Classical and noncommutative ergodic theorems

Sahiba Arora (TU Dresden)
Sam Johnson (NIT Karnataka)
Yassine Kharou (Ibn Zohr University, Agadir)
Marianna Porfido (University of Salerno)
Bjarne Wittlieb (CAU Kiel)

Coordinators: Markus Haase and Sascha Trostorff (CAU Kiel)

Theorem

Let $(M, \mathcal{J})$ be a dynamical system. If there exists a faithful family Ω of normal states on M satisfying

$$\omega((Tx)^*(Tx)) \leq \omega(x^*x) \quad \text{for all} \quad \omega \in \Omega,\ T \in \mathcal{J},\ x \in M.$$

Then $\mathcal{J}$ is weak* mean ergodic.

Theorem

Let $(M, \mathcal{J})$ be a dynamical system. If there exists a faithful family Ω of normal states on M satisfying

$$\omega((Tx)^*(Tx)) \leq \omega(x^*x) \quad \text{for all} \quad \omega \in \Omega,\ T \in \mathcal{J},\ x \in M.$$

Then $\mathcal{J}$ is weak* mean ergodic.

B. Kümmerer, R. Nagel, *Mean ergodic semigroups on W^*-algebras*, Acta Sci. Math., 41 (1979), 151-159.

Von Neumann's Mean Ergodic Theorem

Let T be a linear operator on a vector space E and let

$$A_n := \frac{1}{n} \sum_{j=0}^{n-1} T^j \qquad (n \in \mathbb{N} \setminus \{0\}),$$

Let T be a linear operator on a vector space E and let

$$A_n := \frac{1}{n} \sum_{j=0}^{n-1} T^j \qquad (n \in \mathbb{N} \setminus \{0\}),$$

$$\operatorname{fix}(T) := \{f \in E : Tf = f\} = \ker(I - T).$$

$$A_n := \frac{1}{n}\sum_{j=0}^{n-1} T^j \qquad (n \in \mathbb{N} \setminus \{0\}),$$

$$\operatorname{fix}(T) := \{f \in E : Tf = f\} = \ker(I - T).$$

$$A_n := \frac{1}{n}\sum_{j=0}^{n-1} T^j \qquad (n \in \mathbb{N} \setminus \{0\}),$$

$$\operatorname{fix}(T) := \{f \in E : Tf = f\} = \ker(I - T).$$

Lemma

Let E be a Banach space and let $T \in \mathcal{B}(E)$.

$$A_n := \frac{1}{n}\sum_{j=0}^{n-1} T^j \qquad (n \in \mathbb{N} \setminus \{0\}),$$

$$\operatorname{fix}(T) := \{f \in E : Tf = f\} = \ker(I - T).$$

Lemma

Let E be a Banach space and let $T \in \mathcal{B}(E)$. Then, the following assertions hold:

1. *If $f \in \operatorname{fix}(T)$, then $A_n f = f$ for all $n \in \mathbb{N}$, and hence $A_n f \to f$;*

$$A_n := \frac{1}{n}\sum_{j=0}^{n-1} T^j \qquad (n \in \mathbb{N}\setminus\{0\}),$$

$$\mathrm{fix}\,(T) := \{f \in E : Tf = f\} = \ker(I - T).$$

Lemma

Let E be a Banach space and let $T \in \mathcal{B}(E)$. Then, the following assertions hold:

1. *If $f \in \mathrm{fix}\,(T)$, then $A_n f = f$ for all $n \in \mathbb{N}$, and hence $A_n f \to f$;*
2. *If $\frac{1}{n} T^n f \to 0$ for all $f \in E$, then $A_n f \to 0$ for all $f \in \mathrm{ran}(I - T)$.*

$$A_n := \frac{1}{n}\sum_{j=0}^{n-1} T^j \qquad (n \in \mathbb{N} \setminus \{0\}),$$

$$A_n := \frac{1}{n} \sum_{j=0}^{n-1} T^j \qquad (n \in \mathbb{N} \setminus \{0\}),$$

Proof:

For $n \in \mathbb{N} \setminus \{0\}$, we have

$$A_n(I - T) = \frac{1}{n} \sum_{j=0}^{n-1} \left(T^j - T^{j+1} \right) = \frac{1}{n} (I - T^n) \, .$$

$$A_n := \frac{1}{n}\sum_{j=0}^{n-1} T^j \qquad (n \in \mathbb{N} \setminus \{0\}),$$

Proof:

For $n \in \mathbb{N} \setminus \{0\}$, we have

$$A_n(I - T) = \frac{1}{n}\sum_{j=0}^{n-1}\left(T^j - T^{j+1}\right) = \frac{1}{n}(I - T^n).$$

Hence, if $\frac{1}{n}T^n f \to 0$ for all $f \in E$, then $A_n f \to 0$ for all $f \in \mathrm{ran}(I - T)$.

Lemma

Let H be a Hilbert space and $T \in \mathcal{B}(H)$ be a contraction, i.e., $||T|| \leq 1$. Then $\operatorname{fix}(T) = \operatorname{fix}(T^*)$.

Lemma

Let H be a Hilbert space and $T \in \mathcal{B}(H)$ be a contraction, i.e., $||T|| \leq 1$. Then $\operatorname{fix}(T) = \operatorname{fix}(T^*)$.

Proof:

Let $f \in \operatorname{fix}(T^*)$.

Lemma

Let H be a Hilbert space and $T \in \mathcal{B}(H)$ be a contraction, i.e., $||T|| \leq 1$. Then $\operatorname{fix}(T) = \operatorname{fix}(T^*)$.

Proof:

Let $f \in \operatorname{fix}(T^*)$. Then $\langle Tf, f\rangle = \langle f, T^*f\rangle = ||f||^2$.

Lemma

Let H be a Hilbert space and $T \in \mathcal{B}(H)$ be a contraction, i.e., $||T|| \leq 1$. Then $\text{fix}\,(T) = \text{fix}\,(T^*)$.

Proof:

Let $f \in \text{fix}\,(T^*)$. Then $\langle Tf, f \rangle = \langle f, T^* f \rangle = ||f||^2$. Since T is a contraction,

$$||Tf - f||^2 = ||Tf||^2 - 2\mathrm{Re}\langle f, Tf \rangle + ||f||^2 = ||Tf||^2 - ||f||^2 \leq 0.$$

Lemma

Let H be a Hilbert space and $T \in \mathcal{B}(H)$ be a contraction, i.e., $||T|| \leq 1$. Then $\mathrm{fix}\,(T) = \mathrm{fix}\,(T^*)$.

Proof:

Let $f \in \mathrm{fix}\,(T^*)$. Then $\langle Tf, f\rangle = \langle f, T^* f\rangle = ||f||^2$. Since T is a contraction,

$$||Tf - f||^2 = ||Tf||^2 - 2\mathrm{Re}\langle f, Tf\rangle + ||f||^2 = ||Tf||^2 - ||f||^2 \leq 0.$$

Consequently, $Tf = f$, i.e., $f \in \mathrm{fix}\,(T)$.

Lemma

Let H be a Hilbert space and $T \in \mathcal{B}(H)$ be a contraction, i.e., $||T|| \leq 1$. Then $\mathrm{fix}\,(T) = \mathrm{fix}\,(T^*)$.

Proof:

Let $f \in \mathrm{fix}\,(T^*)$. Then $\langle Tf, f\rangle = \langle f, T^* f\rangle = ||f||^2$. Since T is a contraction,

$$||Tf - f||^2 = ||Tf||^2 - 2\mathrm{Re}\langle f, Tf\rangle + ||f||^2 = ||Tf||^2 - ||f||^2 \leq 0.$$

Consequently, $Tf = f$, i.e., $f \in \mathrm{fix}\,(T)$.

Finally, $\mathrm{fix}\,(T) = \mathrm{fix}\,(T^*)$.

Definition

A *measure-preserving system* is a pair (X, ϕ) such that (X, Σ_X, μ_X) is a probability space, $\phi : X \to X$ is measurable and μ_X is ϕ-invariant.

Definition

A *measure-preserving system* is a pair (X, ϕ) such that (X, Σ_X, μ_X) is a probability space, $\phi : X \to X$ is measurable and μ_X is ϕ-invariant.

Theorem (Von Neumann's Theorem)

Let (X, ϕ) be a measure-preserving system and consider the Koopman operator $T = T_\phi := (f \longmapsto f \circ \phi)$, where $f : X \longrightarrow \mathbb{R}$ is a function.

Definition

A *measure-preserving system* is a pair (X, ϕ) such that (X, Σ_X, μ_X) is a probability space, $\phi : X \to X$ is measurable and μ_X is ϕ-invariant.

