

[35] F. J. Murray and J. Von Neumann. On rings of operators. *Ann. of Math. (2)*, 37(1):116–229, 1936.

[36] G. K. Pedersen. *Analysis now*, volume 118 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1989.

[37] Gert K. Pedersen. *C^*-algebras and their automorphism groups*. Pure and Applied Mathematics (Amsterdam). Academic Press, London, 2018. Second edition of [MR0548006], Edited and with a preface by Søren Eilers and Dorte Olesen.

[38] M. Pimsner and D. Voiculescu. Imbedding the irrational rotation C^*-algebra into an AF-algebra. *J. Operator Theory*, 4(2):201–210, 1980.

[39] Robert T. Powers. Simplicity of the C^*-algebra associated with the free group on two generators. *Duke Math. J.*, 42:151–156, 1975.

[40] Ian Putnam, Klaus Schmidt, and Christian Skau. C^*-algebras associated with Denjoy homeomorphisms of the circle. *J. Operator Theory*, 16(1):99–126, 1986.

[41] Ian F. Putnam. The C^*-algebras associated with minimal homeomorphisms of the Cantor set. *Pacific J. Math.*, 136(2):329–353, 1989.

[42] Ian F. Putnam. On the topological stable rank of certain transformation group C^*-algebras. *Ergodic Theory Dynam. Systems*, 10(1):197–207, 1990.

[43] Ian F. Putnam. Orbit equivalence of Cantor minimal systems: a survey and a new proof. *Expo. Math.*, 28(2):101–131, 2010.

[44] Ian F. Putnam. *Cantor minimal systems*, volume 70 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2018.

[45] I. Raeburn. Dynamical systems and operator algebras. In *National Symposium on Functional Analysis, Optimization and Applications*, pages 109–119, Canberra AUS, 1999. Centre for Mathematics and its Applications, Mathematical Sciences Institute, The Australian National University.

[46] Iain Raeburn. *Graph algebras*, volume 103 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2005.

[47] Marc A. Rieffel. Deformation quantization for actions of $\mathbf{R}^d$. *Mem. Amer. Math. Soc.*, 106(506):x+93, 1993.

[48] M. Rørdam. Classification of nuclear, simple C^*-algebras. In *Classification of nuclear C^*-algebras. Entropy in operator algebras*, volume 126 of *Encyclopaedia Math. Sci.*, pages 1–145. Springer, Berlin, 2002.

[49] Walter Rudin. *Functional analysis*. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, second edition, 1991.

[50] Shôichirô Sakai. *C^*-algebras and W^*-algebras*. Classics in Mathematics. Springer-Verlag, Berlin, 1998. Reprint of the 1971 edition.

[51] M. Takesaki. *Theory of operator algebras. I*, volume 124 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2002. Reprint of the first (1979) edition, Operator Algebras and Non-commutative Geometry, 5.

[52] M. Takesaki. *Theory of operator algebras. II*, volume 125 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2003. Operator Algebras and Non-commutative Geometry, 6.

[53] M. Takesaki. *Theory of operator algebras. III*, volume 127 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2003. Operator Algebras and Non-commutative Geometry, 8.

[54] Aaron Tikuisis, Stuart White, and Wilhelm Winter. Quasidiagonality of nuclear C^*-algebras. *Ann. of Math. (2)*, 185(1):229–284, 2017.

[55] M. Weber. Quantum symmetry. *Snapshots of modern mathematics from Oberwolfach*, 5:16p., 2020.

[56] Moritz Weber. On C^*-algebras generated by isometries with twisted commutation relations. *J. Funct. Anal.*, 264(8):1975–2004, 2013.

[57] N. E. Wegge-Olsen. *K-theory and C^*-algebras*. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1993. A friendly approach.

[58] Wilhelm Winter. Structure of nuclear C*-algebras: from quasidiagonality to classification and back again. In *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. III. Invited lectures*, pages 1801–1823. World Sci. Publ., Hackensack, NJ, 2018.