Universal C^*-algebra of two projections

ISem 24 - Project 10

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Overview

1. Two subspaces in generic position of a Hilbert space $\mathcal{H}$: $M \cap N = M \cap N^{\perp} = M^{\perp} \cap N = M^{\perp} \cap N^{\perp} = \{0\}$ [1]
2. Existence of a C^*-algebra $C^*(p,q)$, s.t. $\exists$ a representation π of $C^*(p,q)$ with $\pi(p) = P$ and $\pi(q) = Q$ for all projections $P, Q \in \mathcal{H}$ [2]
3. Unitary equivalence of projections P and Q in a van-Neumann-algebra M: Find some unitary $U \in M$ satisfying $UPU^* = Q$ and minimising $\|1 - U\|$ [2]
4. Unitary equivalence of pairs of projections $\{P, Q\}$ and $\{P', Q'\}$ [2]

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[1] P. R. Halmos, *Two subspaces*, Trans. Amer. Math. Soc. 144 (1969), 381-389.

[2] I. Raeburn, A. M. Sinclair, The *C^*-algebra generated by two projections*, MATHEMATICA SCANDINAVICA 65 (1989), 278–290.

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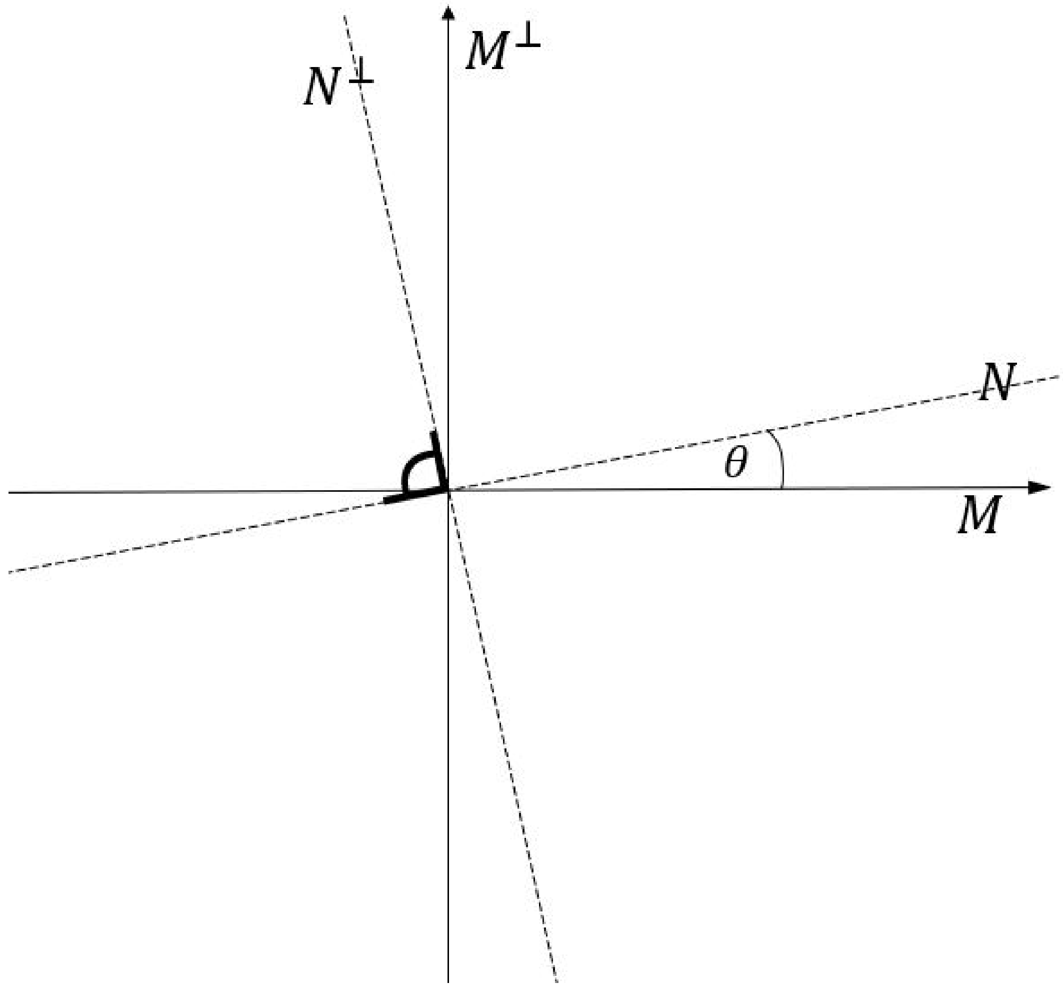
Two subspaces in generic position of a Hilbert space $\mathcal{H}$:
$M \cap N = M \cap N^{\perp} = M^{\perp} \cap N = M^{\perp} \cap N^{\perp} = \{0\}$