Theorem (Von Neumann's Theorem)

Let (X, ϕ) be a measure-preserving system and consider the Koopman operator $T = T_\phi := (f \longmapsto f \circ \phi)$, where $f : X \longrightarrow \mathbb{R}$ is a function. For each $f \in L^2(X)$, the limit

$$\lim_{n\to\infty} A_n f = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} T^j f$$

exists in the L^2-sense and is a fixed point of T.

Theorem (Mean Ergodic Theorem on Hilbert Spaces)

Let H be a Hilbert space and let $T \in \mathcal{B}(H)$ be a contraction, i.e., $||T|| \leq 1$.

Theorem (Mean Ergodic Theorem on Hilbert Spaces)

Let H be a Hilbert space and let $T \in \mathcal{B}(H)$ be a contraction, i.e., $||T|| \leq 1$. Then the following assertions hold:

1. $H = \mathrm{fix}\,(T) \oplus \overline{\mathrm{ran}}(I - T)$ *is an orthogonal decomposition;*

Theorem (Mean Ergodic Theorem on Hilbert Spaces)

Let H be a Hilbert space and let $T \in \mathcal{B}(H)$ be a contraction, i.e., $||T|| \leq 1$. Then the following assertions hold:

1. $H = \operatorname{fix}(T) \oplus \overline{\operatorname{ran}}(I - T)$ *is an orthogonal decomposition;*
2. $Pf := \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} T^j f$ *exists for every* $f \in H$*. And* P *is the orthogonal projection onto* $\operatorname{fix}(T)$*.*

Proof:

1. $H = \mathrm{fix}\,(T) \oplus \overline{\mathrm{ran}}(I - T)$ is an orthogonal decomposition.

Proof:

1. $H = \mathrm{fix}\,(T) \oplus \overline{\mathrm{ran}}(I - T)$ is an orthogonal decomposition. For $S \in \mathcal{B}(H)$, we have:

$$(\mathrm{ran}(S^*))^\perp = \ker(S).$$

Proof:

1. $H = \mathrm{fix}\,(T) \oplus \overline{\mathrm{ran}}(I - T)$ is an orthogonal decomposition. For $S \in \mathcal{B}(H)$, we have:

$$(\mathrm{ran}(S^*))^\perp = \ker(S).$$

In fact,

$$\begin{aligned} u \in (\mathrm{ran}(S^*))^\perp &\iff \forall v \in H \quad \langle u | S^* v\rangle = 0 \\ &\iff \forall v \in H \quad \langle Su | v\rangle = 0 \\ &\iff Su = 0. \end{aligned}$$

Proof:

1. $H = \operatorname{fix}(T) \oplus \overline{\operatorname{ran}}(I - T)$ is an orthogonal decomposition. For $S \in \mathcal{B}(H)$, we have:

$$(\operatorname{ran}(S^*))^\perp = \ker(S).$$

In fact,

$$\begin{aligned} u \in (\operatorname{ran}(S^*))^\perp &\iff \forall v \in H \quad \langle u | S^* v \rangle = 0 \\ &\iff \forall v \in H \quad \langle Su | v \rangle = 0 \\ &\iff Su = 0. \end{aligned}$$

Hence,

$$(\operatorname{ran}(I - T))^\perp = \ker(I - T^*) = \operatorname{fix}(T^*) = \operatorname{fix}(T).$$

Proof (continued):

2. $Pf := \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} T^j f$ exists for every $f \in H$. And P is the orthogonal projection onto $\operatorname{fix}(T)$.

Proof (continued):

2. $Pf := \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} T^j f$ exists for every $f \in H$. And P is the orthogonal projection onto $\mathrm{fix}\,(T)$.

   Take $f \in H$, there exist $f_1 \in \mathrm{fix}\,(T)$ and $f_2 \in \overline{\mathrm{ran}}(I - T)$ such that $f = f_1 + f_2$. Then

$$A_n f = A_n f_1 + A_n f_2.$$

Mean Ergodic Semigroups

Definition

E: Banach space and $\mathcal{J} \subseteq \mathcal{B}(E)$: semigroup, i.e.,

$$\mathcal{J} \cdot \mathcal{J} := \{ST : S, T \in \mathcal{J}\} \subseteq \mathcal{J}.$$

Definition

E: Banach space and $\mathcal{J} \subseteq \mathcal{B}(E)$: semigroup, i.e.,

$$\mathcal{J} \cdot \mathcal{J} := \{ST : S, T \in \mathcal{J}\} \subseteq \mathcal{J}.$$

$\mathcal{J}$ is called mean ergodic if $\exists\ P \in \mathcal{B}(E)$:

Definition

E: Banach space and $\mathcal{J} \subseteq \mathcal{B}(E)$: semigroup, i.e.,

$$\mathcal{J} \cdot \mathcal{J} := \{ST : S, T \in \mathcal{J}\} \subseteq \mathcal{J}.$$

$\mathcal{J}$ is called mean ergodic if $\exists\ P \in \mathcal{B}(E)$:

(a) $TP = PT = P\ \forall\ T \in \mathcal{J}$ and

Definition

E: Banach space and $\mathcal{J} \subseteq \mathcal{B}(E)$: semigroup, i.e.,

$$\mathcal{J} \cdot \mathcal{J} := \{ST : S, T \in \mathcal{J}\} \subseteq \mathcal{J}.$$

$\mathcal{J}$ is called mean ergodic if $\exists\ P \in \mathcal{B}(E)$:

(a) $TP = PT = P \ \forall\ T \in \mathcal{J}$ and

(b) $Pf \in \overline{\operatorname{conv}}\,\{\mathcal{J} f\} := \overline{\operatorname{conv}}\,\{Tf : T \in \mathcal{J}\} \ \forall\ f \in E$.

Definition

E: Banach space and $\mathcal{J} \subseteq \mathcal{B}(E)$: semigroup, i.e.,

$$\mathcal{J} \cdot \mathcal{J} := \{ST : S, T \in \mathcal{J}\} \subseteq \mathcal{J}.$$

$\mathcal{J}$ is called mean ergodic if $\exists\ P \in \mathcal{B}(E)$:

(a) $TP = PT = P\ \forall\ T \in \mathcal{J}$ and

(b) $Pf \in \overline{\operatorname{conv}}\{\mathcal{J}f\} := \overline{\operatorname{conv}}\{Tf : T \in \mathcal{J}\}\ \forall\ f \in E$.

P is called the mean ergodic projection.

Recall

$\mathcal{J}$ is called mean ergodic if $\exists\ P \in \mathcal{B}(H)$:

(a) $TP = PT = P\ \forall\ T \in \mathcal{J}$ and

(b) $Pf \in \overline{\text{conv}}\,\{\mathcal{J}f\}\,\forall\ f \in H$.

T is contraction on H (Hilbert)

Recall

$\mathcal{J}$ is called mean ergodic if $\exists\ P \in \mathcal{B}(H)$:

(a) $TP = PT = P \ \forall\ T \in \mathcal{J}$ and

(b) $Pf \in \overline{\mathrm{conv}}\,\{\mathcal{J}f\} \,\forall\ f \in H$.

T is contraction on H (Hilbert) $\Rightarrow \{T^n : n \in \mathbb{N}_0\}$ is mean ergodic.

Example

Recall

$\mathcal{J}$ is called mean ergodic if $\exists\ P \in \mathcal{B}(H)$:

(a) $TP = PT = P\ \forall\ T \in \mathcal{J}$ and

(b) $Pf \in \overline{\operatorname{conv}}\,\{\mathcal{J}f\}\,\forall\ f \in H$.

T is contraction on H (Hilbert) $\Rightarrow \{T^n : n \in \mathbb{N}_0\}$ is mean ergodic.

T is a contraction

Recall

$\mathcal{J}$ is called mean ergodic if $\exists\ P \in \mathcal{B}(H)$:

(a) $TP = PT = P\ \forall\ T \in \mathcal{J}$ and

(b) $Pf \in \overline{\text{conv}}\,\{\mathcal{J}f\}\,\forall\ f \in H$.

T is contraction on H (Hilbert) $\Rightarrow \{T^n : n \in \mathbb{N}_0\}$ is mean ergodic.

$$T \text{ is a contraction} \Rightarrow Pf := \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} T^j f \text{ exists } \forall\ f \in H$$

Example

Recall

$\mathcal{J}$ is called mean ergodic if $\exists\ P \in \mathcal{B}(H)$:

(a) $TP = PT = P\ \forall\ T \in \mathcal{J}$ and

(b) $Pf \in \overline{\text{conv}}\,\{\mathcal{J}f\}\,\forall\ f \in H$.

T is contraction on H (Hilbert) $\Rightarrow \{T^n : n \in \mathbb{N}_0\}$ is mean ergodic.

$$T \text{ is a contraction} \Rightarrow Pf := \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} T^j f \text{ exists } \forall\ f \in H$$

and P is a projection onto $\text{fix}\,(T)$

Recall

$\mathcal{J}$ is called mean ergodic if $\exists\ P \in \mathcal{B}(H)$:

(a) $TP = PT = P\ \forall\ T \in \mathcal{J}$ and

(b) $Pf \in \overline{\operatorname{conv}}\,\{\mathcal{J}f\}\,\forall\ f \in H$.

T is contraction on H (Hilbert) $\Rightarrow \{T^n : n \in \mathbb{N}_0\}$ is mean ergodic.

$$T \text{ is a contraction} \Rightarrow Pf := \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} T^j f \text{ exists } \forall\ f \in H$$

and P is a projection onto $\operatorname{fix}(T)$

$\Rightarrow\ \{T^n : n \in \mathbb{N}_0\}$ is mean ergodic.

Recall

$\mathcal{J}$ is mean ergodic iff $\exists\ P \in \mathcal{B}(E)$: $TP = PT = P\ \forall\ T \in \mathcal{J}$ and $Pf \in \overline{\mathrm{conv}}\,\{\mathcal{J}f\}\,\forall\ f \in E$.

Recall

$\mathcal{J}$ is mean ergodic iff $\exists\ P \in \mathcal{B}(E)$: $TP = PT = P\ \forall\ T \in \mathcal{J}$ and $Pf \in \overline{\mathrm{conv}}\,\{\mathcal{J}f\}\,\forall\ f \in E$.

- P is a projection.

Observations

Recall

$\mathcal{J}$ is mean ergodic iff $\exists\ P \in \mathcal{B}(E)$: $TP = PT = P\ \forall\ T \in \mathcal{J}$ and $Pf \in \overline{\text{conv}}\,\{\mathcal{J}f\}\,\forall\ f \in E$.

- P is a projection.

$$TP = P\ \forall\ T \in \mathcal{J}$$

Observations

Recall

$\mathcal{J}$ is mean ergodic iff $\exists\ P \in \mathcal{B}(E)$: $TP = PT = P\ \forall\ T \in \mathcal{J}$ and $Pf \in \overline{\text{conv}}\,\{\mathcal{J}f\}\,\forall\ f \in E$.

- P is a projection.

$$TP = P\ \forall\ T \in \mathcal{J} \Rightarrow \operatorname{Rg} P \subseteq \operatorname{fix}(\mathcal{J}) := \bigcap_{T\in\mathcal{J}} \operatorname{fix}(T)$$

Recall

$\mathcal{J}$ is mean ergodic iff $\exists\ P \in \mathcal{B}(E)$: $TP = PT = P\ \forall\ T \in \mathcal{J}$ and $Pf \in \overline{\operatorname{conv}}\{\mathcal{J}f\}\,\forall\ f \in E$.

- P is a projection.

$$TP = P\ \forall\ T \in \mathcal{J} \Rightarrow \operatorname{Rg} P \subseteq \operatorname{fix}(\mathcal{J}) := \bigcap_{T\in\mathcal{J}} \operatorname{fix}(T)$$

and

$$Pf \in \overline{\operatorname{conv}}\{\mathcal{J}f\}\ \forall\ f \in E$$

Recall

$\mathcal{J}$ is mean ergodic iff $\exists\ P \in \mathcal{B}(E)$: $TP = PT = P\ \forall\ T \in \mathcal{J}$ and $Pf \in \overline{\mathrm{conv}}\,\{\mathcal{J}f\}\,\forall\ f \in E$.

- P is a projection.

$$TP = P\ \forall\ T \in \mathcal{J} \Rightarrow \operatorname{Rg} P \subseteq \operatorname{fix}(\mathcal{J}) := \bigcap_{T \in \mathcal{J}} \operatorname{fix}(T)$$

and

$$Pf \in \overline{\mathrm{conv}}\,\{\mathcal{J}f\}\ \forall\ f \in E \Rightarrow P|_{\operatorname{fix}(\mathcal{J})} = I,$$

Recall

$\mathcal{J}$ is mean ergodic iff $\exists\ P \in \mathcal{B}(E)$: $TP = PT = P\ \forall\ T \in \mathcal{J}$ and $Pf \in \overline{\text{conv}}\{\mathcal{J}f\}\,\forall\ f \in E$.

- *P is a projection.*

$$TP = P\ \forall\ T \in \mathcal{J} \Rightarrow \operatorname{Rg} P \subseteq \operatorname{fix}(\mathcal{J}) := \bigcap_{T \in \mathcal{J}} \operatorname{fix}(T)$$

and

$$Pf \in \overline{\text{conv}}\{\mathcal{J}f\}\ \forall\ f \in E \Rightarrow P|_{\operatorname{fix}(\mathcal{J})} = I,$$

therefore,

$$P^2 = P.$$

Recall

$\mathcal{J}$ is mean ergodic iff $\exists\ P \in \mathcal{B}(E)$: $TP = PT = P\ \forall\ T \in \mathcal{J}$ and $Pf \in \overline{\text{conv}}\{\mathcal{J}f\}\ \forall\ f \in E$.

- P is unique.

Observations

Recall

$\mathcal{J}$ is mean ergodic iff $\exists\ P \in \mathcal{B}(E)$: $TP = PT = P\ \forall\ T \in \mathcal{J}$ and $Pf \in \overline{\text{conv}}\,\{\mathcal{J}f\}\,\forall\ f \in E$.

- P is unique.

  If Q is another mean ergodic projection, then for all $f \in E$:

Recall

$\mathcal{J}$ is mean ergodic iff $\exists\ P \in \mathcal{B}(E)$: $TP = PT = P\ \forall\ T \in \mathcal{J}$ and $Pf \in \overline{\text{conv}}\,\{\mathcal{J}f\}\,\forall\ f \in E$.

- P is unique.

  If Q is another mean ergodic projection, then for all $f \in E$:

$$PQf$$

Recall

$\mathcal{J}$ is mean ergodic iff $\exists\ P \in \mathcal{B}(E)$: $TP = PT = P\ \forall\ T \in \mathcal{J}$ and $Pf \in \overline{\mathrm{conv}}\,\{\mathcal{J}f\}\,\forall\ f \in E$.

- *P is unique.*

  If Q is another mean ergodic projection, then for all $f \in E$:

$$\begin{aligned} &\overline{\mathrm{conv}}\,\{TQf : T \in \mathcal{J}\} \\ \ni\ &PQf \end{aligned}$$

Recall

$\mathcal{J}$ is mean ergodic iff $\exists\ P \in \mathcal{B}(E)$: $TP = PT = P\ \forall\ T \in \mathcal{J}$ and $Pf \in \overline{\text{conv}}\,\{\mathcal{J}f\}\,\forall\ f \in E$.

- *P* is unique.

  If Q is another mean ergodic projection, then for all $f \in E$:

$$\begin{aligned}\{Qf\} &= \overline{\text{conv}}\,\{TQf : T \in \mathcal{J}\} \\ &\ni PQf\end{aligned}$$

Recall

$\mathcal{J}$ is mean ergodic iff $\exists\ P \in \mathcal{B}(E)$: $TP = PT = P\ \forall\ T \in \mathcal{J}$ and $Pf \in \overline{\mathrm{conv}}\,\{\mathcal{J}f\}\,\forall\ f \in E$.

- *P is unique.*

  If Q is another mean ergodic projection, then for all $f \in E$:

$$\begin{aligned}\{Qf\} &= \overline{\mathrm{conv}}\,\{TQf : T \in \mathcal{J}\} \\ &\ni PQf \\ &\in \overline{\mathrm{conv}}\,\{PTf : T \in \mathcal{J}\}\end{aligned}$$

Recall

$\mathcal{J}$ is mean ergodic iff $\exists\ P \in \mathcal{B}(E)$: $TP = PT = P\ \forall\ T \in \mathcal{J}$ and $Pf \in \overline{\text{conv}}\,\{\mathcal{J}f\}\,\forall\ f \in E$.

- *P is unique.*

  If Q is another mean ergodic projection, then for all $f \in E$:

$$\begin{aligned}\{Qf\} &= \overline{\text{conv}}\,\{TQf : T \in \mathcal{J}\}\\ &\ni PQf\\ &\in \overline{\text{conv}}\,\{PTf : T \in \mathcal{J}\}\\ &= \{Pf\}.\end{aligned}$$

Recall

$\mathcal{J}$ is mean ergodic iff $\exists\ P \in \mathcal{B}(E)$: $TP = PT = P\ \forall\ T \in \mathcal{J}$ and $Pf \in \overline{\text{conv}}\,\{\mathcal{J}f\}\,\forall\ f \in E$.