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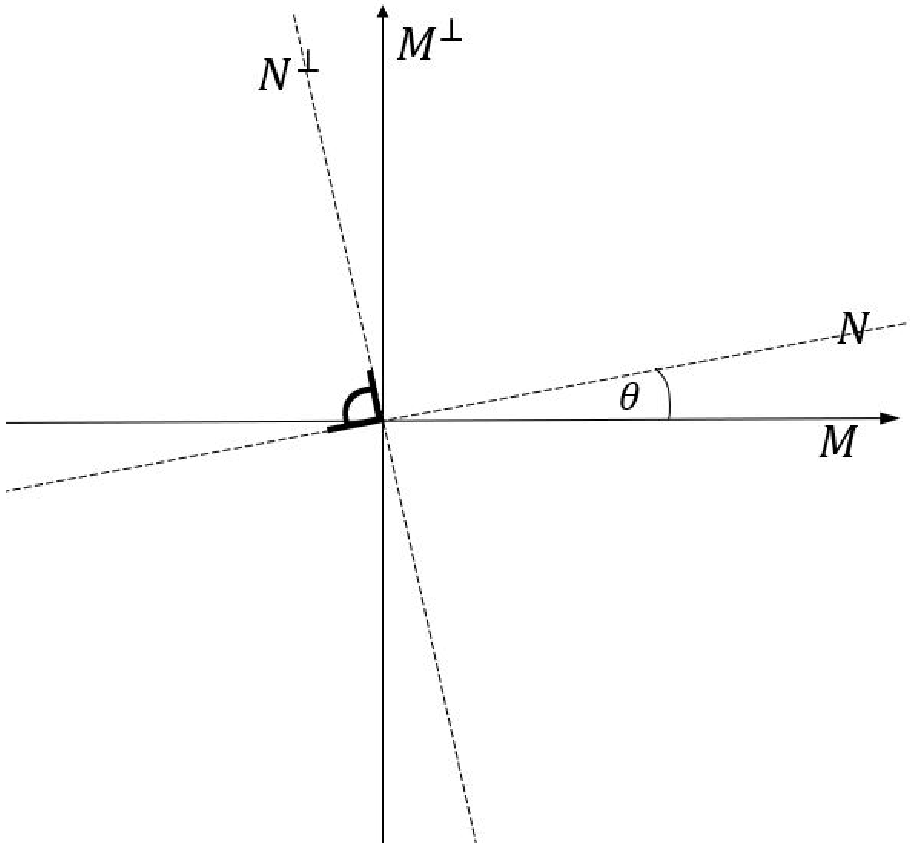


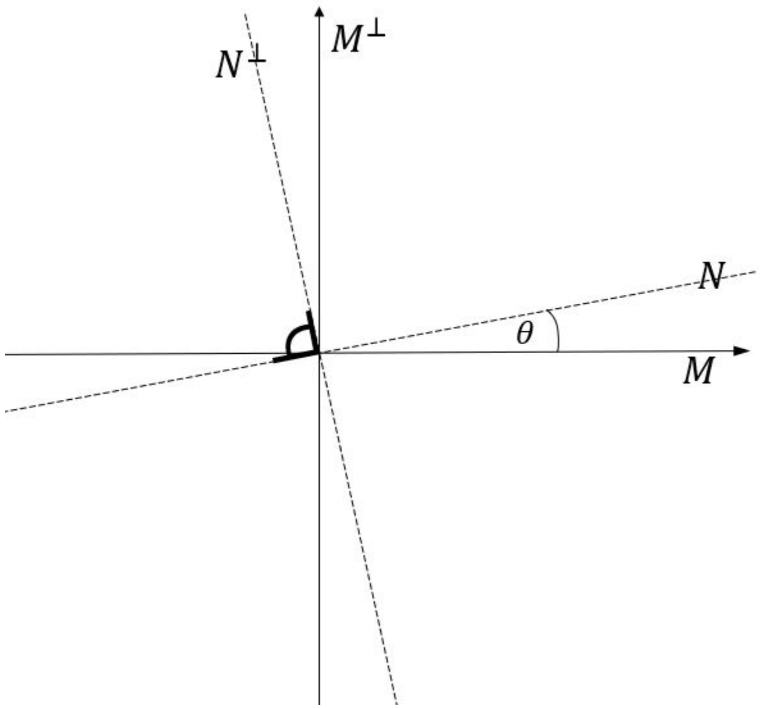
$$
\begin{aligned}
M &\leftrightarrow P \\
M^{\perp} &\leftrightarrow 1 - P \\
N &\leftrightarrow Q \\
N^{\perp} &\leftrightarrow 1 - Q
\end{aligned}
$$