- P is unique.

  If Q is another mean ergodic projection, then for all $f \in E$:

$$\begin{aligned}\{Qf\} &= \overline{\text{conv}}\,\{TQf : T \in \mathcal{J}\}\\ &\ni PQf\\ &\in \overline{\text{conv}}\,\{PTf : T \in \mathcal{J}\}\\ &= \{Pf\}.\end{aligned}$$

  Hence, $Q = P$.

Theorem

A contraction semigroup $\mathcal{J}$ on H (Hilbert) is mean ergodic.

Theorem

A contraction semigroup $\mathcal{J}$ on H (Hilbert) is mean ergodic. The mean ergodic projection P is the projection onto $\operatorname{fix}(\mathcal{J})$.

Theorem

A contraction semigroup $\mathcal{J}$ on H (Hilbert) is mean ergodic. The mean ergodic projection P is the projection onto $\operatorname{fix}(\mathcal{J})$.

For each $f \in H$, Pf is the unique element of $\overline{\operatorname{conv}}\{\mathcal{J}f\}$ with minimal norm.

Theorem

A contraction semigroup $\mathcal{J}$ on H (Hilbert) is mean ergodic. The mean ergodic projection P is the projection onto $\operatorname{fix}(\mathcal{J})$.

For each $f \in H$, Pf is the unique element of $\overline{\operatorname{conv}}\{\mathcal{J}f\}$ with minimal norm.

Proof:

Bounded mean ergodic semigroups

Theorem (Nagel)

Let $\mathcal{J}$: bounded semigroup on E (Banach). Equivalent:

Theorem (Nagel)

Let $\mathcal{J}$: bounded semigroup on E (Banach). Equivalent:

(i) $\mathcal{J}$ *is mean ergodic.*

Theorem (Nagel)

Let $\mathcal{J}$: bounded semigroup on E (Banach). Equivalent:

(i) $\mathcal{J}$ *is mean ergodic.*

(ii) $\overline{\mathrm{conv}}\,\{\mathcal{J}f\} \cap \mathrm{fix}\,(\mathcal{J})$ *is a singleton* $\forall\, f \in E$.

Theorem (Nagel)

Let $\mathcal{J}$: bounded semigroup on E (Banach). Equivalent:

(i) $\mathcal{J}$ *is mean ergodic.*

(ii) $\overline{\mathrm{conv}}\,\{\mathcal{J}f\} \cap \mathrm{fix}\,(\mathcal{J})$ *is a singleton* $\forall\ f \in E$.

(iii) $\mathrm{fix}\,(\mathcal{J})$ *separates* $\mathrm{fix}\,(\mathcal{J}^*)$ *and*
$\overline{\mathrm{conv}}^{w^*}\{\mathcal{J}^* f^*\} \cap \mathrm{fix}\,(\mathcal{J}^*) \neq \emptyset\ \forall\ f^* \in E'$.

Theorem (Nagel)

Let $\mathcal{J}$: bounded semigroup on E (Banach). Equivalent:

(i) $\mathcal{J}$ *is mean ergodic.*

(ii) $\overline{\mathrm{conv}}\,\{\mathcal{J} f\} \cap \mathrm{fix}\,(\mathcal{J})$ *is a singleton* $\forall\ f \in E$.

(iii) $\mathrm{fix}\,(\mathcal{J})$ *separates* $\mathrm{fix}\,(\mathcal{J}^*)$ *and* $\overline{\mathrm{conv}}^{w^*}\{\mathcal{J}^* f^*\} \cap \mathrm{fix}\,(\mathcal{J}^*) \neq \emptyset\ \forall\ f^* \in E'$.

(iv) $\overline{\mathrm{conv}}\,\{\mathcal{J} f\} \cap \mathrm{fix}\,(\mathcal{J}) \neq \emptyset\ \forall\ f \in E$ *and* $\overline{\mathrm{conv}}^{w^*}\{\mathcal{J}^* f^*\} \cap \mathrm{fix}\,(\mathcal{J}^*) \neq \emptyset\ \forall\ f^* \in E'$.

Theorem (Nagel)

Let $\mathcal{J}$: bounded semigroup on E (Banach). Equivalent:

(i) $\mathcal{J}$ *is mean ergodic.*

(ii) $\overline{\operatorname{conv}}\,\{\mathcal{J}f\} \cap \operatorname{fix}(\mathcal{J})$ *is a singleton* $\forall\, f \in E$.

(iii) $\operatorname{fix}(\mathcal{J})$ *separates* $\operatorname{fix}(\mathcal{J}^*)$ *and*
$\overline{\operatorname{conv}}^{w^*}\{\mathcal{J}^* f^*\} \cap \operatorname{fix}(\mathcal{J}^*) \neq \emptyset \ \forall\, f^* \in E'$.

(iv) $\overline{\operatorname{conv}}\,\{\mathcal{J}f\} \cap \operatorname{fix}(\mathcal{J}) \neq \emptyset \ \forall\, f \in E$ *and*
$\overline{\operatorname{conv}}^{w^*}\{\mathcal{J}^* f^*\} \cap \operatorname{fix}(\mathcal{J}^*) \neq \emptyset \ \forall\, f^* \in E'$.

In this case, $\overline{\operatorname{conv}}\,\{\mathcal{J}f\} \cap \operatorname{fix}(\mathcal{J}) = \{Pf\} \ \forall\, f \in E$.

Preliminaries on von Neumann Algebras

Theorem

Let $(M, \mathcal{J})$ *be a dynamical system. If there exists a faithful normal state* ω *on* M *satisfying*

$$\omega((Tx)^*(Tx)) \leq \omega(x^*x) \text{ for all } T \in \mathcal{J}, x \in M,$$

then $\mathcal{J}$ *is sot-ergodic.*

Theorem

Let (M,$\mathcal{J}$) be a dynamical system. If there exists a faithful normal state ω on M satisfying

$$\omega((Tx)^*(Tx)) \leq \omega(x^*x) \text{ for all } T \in \mathcal{J}, x \in M,$$

then $\mathcal{J}$ is sot-ergodic.

M: von Neumann Algebra

$\mathcal{J}$: bounded semigroup of linear operators on M

ABSTRACT DEFINITION

CONCRETE REPRESENTATION

ABSTRACT DEFINITION

C^*-algebra M with a predual M_*

$$M = (M_*)^*$$

CONCRETE REPRESENTATION

ABSTRACT DEFINITION

C^*-algebra M with a predual M_*

$$M = (M_*)^*$$

CONCRETE REPRESENTATION

*-subalgebra M of $\mathcal{B}(H)$ with

- M closed (w.r.t. sot/wot)
- $Id_H \in M$

Strong operator topology

The sot is the coarsest topology on $\mathcal{B}(H)$ such that all evaluation mappings

$$T \in \mathcal{B}(H) \mapsto Tu \in H \quad (u \in H)$$

are continuous,

Strong operator topology

The sot is the coarsest topology on $\mathcal{B}(H)$ such that all evaluation mappings

$$T \in \mathcal{B}(H) \mapsto Tu \in H \quad (u \in H)$$

are continuous, i.e., given a net $\{T_i\} \subset \mathcal{B}(H)$

$$T_i \overset{sot}{\to} T \iff T_i u \to Tu \quad \forall u \in H.$$

Weak operator topology

The wot is the coarsest topology on $\mathcal{B}(H)$ such that all evaluation mappings

$$T \in \mathcal{B}(H) \mapsto \langle Tu, v \rangle \in H \quad (u, v \in H)$$

are continuous,

Weak operator topology

The wot is the coarsest topology on $\mathcal{B}(H)$ such that all evaluation mappings

$$T \in \mathcal{B}(H) \mapsto \langle Tu, v\rangle \in H \quad (u, v \in H)$$

are continuous, i.e., given a net $\{T_i\} \subset \mathcal{B}(H)$

$$T_i \overset{wot}{\to} T \iff \langle T_i u, v\rangle \to \langle Tu, v\rangle \quad \forall u, v \in H.$$

ABSTRACT DEFINITION

C^*-algebra M with a predual M_*

$$M = (M_*)^*$$

CONCRETE REPRESENTATION

*-subalgebra M of $\mathcal{B}(H)$ with

- **M closed (w.r.t. sot/wot)**
- $Id_H \in M$

Theorem

Let M be a unital self-adjoint subalgebra of $\mathcal{B}(H)$. The following conditions are equivalent:

(i) $M = M''$;