- Topic arose in the study of invariant subspaces of operators
- Rotation of eigenvectors by perturbation

Motivation

- Perturbation theory
- K-theory of C^*-algebras
- Quantum mechanics

Two subspaces



1. T be a linear transformation on a dense subset of $\mathcal{K}$ (closed graph, zero kernel, dense range)
2. Write $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}$
3. M be the "horizontal axis" $\rightarrow \langle f, 0 \rangle$
4. N be the graph of $T \rightarrow \langle f, Tf \rangle$

$\boxed{M \cap N}$: $\langle f, 0 \rangle = \langle g, Tg \rangle \Rightarrow Tg = 0 \Rightarrow g = 0 \Rightarrow f = 0$

$\Rightarrow M \cap N = \{0\}$

$\rightsquigarrow M \cap N^{\perp} = M^{\perp} \cap N = M^{\perp} \cap N^{\perp} = \{0\}$

Theorem 1 *(Halmos)*: Let M and N be subspaces in generic position in a Hilbert space $\mathcal{H}$.

- $\exists$ Hilbert space $\mathcal{K}$,
- $\exists$ linear transformation T on $\mathcal{K}$ (closed graph, zero kernel and dense range), s.t.

$$\langle \mathcal{K}\oplus 0, \text{graph } T\rangle \sim \langle M, N\rangle \quad \text{(unitary equivalence)}$$

Unitary equivalence: $\langle M_1, N_1\rangle \sim \langle M_2, N_2\rangle$, if there exists a unitary operator U, s.t. $UM_1 = M_2$ and $UN_1 = N_2$.

Two subspaces

Proof of Theorem 1 *(Halmos)*

Let P be the orthogonal projection with range M and $P|_N$ be the restriction of P to N

- $P|_N$ is dense in M and has zero kernel
  - For all $g \in N$ be $Pg = 0$
    $\Rightarrow g \in M^{\perp} \cap N$
    $\Rightarrow g = 0$
  - Be $f \in M$ and $f \perp P|_N$
    $\Rightarrow$ If $g \in N$, then
    $0 = (f, Pg) = (Pf, g) = (f, g)$
    $\Rightarrow f \in M \cap N^{\perp}$
    $\Rightarrow f = 0$
- All spaces $M, N, M^{\perp}, N^{\perp}$ have the same dimension:
  $\exists$ isomerty $V : M \mapsto M^{\perp}$

Two subspaces

Proof of Theorem 1 *(Halmos)*

- $\boxed{\mathcal{K} := M}$
- Define T on the dense subset $P|_N$ of M by
  $\boxed{TPg = V^{-1}(1-P)g} \quad (g \in N)$
  - Idea: $\langle f, Tf \rangle$ in the decomposition $\mathcal{H} = M \oplus M^{\perp}$ would be Pg and $(1-P)g$. The "closesest" Tf can come to $(1-P)g$ is $V^{-1}(1-P)g$
- If f and g are in $\mathcal{K}\,(= M)$: $\boxed{U\langle f, g\rangle = f + Vg}$

It remains to show, that K, T and U have the required properties.

Theorem 2 *(Halmos)*:

If M and N are subspaces in generic position in $\mathcal{H}$, with respective projections P and Q, then

- $\exists$ a Hilbert space $\mathcal{K}$ and
- $\exists$ positive contractions S and C on $\mathcal{K}$, with
- $S^2 + C^2 = 1$ and $ker\, S = ker\, C = 0$, s.t.

$$P \sim \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Q \sim \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix} \quad \text{respectively.}$$

Two subspaces

Proof (sketch) of Theorem 2 *(Halmos)*

Assertion:

$$P \sim \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Q \sim \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix}$$

- Identify $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}$
- M and $M^{\perp}$ are the axis $\mathcal{K} \oplus 0$ and $0 \oplus \mathcal{K}$
- Idea:
  projection of rank 1 acting on a space of dimension 2, whose range is line of inclination with angle θ is

$$\begin{pmatrix} cos^2\theta & cos\theta\, sin\theta \\ cos\theta\, sin\theta & sin^2\theta \end{pmatrix}$$

Theorem 3 *(Halmos)*: If M and N are subspaces in generic position in a Hilbert space $\mathcal{H}$, then there exists a Hilbert space $\mathcal{K}$, and there exists a positive contraction T_0 on $\mathcal{K}$, with

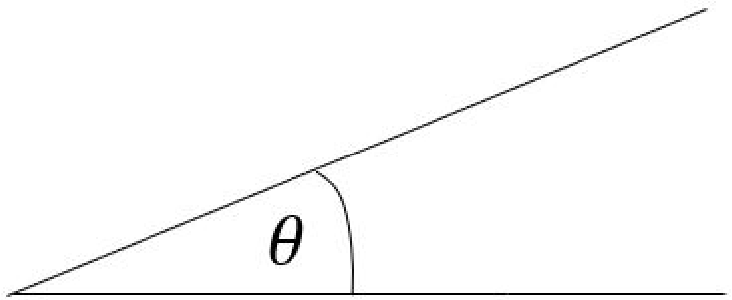
$$ker\ T_0 = ker(1 - T_0) = 0, \text{ such that}$$

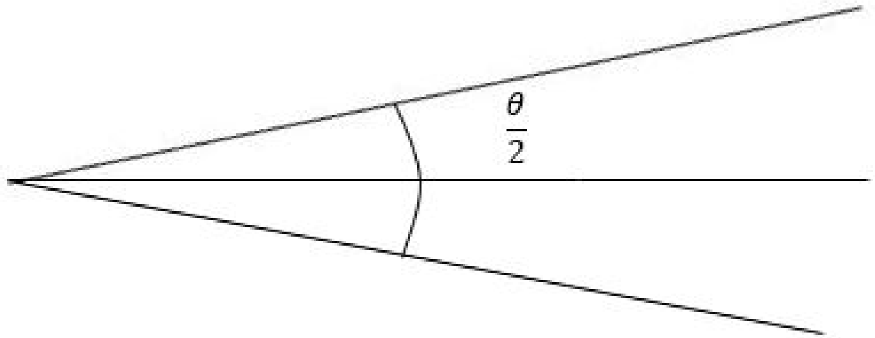
$$\langle M, N\rangle \sim \langle graph\, T_0, graph\,(-T_0)\rangle$$

Idea of the proof:
Underlying geometric fact: $\measuredangle$ rotated by $\theta/2$

Two subspaces

Theorem 3 *(Halmos)*





Corollary:
The unitary equivalence class of $\langle M, N\rangle$ is the one of the Hermitian operator $P+Q$.

Proof (sketch):

- Use Theorem 3 to see:

$$P = \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} C^2 & -CS \\ -CS & S^2 \end{pmatrix}$$ [1]

- Consider $R = P + Q - 1 = \begin{pmatrix} \hat{C} & 0 \\ 0 & -\hat{C} \end{pmatrix}$

- Unitary equivalence class of R determines that of $\hat{C}$, and thence that of P and Q

[1] $C^2 + C^2 = C^2 + (1 - S^2) := \hat{C} + 1$ and $S^2 + S^2 = S^2 + (1 - C^2) = -\hat{C} + 1$

Iain RAEBURN/Allan M. SINCLAIR: The C^*-Algebra Generated by Two Projections, Math.Scand. 65(1989), 278-290

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What is it about?

1. The existence of a free algebra generated by two projections.

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1. The existence of a free algebra generated by two projections.
2. A very convenient isomorphic version.

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What is it about?

1. The existence of a free algebra generated by two projections.
2. A very convenient isomorphic version.
3. A decomposition of a representation of the algebra.

Raeburn/Sinclair §1

1. The existence

"There is a unital C*-algebra A generated by two projections p, q with the following universal property: whenever P, Q are a pair of projections in a unital C*-algebra B, there is a unital homomorphism

$$\Phi : A \to B \text{ such that } \Phi(p) = P \text{ and } \Phi(q) = Q \text{ ."}$$

(Proposition 1.1. in Raeburn/Sinclair)

Raeburn/Sinclair §1

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- The algebra A is given as the concrete object $C^*(\mathbb{Z}_2 * \mathbb{Z}_2)$ (i.e. the free product of two copies of $\mathbb{Z}_2$).

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	1	u
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- $\mathbb{Z}_2$ is generated by a self-adjoint unitary $u \neq 1$ (resp. v for the second copy).
- If you take $p = \frac{1-u}{2}$, $q = \frac{1-v}{2}$, you obtain the two projections, which generate the same C^*-algebra.