(ii) *M is weakly closed;*

(iii) *M is strongly closed.*

Theorem

Let M be a unital self-adjoint subalgebra of $\mathcal{B}(H)$. The following conditions are equivalent:

(i) $M = M''$;

(ii) M *is weakly closed;*

(iii) M *is strongly closed.*

Remark

The commutant and the bicommutant are defined as follows:

$$M' := \{x \in \mathcal{B}(H) \mid xy = yx \text{ for all } y \in M\}, \quad M'' = (M')'.$$

Theorem

Let $(M, \mathcal{J})$ be a dynamical system. If there exists a ***faithful normal state*** ω *on* M *satisfying*

$$\omega((Tx)^*(Tx)) \leq \omega(x^*x) \text{ for all } T \in \mathcal{J}, x \in M,$$

then $\mathcal{J}$ *is sot-ergodic.*

Definition

A positive linear functional ω on M is *normal* if for every bounded increasing net $\{x_i\}$ of positive elements in M, we have

$$\omega(\sup_i x_i) = \sup_i \omega(x_i).$$

ABSTRACT DEFINITION

C^*-algebra M with a predual M_*

CONCRETE REPRESENTATION

*-subalgebra M of $\mathcal{B}(H)$ with

- M closed (w.r.t. sot/wot)
- $Id_H \in M$

$$M_* = \mathrm{span}\{\omega \mid \omega \text{ positive normal linear functional on } M\}$$

Definition

Let ω be a positive normal linear functional on M. We say that ω is a *faithful normal state* if it is

Definition

Let ω be a positive normal linear functional on M. We say that ω is a *faithful normal state* if it is

- faithful, i.e., ($x \geq 0$, $\omega(x) = 0 \Rightarrow x = 0$)

Definition

Let ω be a positive normal linear functional on M. We say that ω is a *faithful normal state* if it is

- faithful, i.e., $(x \geq 0, \omega(x) = 0 \Rightarrow x = 0)$
- a state, i.e., $\omega(1) = 1$

weak*-topology

The weak*-topology is the weakest topology in M with respect to which all $\omega \in M_*$ are continuous,

weak*-topology

The weak*-topology is the weakest topology in M with respect to which all $\omega \in M_*$ are continuous, i.e., given a net $\{x_i\}$

$$x_i \stackrel{w^*}{\to} x \Longleftrightarrow \omega(x_i) \to \omega(x) \; \forall \; \omega \in M_*.$$

On its unit ball, the w.o. and s.o. topologies do not depend on the concrete representation of the von Neumann algebra.

Let A be a C^*-subalgebra of $\mathcal{B}(H)$.

$\overline{A}^{sot}$: the closure of A in sot
$\overline{A}^{wot}$: the closure of A in wot

A_1 : the unit ball of A

A_{sa} : the set of self-adjoint operators in A

Let A be a C^*-subalgebra of $\mathcal{B}(H)$.

$\overline{A}^{sot}$: the closure of A in sot
$\overline{A}^{wot}$: the closure of A in wot
A_1 : the unit ball of A
A_{sa} : the set of self-adjoint operators in A

Theorem

Let A be a C^*-subalgebra of $\mathcal{B}(H)$ and let $M = \overline{A}^{wot}$. Then

1. $M_1 = \overline{A_1}^{sot}$
2. $M_{sa} = \overline{A}_{sa}^{sot}$

Corollary

Let M be a C^*-subalgebra of $\mathcal{B}(H)$ with $Id_{\mathcal{H}} \in M$. Then M is a von Neumann algebra iff its unit ball is compact (or equivalently closed) in wot.

Corollary

Let M be a C^*-subalgebra of $\mathcal{B}(H)$ with $Id_{\mathcal{H}} \in M$. Then M is a von Neumann algebra iff its unit ball is compact (or equivalently closed) in wot.

Proof: We have $M_1 = M \cap \mathcal{B}(H)_1$.

Corollary

Let M be a C^*-subalgebra of $\mathcal{B}(H)$ with $Id_{\mathcal{H}} \in M$. Then M is a von Neumann algebra iff its unit ball is compact (or equivalently closed) in wot.

Proof: We have $M_1 = M \cap \mathcal{B}(H)_1$. Suppose M is a von Neumann algebra.

Corollary

Let M be a C^*-subalgebra of $\mathcal{B}(H)$ with $Id_{\mathcal{H}} \in M$. Then M is a von Neumann algebra iff its unit ball is compact (or equivalently closed) in wot.

Proof: We have $M_1 = M \cap \mathcal{B}(H)_1$. Suppose M is a von Neumann algebra. Then the unit ball of M is compact in wot.

Corollary

Let M be a C^*-subalgebra of $\mathcal{B}(H)$ with $Id_{\mathcal{H}} \in M$. Then M is a von Neumann algebra iff its unit ball is compact (or equivalently closed) in wot.

Proof: We have $M_1 = M \cap \mathcal{B}(H)_1$. Suppose M is a von Neumann algebra. Then the unit ball of M is compact in wot.

Conversely, suppose that the unit ball of M is compact in wot.

Corollary

Let M be a C^*-subalgebra of $\mathcal{B}(H)$ with $Id_{\mathcal{H}} \in M$. Then M is a von Neumann algebra iff its unit ball is compact (or equivalently closed) in wot.

Proof: We have $M_1 = M \cap \mathcal{B}(H)_1$. Suppose M is a von Neumann algebra. Then the unit ball of M is compact in wot.

Conversely, suppose that the unit ball of M is compact in wot.

Let $x \in \overline{M}^{wot}$. We may assume $\|x\| \leq 1$.

Corollary

Let M be a C^*-subalgebra of $\mathcal{B}(H)$ with $Id_{\mathcal{H}} \in M$. Then M is a von Neumann algebra iff its unit ball is compact (or equivalently closed) in wot.

Proof: We have $M_1 = M \cap \mathcal{B}(H)_1$. Suppose M is a von Neumann algebra. Then the unit ball of M is compact in wot.

Conversely, suppose that the unit ball of M is compact in wot.

Let $x \in \overline{M}^{wot}$. We may assume $\|x\| \leq 1$. By Kaplansky density theorem, there exists (x_α) in M_1 converging to x in the wot. Hence $x \in M$.

Characterizations of normality

We recall that a positive linear functional ω on a von Neumann algebra M is normal if for every bounded increasing net $\{x_\alpha\}$ of positive elements in M, we have $\omega(\sup_\alpha x_\alpha) = \sup_\alpha \omega(x_\alpha)$.

We recall that a positive linear functional ω on a von Neumann algebra M is normal if for every bounded increasing net $\{x_\alpha\}$ of positive elements in M, we have $\omega(\sup_\alpha x_\alpha) = \sup_\alpha \omega(x_\alpha)$.

Theorem

Let $\omega \geq 0$ be a linear functional on M. The following conditions are equivalent :

1. *ω is normal;*
2. *$\omega|_{M_1}$ is sot-continuous;*
3. *$\omega|_{M_1}$ is wot-continuous.*

Concrete representation of M

Lemma

Let $M_* = \text{span}\{\omega : \omega \geq 0,\ \text{normal}\}$ and τ_{M_*} be the topology generated by M_*. Then $\tau_{M_*} = \text{wot}$ on M_1.

Lemma

Let $M_* = \text{span}\{\omega : \omega \geq 0, \text{ normal}\}$ and τ_{M_*} be the topology generated by M_*. Then $\tau_{M_*} = \text{wot}$ on M_1.

Proof: Consider the mapping $\mathbb{I} : (M_1, wot) \to (M_1, \tau_{M_*})$.

Lemma

Let $M_* = \text{span}\{\omega : \omega \geq 0, \text{ normal}\}$ and τ_{M_*} be the topology generated by M_*. Then $\tau_{M_*} = \text{wot}$ on M_1.

Proof: Consider the mapping $\mathbb{I} : (M_1, wot) \to (M_1, \tau_{M_*})$.

We have: ω is normal $\iff$ $\omega|_{M_1}$ is wot-continuous.

Lemma

Let $M_* = \text{span}\{\omega : \omega \geq 0, \text{ normal}\}$ and τ_{M_*} be the topology generated by M_*. Then $\tau_{M_*} = \text{wot}$ on M_1.

Proof: Consider the mapping $\mathbb{I} : (M_1, wot) \to (M_1, \tau_{M_*})$.

We have: ω is normal $\iff \omega\big|_{M_1}$ is wot-continuous.

If $M_1 \ni x_\alpha \to x \in M_1($ wot$)$, then $\omega(x_\alpha) \to \omega(x)$ is wot-continuous. So, the map $\mathbb{I}$ is continuous.

Concrete representation of M

Moreover, (M_1, wot) is compact and (M_1, τ_{M_*}) is Hausdorff.