	1	u
1	1	u
u	u	1

Raeburn/Sinclair §1

2. The isomorphic version

"There is an isomorphism of the C*-algebra $C^*(p, q)$ generated by two projections onto

$$A = \{f \in C([0,1], M_2(\mathbb{C})) : f(0), f(1) \text{ are diagonal}\}."$$

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(Theorem 1.3. in Raeburn/Sinclair)

- The elements of A are continuous functions with the matrix-multiplication as the multiplication, i.e. for $x \in [0,1]$, $f, g \in A$:

$$(f \underbrace{\cdot}_{\text{mult. in } A} g)(x) = f(x) \underbrace{\cdot}_{\text{mult. in } M_2(\mathbb{C})} g(x)$$

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- $$p(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad q(x) = \begin{pmatrix} x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & 1-x \end{pmatrix}$$

[cf. $x = cos^2\theta$ in the Halmos-paper]

Raeburn/Sinclair §1

2. The isomorphic version

Remember: $A = \{f \in C([0,1], M_2(\mathbb{C})) : f(0), f(1) \text{ are diagonal}\}$
$\exists\, \Phi : C^*(p,q) \to A$ with

$$\Phi(p) = \left(x \mapsto p(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) \quad \text{and}$$

$$\Phi(q) = \left(x \mapsto q(x) = \begin{pmatrix} x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & 1-x \end{pmatrix}\right).$$

Let $B := \Phi(C^*(p,q))$.

Φ is surjective (idea of proof):

$$p(x) \cdot q(x) \cdot p(x) = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \in B$$

$$\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}^n = \begin{pmatrix} x^n & 0 \\ 0 & 0 \end{pmatrix} \in B \qquad (n \in \mathbb{N})$$

$$\Rightarrow \forall f_{11} \text{ polynomial: } \begin{pmatrix} f_{11}(x) & 0 \\ 0 & 0 \end{pmatrix} \in B$$

Raeburn/Sinclair §1

2. The isomorphic version

Remember: $A = \{f \in C([0,1], M_2(\mathbb{C})) : f(0), f(1) \text{ are diagonal}\}$
$\Phi : C^*(p,q) \to A$

Φ is injective:

This is (not) shown in three steps.

- Every irreducible representation of $C^*(p,q)$ must be one- or two-dimensional.
- Every irreducible representation of $C^*(p,q)$ factors through A.
- Φ is injective.

$$\begin{tikzcd}
C^*(p,q) \arrow[r, "\pi \text{ irr.}"] \arrow[d, "\Phi"'] & B(H) \\
A \arrow[ur] &
\end{tikzcd}$$

$$\Phi(f) = \Phi(g) \Rightarrow \pi(f) = \pi(g)$$

- Every irreducible representation of $C^*(p,q)$ factors through A.

  $\Downarrow$

- Φ is injective.

$$\begin{tikzcd} C^*(p,q) \arrow[r, "\pi \text{ irr.}"] \arrow[d, "\Phi"'] & B(H) \\ A \arrow[ur] & \end{tikzcd}$$

$$\Phi(f) = \Phi(g) \Rightarrow \pi(f) = \pi(g)$$

Assume Φ is not injective. Then there is a $0 \neq a \in C^*(p,q)$ with $\Phi(a) = 0$. By Isem-lecture notes Rem. 5.31 then there is an irreducible representation π of $C^*(p,q)$ with $\|\pi(a)\| = \|a\|$. But $\Phi(a) = 0 = \Phi(0)$ entails by the last item that $\pi(a) = \pi(0) = 0$, so $a = 0$ and we have a contradiction.

Raeburn/Sinclair §1

3. The decomposition

" π has a direct sum decomposition $\pi = \pi_c \oplus \pi_0 \oplus \pi_1$, in which π_c is nondegenerate on the ideal I, π_0 factors through the map $f \to f(0)$, and π_1 factors through $f \to f(1)$. Further we can identify the summands as follows:

(1) If $\{f_n\}$ is an approximate identity in I, then $\pi(f_n)$ converges strongly to the projection onto $H_c = H(\pi_c) = \overline{\pi(I)H}$;

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(3) H_1 is the direct sum of the subspaces $H_1^p = \pi_1(p)H = \pi_1(q)H$ and $H_1^{1-p} = \pi_1(1-p)H = \pi_1(1-q)H$."

(Lemma 1.8. in Raeburn/Sinclair)

$$A = \{f \in C([0,1], M_2(\mathbb{C})) \,:\, f(0), f(1) \text{ are diagonal}\}$$

$$I := \{f \in A \,:\, f(0) = f(1) = 0\}$$

I is an ideal in A inducing the equivalence relation

$$f \sim g \Leftrightarrow f - g \in I \Leftrightarrow f(0) = g(0) \text{ and } f(1) = g(1)$$

Let $\pi : A \to B(H)$ be a representation.