Moreover, (M_1, wot) is compact and (M_1, τ_{M_*}) is Hausdorff.

$A \subseteq (M_1, wot)$ closed

Moreover, (M_1, wot) is compact and (M_1, τ_{M_*}) is Hausdorff.

$A \subseteq (M_1, wot)$ closed $\implies$ A is wot-compact

Moreover, (M_1, wot) is compact and (M_1, τ_{M_*}) is Hausdorff.

$A \subseteq (M_1, wot)$ closed $\implies$ A is wot-compact $\implies$ A is τ_{M_*}-compact (since $\mathbb{I}$ is continuous)

Moreover, (M_1, wot) is compact and (M_1, τ_{M_*}) is Hausdorff.

$A \subseteq (M_1, wot)$ closed $\implies$ A is wot-compact $\implies$ A is τ_{M_*}-compact (since $\mathbb{I}$ is continuous) $\implies$ A is τ_{M_*}-closed (since τ_{M_*} is Hausdorff).

Moreover, (M_1, wot) is compact and (M_1, τ_{M_*}) is Hausdorff.

$A \subseteq (M_1, wot)$ closed $\implies$ A is wot-compact $\implies$ A is τ_{M_*}-compact (since $\mathbb{I}$ is continuous) $\implies$ A is τ_{M_*}-closed (since τ_{M_*} is Hausdorff).

Thus $\mathbb{I}$ is homeomorphism.

Moreover, (M_1, wot) is compact and (M_1, τ_{M_*}) is Hausdorff.

$A \subseteq (M_1, wot)$ closed $\implies$ A is wot-compact $\implies$ A is τ_{M_*}-compact (since $\mathbb{I}$ is continuous) $\implies$ A is τ_{M_*}-closed (since τ_{M_*} is Hausdorff).

Thus $\mathbb{I}$ is homeomorphism.

That is, wot$=\tau_{M_*}$ on M_1.

Moreover, (M_1, wot) is compact and (M_1, τ_{M_*}) is Hausdorff.

$A \subseteq (M_1, wot)$ closed $\implies$ A is wot-compact $\implies$ A is τ_{M_*}-compact (since $\mathbb{I}$ is continuous) $\implies$ A is τ_{M_*}-closed (since τ_{M_*} is Hausdorff).

Thus $\mathbb{I}$ is homeomorphism.

That is, wot$=\tau_{M_*}$ on M_1.

Corollary

(M_1, wot) does not depend on the concrete representation of M.

Lemma

Let $(x_\alpha)_\alpha$ be a net in M. Then $x_\alpha \to 0$ in the sot iff $x_\alpha^* x_\alpha \to 0$ in the wot.

Lemma

Let $(x_\alpha)_\alpha$ be a net in M. Then $x_\alpha \to 0$ in the sot iff $x_\alpha^* x_\alpha \to 0$ in the wot.

Proof: The lemma follows from the polarization identity:

$$\langle x_\alpha^* x_\alpha \xi, \eta \rangle = \frac{1}{4} \sum_{k=0}^{3} i^k \| x_\alpha(\xi + i^k \eta) \|^2,$$

where $\xi, \eta \in H$ are arbitrary vectors.

Lemma

Let $(x_\alpha)_\alpha$ be a net in M. Then $x_\alpha \to 0$ in the sot iff $x_\alpha^* x_\alpha \to 0$ in the wot.

Proof: The lemma follows from the polarization identity:

$$\langle x_\alpha^* x_\alpha \xi, \eta \rangle = \frac{1}{4} \sum_{k=0}^{3} i^k \| x_\alpha(\xi + i^k \eta) \|^2,$$

where $\xi, \eta \in H$ are arbitrary vectors.

Corollary

(M_1, sot) does not depend on the concrete representation of M.

GNS representation

We recall that a positive normal linear functional ω on M is a faithful normal state if

$$x \geq 0,\ \omega(x) = 0 \quad \Rightarrow \; x = 0.$$

We recall that a positive normal linear functional ω on M is a faithful normal state if

$$x \geq 0,\ \omega(x) = 0 \quad \Rightarrow \ x = 0.$$

Definition

Let $\omega \geq 0$ be a faithful normal state on M.

Define $\quad \langle x, y\rangle_\omega := \omega(y^* x), \quad \|x\|_\omega^2 := \omega(x^* x).$

We recall that a positive normal linear functional ω on M is a faithful normal state if

$$x \geq 0,\ \omega(x) = 0 \quad \Rightarrow \ x = 0.$$

Definition

Let $\omega \geq 0$ be a faithful normal state on M.

Define $\quad \langle x, y \rangle_\omega := \omega(y^* x), \quad \|x\|_\omega^2 := \omega(x^* x).$

$\| \cdot \|_\omega$ is a norm since $N_\omega = \{x : \omega(x^* x) = 0\} = 0.$

We recall that a positive normal linear functional ω on M is a faithful normal state if

$$x \geq 0,\ \omega(x) = 0 \quad \Rightarrow\ x = 0.$$

Definition

Let $\omega \geq 0$ be a faithful normal state on M.

Define $\quad \langle x, y\rangle_\omega := \omega(y^* x),\ \ \|x\|_\omega^2 := \omega(x^* x).$

$\|\cdot\|_\omega$ is a norm since $N_\omega = \{x : \omega(x^* x) = 0\} = 0$.

We define M_ω as the completion of M with respect to the inner product $\langle ., .\rangle_\omega$.

$\pi_\omega(M)$ is a von Neumann algebra.

Lemma

For $x, y \in M$, $\pi_\omega : M \to \mathcal{B}(M_\omega),\ \pi_\omega(x)y = xy$. Then $\pi_\omega(M)$ is a von Neumann algebra on M_ω.

$\pi_\omega(M)$ is a von Neumann algebra.

Lemma

For $x, y \in M$, $\pi_\omega : M \to \mathcal{B}(M_\omega)$, $\pi_\omega(x)y = xy$. Then $\pi_\omega(M)$ is a von Neumann algebra on M_ω.

Proof: Let $a, b \in M$.

$x \mapsto \omega(a^* x b) = \langle xb, a \rangle_\omega$ is wot-continuous on M_1, by normality.

Lemma

For $x, y \in M$, $\pi_\omega : M \to \mathcal{B}(M_\omega),\ \pi_\omega(x)y = xy$. Then $\pi_\omega(M)$ is a von Neumann algebra on M_ω.

Proof: Let $a, b \in M$.

$x \mapsto \omega(a^* x b) = \langle xb, a \rangle_\omega$ is wot-continuous on M_1, by normality.

By the density of M in M_ω, we have $x \mapsto \langle xh, k \rangle_\omega$ is wot-continuous on M_1 for all $h, k \in M_\omega$.

Lemma

For $x, y \in M$, $\pi_\omega : M \to \mathcal{B}(M_\omega),\ \pi_\omega(x)y = xy$. Then $\pi_\omega(M)$ is a von Neumann algebra on M_ω.

Proof: Let $a, b \in M$.

$x \mapsto \omega(a^* x b) = \langle xb, a\rangle_\omega$ is wot-continuous on M_1, by normality.

By the density of M in M_ω, we have $x \mapsto \langle xh, k\rangle_\omega$ is wot-continuous on M_1 for all $h, k \in M_\omega$.

$\implies \quad (M_1, \text{wot}) \overset{\pi_\omega}{\to} (\mathcal{B}(M_\omega), \text{wot})$ is continuous.

$\pi_\omega(M)$ is a von Neumann algebra.

Since π_ω is injective, hence π_ω is isometric.

Since π_ω is injective, hence π_ω is isometric.

Hence $\quad \pi_\omega(M_1) = (\pi_\omega(M))_1$ is wot-compact.

$\pi_\omega(M)$ is a von Neumann algebra.

Since π_ω is injective, hence π_ω is isometric.

Hence $\quad \pi_\omega(M_1) = (\pi_\omega(M))_1$ is wot-compact.

By Kaplansky density theorem, $\pi_\omega(M)$ is a von Neumann algebra on M_ω.

Proof of the main theorem

Proposition

Let $(M, \mathcal{J})$ be a dynamical system. Let ω be a faithful normal state on M satisfying

$$\omega((Tx)^*(Tx)) \leq \omega(x^*x) \text{ for all } T \in \mathcal{J}, x \in M.$$

Proposition

Let $(M, \mathcal{J})$ be a dynamical system. Let ω be a faithful normal state on M satisfying

$$\omega((Tx)^*(Tx)) \leq \omega(x^*x) \text{ for all } T \in \mathcal{J}, x \in M.$$

Then there exists a $P \in \mathcal{B}(H)$ with $P\hat{T} = \hat{T}P = P$ for all $\hat{T} \in \hat{\mathcal{J}}$, where $\hat{\mathcal{J}}$ is the extended semigroup of $\mathcal{J}$ on $B(H)$.
$\forall x \in M \ Px \in \overline{\mathrm{conv}}^{sot} \mathcal{J} x$.