Raeburn/Sinclair §1

3. The decomposition

$$f \in A = C^*(p,q), \ \pi(f) \in B(H)$$

$$I = \{f \in A : f(0) = f(1) = 0\}$$

$$H_c := \overline{span(\pi(I)H)}$$

$$\pi(f)|_H = \begin{pmatrix} \pi(f)|_{H_c} & 0 \\ 0 & \pi(f)|_{H_c^\perp} \end{pmatrix}$$

Raeburn/Sinclair §1

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$$\pi(f)|_H = \begin{pmatrix} \pi(f)|_{H_c} & 0 \\ 0 & \pi(f)|_{H_c^\perp} \end{pmatrix}$$

$$\begin{tikzcd} A \arrow[rr, "\pi"] \arrow[d, "f \mapsto f(0) \oplus f(1)"'] & & B(H_c^\perp) \\ \mathbb{C}^2 \oplus \mathbb{C}^2 \arrow[urr, "\phi"'] & & \end{tikzcd}$$

$\pi|_I = 0$ on $H_c^\perp$ $\qquad$ I ist the kernel of $f \mapsto f(0) \oplus f(1)$

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$$\begin{tikzcd} f \arrow[d, mapsto] & A \arrow[r, "\pi"] \arrow[d] & B(H_c^\perp) \\ f(0) \oplus f(1) & \mathbb{C}^2 \oplus \mathbb{C}^2 \arrow[ur, "\Phi"] & \end{tikzcd}$$

$\pi|_I = 0$ on $H_c^\perp$ $\qquad$ I ist the kernel of $f \mapsto f(0) \oplus f(1)$

$$\pi(f) = \pi_c(f) \oplus \Phi(f(0) \oplus f(1)) = \pi_c(f) \oplus \pi_0(f) \oplus \pi_1(f)$$

Raeburn/Sinclair §1

3. The decomposition

$$
\begin{array}{ccccccccc}
A \ni & \mathrm{f} & & \text{on } I & & \text{on } f(0) & & \text{on } f(1) \\
& \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
B(H) \ni & \pi(f) & = & \pi_c(f) & \oplus & \pi_0(f) & \oplus & \pi_1(f) \\
& \Downarrow & & & & & & \\
& H & = & H_c & \oplus & & H_c^{\perp} & \\
& & = & H_c & \oplus & H_0 & \oplus & H_1
\end{array}
$$

Raeburn/Sinclair §1

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& \Downarrow & & & & & & \\
& H & = & H_c & \oplus & & H_c^{\perp} & \\
& & = & H_c & \oplus & \overbrace{}^{H_0} & \oplus & \overbrace{\phantom{H_1^p \oplus H_1^{1-p}}}^{H_1} \\
& & = & H_c & \oplus & H_0^p \oplus H_0^q & \oplus & H_1^p \oplus H_1^{1-p}
\end{array}
$$

$$
\begin{aligned}
H_0^p &= \pi_0(p)H \\
H_0^q &= \pi_0(q)H
\end{aligned}
\qquad\qquad
\begin{aligned}
H_1^p &= \pi_1(p)H \\
&= \pi_1(q)H \\
H_1^{1-p} &= \pi_1(1-p)H \\
&= \pi_1(1-q)H
\end{aligned}
$$

Raeburn/Sinclair §1

Comparison with Halmos

$$
\begin{aligned}
H_1^p &= \pi_1(p)H = \pi_1(q)H = \operatorname{ran}\pi(p) \cap \operatorname{ran}\pi(q) =: M \cap N \\
H_1^{1-p} &= \pi_1(1-p)H = \pi_1(1-q)H = \ker\pi(p) \cap \ker\pi(q) = M^\perp \cap N^\perp \\
H_0^p &= \pi_0(p)H = \operatorname{ran}\pi(p) \cap \ker\pi(q) = M \cap N^\perp \\
H_0^q &= \pi_0(q)H = \ker\pi(p) \cap \operatorname{ran}\pi(q) = M^\perp \cap N
\end{aligned}
$$

So, if M and N are in generic position, all of these spaces will vanish and we will be back in the constellation of Halmos.

State of the art so far...

- Whenever there is a pair of projections P and Q on a Hilbert space in a unital C^*-algebra, there is a unital homomorphism π such as
$$\pi(p) = P \quad and \quad \pi(q) = Q$$
- π has a direct sum decomposition
$$\pi = \pi_c \oplus \pi_0 \oplus \pi_1$$

Unitary equivalence of two projections

Theorem 2.1 (Raeburn/Sinclair)

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Suppose P and Q are projections in a von Neumann algebra M.

If it exists an element W such that :

1. WW^* is the projection onto $\ker P \cap \operatorname{ran} Q$
2. W^*W is the projection onto $\ker Q \cap \operatorname{ran} P$

Then there is a unitary $U \in M$ such that :

a. $UPU^* = Q$

b. U commutes with $\mid P - Q \mid$

c. $\mid 1 - U \mid = \sqrt{2}(1 - (1 - \mid P - Q \mid^2)^{\frac{1}{2}})^{\frac{1}{2}} \leq \sqrt{2} \mid P - Q \mid$

Proof

Let $M \subset B(H)$ and π be the representation of $C^*(p,q)$ such that

$$\pi(p) = P \quad and \quad \pi(q) = Q$$

We consider the ideal

$$J = \{f \in C^*(p,q) : f(0) = 0\}$$

By the decomposition result, π has a direct sum decomposition

$$\pi = \pi_c' \oplus \pi_0 \oplus \pi_1$$

- π_c' is nondegenerate on the ideal J
- π_0 and π_1 factors through the map $f \to f(0)$ and $f \to f(1)$ respectively

Proof

We take

$$\pi_c = \pi_c' \oplus \pi_1$$

We get

$$\pi = \pi_c \oplus \pi_0$$

- π_c is nondegenerate on the ideal J
- π_0 factors through the map $f \to f(0)$

We first solve the problem in $C([0,1], M_2(\mathbb{C}))$ and then transfer the solution to the von Neumann algebra $M \subset B(H)$.

Proof

- We verify the required properties (a, b and c) relative to the projections $\pi_c(p)$ and $\pi_c(q)$.

$$p(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad q(x) = \begin{pmatrix} x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & 1-x \end{pmatrix}$$

For each pair $(p(x), q(x))$ we set

$$u(x) = \begin{pmatrix} \sqrt{x} & -\sqrt{1-x} \\ \sqrt{1-x} & \sqrt{x} \end{pmatrix}$$

Proof

Property a. p and q are unitary equivalent

By direct calculation we get

$$u(x)p(x)u^*(x) = q(x)$$

where

$$u^*(x) = \begin{pmatrix} \sqrt{x} & \sqrt{1-x} \\ -\sqrt{1-x} & \sqrt{x} \end{pmatrix}$$

Property b. $u(x)$ commutes with $\mid p(x) - q(x) \mid$

$$\mid p(x) - q(x) \mid = \sqrt{1-x}\mathbf{1}$$

Proof

Property c. minimizing $\mid 1 - u(x) \mid$

$$\mid 1 - u(x) \mid = \sqrt{2}\sqrt{1 - \sqrt{x}}\mathbf{1}$$

By

$$\sqrt{2}\sqrt{1 - \sqrt{x}} = \sqrt{2}\sqrt{1 - \sqrt{1 - (1 - x)}} \leq \sqrt{2}\sqrt{1 - x}$$

We get

$$\mid 1 - u(x) \mid = \sqrt{2}(1 - (1 - \mid p(x) - q(x) \mid^2)^{\frac{1}{2}})^{\frac{1}{2}} \leq \sqrt{2} \mid p(x) - q(x) \mid$$

Proof

- To complete the proof of **Theorem 2.1** we verify the properties (a, b and c) for H_0

Property a. p and q are unitary equivalent

We set

$$U_1 = W - W^*$$

Where W an element such that

- WW^* is the projection onto $\ker P \cap \operatorname{ran} Q$
- W^*W is the projection onto $\ker Q \cap \operatorname{ran} P$

Using $W = WW^*W$ we get U_1 is unitary $(U_1U_1^* = 1$ and $U_1^*U_1 = 1)$

$$U_1(x)\pi_0(p)U_1^*(x) = \pi_0(q)$$

Proof

Property b. U_1 commutes with $\mid \pi_0(p) - \pi_0(q) \mid$

$$\mid \pi_0(p) - \pi_0(q) \mid = \mid \pi_0(p(0)) - \pi_0(q(0)) \mid = \mid \pi_0(1) \mid = \mathbf{1}$$

Property c. minimizing $\mid 1 - U_1 \mid$

$$\mid 1 - U_1 \mid^2 = (1 - U_1^*)(1 - U_1)$$

$$\mid 1 - U_1 \mid^2 = 1\mathbf{1} + U_1^* U_1 = 2\mathbf{1}$$

Thus U_1 has the required properties for H_0

Proof

For

$$U = \pi_c(u) + U_1$$

a. $UPU^* = Q$
b. U commutes with $\mid P - Q \mid$
c. $\mid 1 - U \mid = \sqrt{2}(1 - (1 - \mid P - Q \mid^2)^{\frac{1}{2}})^{\frac{1}{2}} \leq \sqrt{2} \mid P - Q \mid$

hold for a unitary $U \in M$.

Which completes the proof of **Theorem 2.1**.

Remark 2.3 (Raeburn/Sinclair)

If we suppose that P and Q are projections in a von Neumann algebra M satisfying $\|P - Q\| < 1$ then

- There is a unitary $U \in M$ such that $UPU^* = Q$
- $$\|1 - U\| = \sqrt{2}(1 - (1 - \|P - Q\|^2)^{\frac{1}{2}})^{\frac{1}{2}} \leq \sqrt{2}\|P - Q\|$$

and the constant $\sqrt{2}$ is the best possible constant for P and Q.