Proposition

Let $(M, \mathcal{J})$ be a dynamical system. Let ω be a faithful normal state on M satisfying

$$\omega((Tx)^*(Tx)) \leq \omega(x^*x) \text{ for all } T \in \mathcal{J}, x \in M.$$

Then there exists a $P \in \mathcal{B}(H)$ with $P\hat{T} = \hat{T}P = P$ for all $\hat{T} \in \hat{\mathcal{J}}$, where $\hat{\mathcal{J}}$ is the extended semigroup of $\mathcal{J}$ on $B(H)$. $\forall x \in M$ $Px \in \overline{\text{conv}}^{sot}\mathcal{J}x$. In particular, $Px \in M$ for $x \in M$ and therefore $P \in \pi(M)$.

Lemma

Let ω be a faithful normal state, $H = M_\omega$, π from the GNS construction. Let $(x_\alpha)_\alpha$ be a bounded net in M. Then

$$||x_\alpha||_\omega \to 0 \iff \pi_{x_\alpha} \overset{\text{sot}}{\to} 0$$

$||x_\alpha||_\omega \to 0 \implies \pi_{x_\alpha} \xrightarrow{\textsf{sot}} 0$

We show $||x_\alpha||_\omega \to 0 \implies \pi_{x_\alpha} \overset{\text{sot}}{\to} 0$:

We show $||x_\alpha||_\omega \to 0 \implies \pi_{x_\alpha} \overset{\text{sot}}{\to} 0$:

$$P := P_{\overline{\pi(M)'1}} = \pi_q \quad \text{for a } q \in M$$

since P is in the commutant of $\pi(M)$.

$||x_\alpha||_\omega \to 0 \implies \pi_{x_\alpha} \xrightarrow{\text{sot}} 0$

We show $P = I$:

We show $P = I$:

$$\omega(1-p) = \langle (1-p)1, 1\rangle_\omega$$

We show $P = I$:

$$\omega(1-p) = \langle (1-p)1, 1\rangle_\omega = \langle (I-P)(1), 1\rangle_\omega$$

We show $P = I$:

$$\begin{aligned} \omega(1-p) =& \langle (1-p)1, 1\rangle_\omega = \langle (I-P)(1), 1\rangle_\omega \\ =& \langle (I-P)(1), P(1)\rangle_\omega \end{aligned}$$

We show $P = I$:

$$\begin{aligned}\omega(1-p) =& \langle (1-p)1, 1\rangle_\omega = \langle (I-P)(1), 1\rangle_\omega \\ =& \langle (I-P)(1), P(1)\rangle_\omega = 0\end{aligned}$$

We show $P = I$:

$$\begin{aligned}\omega(1-p) =& \langle (1-p)1, 1\rangle_\omega = \langle (I-P)(1), 1\rangle_\omega \\ =& \langle (I-P)(1), P(1)\rangle_\omega = 0 \\ \implies& 1-p = 0 \quad \text{since } 1-p \geq 0\end{aligned}$$

We show $P = I$:

$$\begin{aligned}\omega(1-p) =& \langle (1-p)1, 1\rangle_\omega = \langle (I-P)(1), 1\rangle_\omega \\ =& \langle (I-P)(1), P(1)\rangle_\omega = 0 \\ \implies& 1-p = 0 \quad \text{since } 1-p \geq 0 \\ \implies& P = I \implies \pi(M)'1 \text{ dense in } H\end{aligned}$$

Let $T \in \pi(M)'$.

$||x_\alpha||_\omega \to 0 \implies \pi_{x_\alpha} \overset{\textbf{sot}}{\to} 0$

Let $T \in \pi(M)'$.

$$||\pi_{x_\alpha} T1||_\omega^2 \leq ||T||^2 ||\pi_{x_\alpha} 1||_\omega^2 = ||T||^2 ||x_\alpha||_\omega^2 \to 0$$

$||x_\alpha||_\omega \to 0 \implies \pi_{x_\alpha} \xrightarrow{\textbf{sot}} 0$

Let $T \in \pi(M)'$.

$$||\pi_{x_\alpha} T1||_\omega^2 \leq ||T||^2 ||\pi_{x_\alpha} 1||_\omega^2 = ||T||^2 ||x_\alpha||_\omega^2 \to 0$$

Since $(x_\alpha)_\alpha$ is bounded, π_{x_α} is uniformly bounded. Let $h \in H$. By uniform boundedness and *strong convergence lemma* we get

Let $T \in \pi(M)'$.

$$||\pi_{x_\alpha} T1||_\omega^2 \leq ||T||^2 ||\pi_{x_\alpha} 1||_\omega^2 = ||T||^2 ||x_\alpha||_\omega^2 \to 0$$

Since $(x_\alpha)_\alpha$ is bounded, π_{x_α} is uniformly bounded. Let $h \in H$. By uniform boundedness and *strong convergence lemma* we get

$$\pi_{x_\alpha} h \overset{\omega}{\to} 0$$

We show $||x_\alpha||_\omega \to 0 \Longleftarrow \pi_{x_\alpha} \overset{\text{sot}}{\to} 0$:

$||x_\alpha||_\omega \to 0 \impliedby \pi_{x_\alpha} \overset{\text{sot}}{\to} 0$

We show $||x_\alpha||_\omega \to 0 \impliedby \pi_{x_\alpha} \overset{\text{sot}}{\to} 0$:

$$\pi_{x_\alpha} h \overset{\omega}{\to} 0 \;\forall h \in H \implies x_\alpha = \pi_{x_\alpha} 1 \overset{\omega}{\to} 0$$

Lemma

Let $(x_\alpha)_\alpha$ be bounded in M, $x_\alpha \xrightarrow{\omega} x$. Then $x \in M$ and $\pi_{x_\alpha} \xrightarrow{\text{sot}} \pi_x$.

Lemma

Let $(x_\alpha)_\alpha$ be bounded in M, $x_\alpha \stackrel{\omega}{\to} x$. Then $x \in M$ and $\pi_{x_\alpha} \stackrel{\text{sot}}{\to} \pi_x$.

Proof.

$$||x_\alpha - x_\beta||_\omega \stackrel{(\alpha,\beta)}{\to} 0 \implies \pi_{x_\alpha} - \pi_{x_\beta} \stackrel{\text{sot}}{\to} 0.$$

sot is complete on bounded subsets of M

$$\implies \exists y \in \mathcal{B}(H),\ \pi_{x_\alpha} \stackrel{\text{sot}}{\to} y.$$

Lemma

Let $(x_\alpha)_\alpha$ be bounded in M, $x_\alpha \stackrel{\omega}{\to} x$. Then $x \in M$ and $\pi_{x_\alpha} \stackrel{\text{sot}}{\to} \pi_x$.

Proof.

$$||x_\alpha - x_\beta||_\omega \stackrel{(\alpha,\beta)}{\to} 0 \implies \pi_{x_\alpha} - \pi_{x_\beta} \stackrel{\text{sot}}{\to} 0.$$

sot is complete on bounded subsets of M

$$\implies \exists y \in \mathcal{B}(H),\ \pi_{x_\alpha} \stackrel{\text{sot}}{\to} y.$$

$$\text{Since } \pi(M) \text{ vN-Algebra } y \in \pi(M) \implies \exists x' \in M\ \pi_{x_\alpha} \stackrel{\text{sot}}{\to} \pi_{x'}$$

$$\implies x_\alpha \stackrel{\omega}{\to} x' \implies x = x'$$

Proposition

Let $(M, \mathcal{J})$ be a dynamical system. Let ω be a faithful normal state on M satisfying

$$\omega((Tx)^*(Tx)) \leq \omega(x^*x) \text{ for all } T \in \mathcal{J}, x \in M.$$

Then there exists a $P \in \mathcal{B}(H)$ with $P\hat{T} = \hat{T}P = P$ for all $\hat{T} \in \hat{\mathcal{J}}$, where $\hat{\mathcal{J}}$ is the extended semigroup of $\mathcal{J}$ on $B(H)$. $\forall x \in M$ $Px \in \overline{conv}^{sot} \mathcal{J}x$. In particular, $Px \in M$ for $x \in M$ and therefore we get $P \in \pi(M)$.

Proof. $\mathcal{J}$ is extended by density via π to $\hat{\mathcal{J}}$, a contraction semigroup on $B(H)$ with norm $||\cdot||_\omega$.

Proof. $\mathcal{J}$ is extended by density via π to $\hat{\mathcal{J}}$, a contraction semigroup on $B(H)$ with norm $||\cdot||_{\omega}$.

By Birkhoff-Alaoglu Ergodic theorem:

Proof. $\mathcal{J}$ is extended by density via π to $\hat{\mathcal{J}}$, a contraction semigroup on $B(H)$ with norm $||\cdot||_\omega$.

By Birkhoff-Alaoglu Ergodic theorem:

$$\exists \hat{P} \in \mathcal{B}(H)\, \forall \hat{T} \in \hat{\mathcal{J}} \quad (\hat{P}\hat{T} = \hat{T}\hat{P} = \hat{P}) \wedge (\hat{P}x \in \overline{\text{conv}}^{||\cdot||_\omega} \hat{\mathcal{J}} x \;\forall x \in H)$$

Proof. $\mathcal{J}$ is extended by density via π to $\hat{\mathcal{J}}$, a contraction semigroup on $B(H)$ with norm $||\cdot||_\omega$.
By Birkhoff-Alaoglu Ergodic theorem:

$$\exists \hat{P} \in \mathcal{B}(H)\, \forall \hat{T} \in \hat{\mathcal{J}} \quad (\hat{P}\hat{T} = \hat{T}\hat{P} = \hat{P}) \wedge (\hat{P}x \in \overline{\mathrm{conv}}^{||\cdot||_\omega} \hat{\mathcal{J}}x \ \forall x \in H)$$

Let $x \in M$. We show that $\hat{P}x$ is in M.

Proof. $\mathcal{J}$ is extended by density via π to $\hat{\mathcal{J}}$, a contraction semigroup on $B(H)$ with norm $||\cdot||_\omega$.
By Birkhoff-Alaoglu Ergodic theorem:

$$\exists \hat{P} \in \mathcal{B}(H)\ \forall \hat{T} \in \hat{\mathcal{J}} \quad (\hat{P}\hat{T} = \hat{T}\hat{P} = \hat{P}) \wedge (\hat{P}x \in \overline{\text{conv}}^{||\cdot||_\omega} \hat{\mathcal{J}}x\ \forall x \in H)$$

Let $x \in M$. We show that $\hat{P}x$ is in M.

$$\hat{T}x \in M\ \forall \hat{T} \in \hat{\mathcal{J}}$$

Proof. $\mathcal{J}$ is extended by density via π to $\hat{\mathcal{J}}$, a contraction semigroup on $B(H)$ with norm $||\cdot||_\omega$.
By Birkhoff-Alaoglu Ergodic theorem:

$$\exists \hat{P} \in \mathcal{B}(H)\ \forall \hat{T} \in \hat{\mathcal{J}} \quad (\hat{P}\hat{T} = \hat{T}\hat{P} = \hat{P}) \wedge (\hat{P}x \in \overline{\mathrm{conv}}^{||\cdot||_\omega} \hat{\mathcal{J}}x\ \forall x \in H)$$

Let $x \in M$. We show that $\hat{P}x$ is in M.

$$\begin{aligned} &\hat{T}x \in M\ \forall \hat{T} \in \hat{\mathcal{J}} \\ \Longrightarrow\ &\hat{T}x \in M\ \forall \hat{T} \in \mathrm{conv}\hat{\mathcal{J}} \end{aligned}$$

We show, that $\overline{\mathrm{conv}}^{\|\cdot\|_\omega}\hat{\mathcal{J}}x \subseteq \overline{\mathrm{conv}}^{\mathrm{sot}}\hat{\mathcal{J}}x$:

We show, that $\overline{\text{conv}}^{||\cdot||_\omega}\hat{\mathcal{J}}x \subseteq \overline{\text{conv}}^{\text{sot}}\hat{\mathcal{J}}x$:

Let $t \in \overline{\text{conv}}^{||\cdot||_\omega}\hat{\mathcal{J}}x$ and find $(t_\alpha)_\alpha \xrightarrow{\omega} t,\ t_\alpha \in \text{conv}\hat{\mathcal{J}}x$.

We show, that $\overline{\text{conv}}^{||\cdot||_\omega}\hat{\mathcal{J}}x \subseteq \overline{\text{conv}}^{\text{sot}}\hat{\mathcal{J}}x$:

Let $t \in \overline{\text{conv}}^{||\cdot||_\omega}\hat{\mathcal{J}}x$ and find $(t_\alpha)_\alpha \xrightarrow{\omega} t,\ t_\alpha \in \text{conv}\hat{\mathcal{J}}x$.

$$\hat{\mathcal{J}} \text{ bounded} \implies (t_\alpha)_\alpha \text{ bounded}$$

We show, that $\overline{\text{conv}}^{||\cdot||_\omega}\hat{\mathcal{J}}x \subseteq \overline{\text{conv}}^{\text{sot}}\hat{\mathcal{J}}x$:

Let $t \in \overline{\text{conv}}^{||\cdot||_\omega}\hat{\mathcal{J}}x$ and find $(t_\alpha)_\alpha \overset{\omega}{\to} t,\ t_\alpha \in \text{conv}\hat{\mathcal{J}}x$.

$$\begin{aligned}\hat{\mathcal{J}} \text{ bounded} &\implies (t_\alpha)_\alpha \text{ bounded} \\ &\implies t_\alpha \overset{\text{sot}}{\to} t \implies t \in \overline{\text{conv}}^{\text{sot}}\hat{\mathcal{J}}x\end{aligned}$$

We show, that $\overline{\text{conv}}^{||\cdot||_\omega}\hat{\mathcal{J}}x \subseteq \overline{\text{conv}}^{\text{sot}}\hat{\mathcal{J}}x$:

Let $t \in \overline{\text{conv}}^{||\cdot||_\omega}\hat{\mathcal{J}}x$ and find $(t_\alpha)_\alpha \overset{\omega}{\to} t,\ t_\alpha \in \text{conv}\hat{\mathcal{J}}x$.

$$\begin{aligned} \hat{\mathcal{J}} \text{ bounded} &\implies (t_\alpha)_\alpha \text{ bounded} \\ &\implies t_\alpha \overset{\text{sot}}{\to} t \implies t \in \overline{\text{conv}}^{\text{sot}}\hat{\mathcal{J}}x \\ \hat{P}x \in \overline{\text{conv}}^{||\cdot||_\omega}\hat{\mathcal{J}}x &\implies \hat{P}x \in \overline{\text{conv}}^{\text{sot}}\hat{\mathcal{J}}x \implies \hat{P}x \in M \end{aligned}$$

Proposition

Let $(M, \mathcal{J})$ be a dynamical system. Let ω be a faithful normal state on M satisfying

$$\omega((Tx)^*(Tx)) \leq \omega(x^*x) \text{ for all } T \in \mathcal{J}, x \in M.$$

Then there exists a $P \in \mathcal{B}(H)$ with $P\hat{T} = \hat{T}P = P$ for all $\hat{T} \in \hat{\mathcal{J}}$, where $\hat{\mathcal{J}}$ is the extended Semigroup of $\mathcal{J}$ on $B(H)$. $\forall x \in M$ $Px \in \overline{\mathrm{conv}}^{sot}\mathcal{J}x$. In particular $Px \in M$ for $x \in M$ and therefore we get $P \in \pi(M)$.

Theorem

Let $(M, \mathcal{J})$ be a dynamical system. If there exists a faithful normal state ω of normal states on M satisfying

$$\omega((Tx)^*(Tx)) \leq \omega(x^*x) \text{ for all } T \in \mathcal{J}, x \in M,$$

then $\mathcal{J}$ is sot-ergodic.

Theorem

Let $(M, \mathcal{J})$ be a dynamical system. If there exists a faithful normal state ω of normal states on M satisfying

$$\omega((Tx)^*(Tx)) \leq \omega(x^*x) \text{ for all } T \in \mathcal{J}, x \in M,$$

then $\mathcal{J}$ is weak*-ergodic.

- J. Dixmier, *Les algèbres d'opérateurs dans l'espace Hilbertien*, Gauthier-Villars (1969).
- S. Sakai, *C*-algebras and W*algebras*, Springer (1971).
- B. Kümmerer, R. Nagel, *Mean ergodic semigroups on W^*-algebras*, Acta Sci. Math., 41 (1979).
- U. Krengel, *Ergodic theorems*, De Gruyter (1985).
- S. Popa, C. Anantharaman, *An introduction to II_1 factors*, Preprint.
- T. Eisner, B. Farkas, M. Haase, R. Nagel, *Operator Theoretic Aspects of Ergodic Theory*, Springer (2015).

Thank you for your attention!