Proof

Let

$$\xi \in \ker P \cap \operatorname{ran} Q$$

then $P\xi = 0$ and $\exists X$, such that $QX = \xi$

Thus $\xi = QX = Q^2X = Q\xi$ and

$$\|(P-Q)\xi\| = \|\xi\| \leq \|P-Q\|\|\xi\|$$

The condition $\|P-Q\| < 1$ implies $\xi = 0$

$$\ker P \cap \operatorname{ran} Q = \ker Q \cap \operatorname{ran} P = \{0\}$$

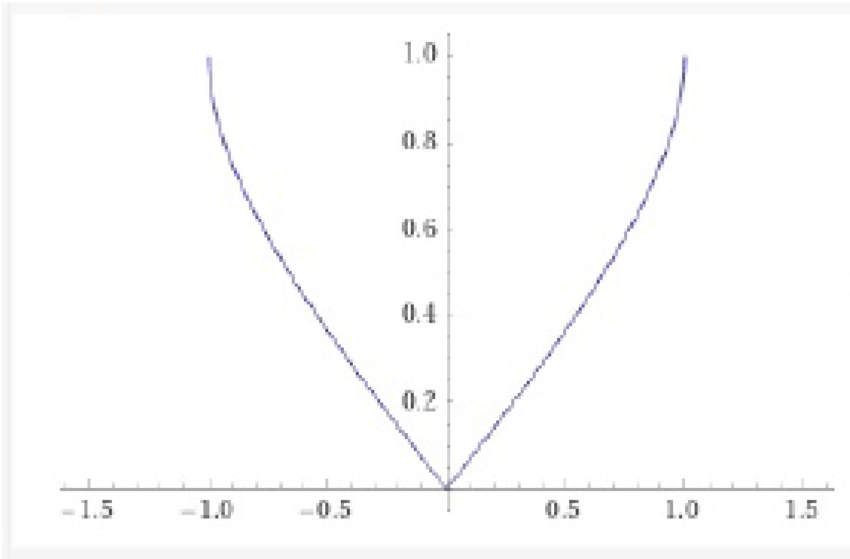
Theorem 2.1 applies for $W = 0$

Proof

Observe that f is increasing on $[0,1]$

$$f(t) = (1-(1-t^2)^{1/2})^{1/2}$$

Therefore $|f(\|S\|)| = f(\|S\|)$ for all operators S with $\|S\| \leq 1$.



$$\|1-U\| = \sqrt{2}(1-(1-\|P-Q\|^2)^{\frac{1}{2}})^{\frac{1}{2}} \leq \sqrt{2}\|P-Q\|$$

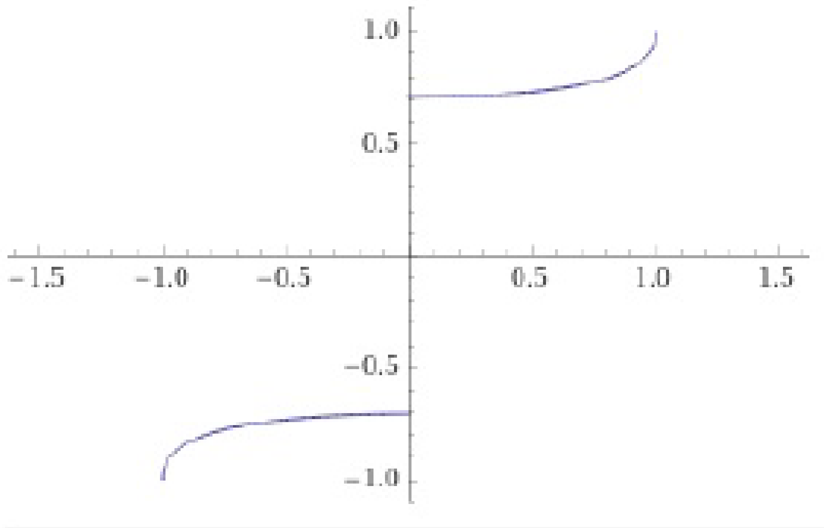
Proof

Besides the inequality $(1-(1-t^2)^{1/2})^{1/2} \leq t$ holds for $t \in [0,1]$

$$g(t) = \frac{(1-(1-t^2)^{1/2})^{1/2}}{t}$$

is also increasing in $[0,1]$, thus for $0 \leq t \leq \delta \leq 1$ we have

$$\frac{(1-(1-t^2)^{1/2})^{1/2}}{t} \leq \frac{(1-(1-\delta^2)^{1/2})^{1/2}}{\delta}$$



Proof

For $\delta < 1$, where $\delta = \sqrt{1-x}$

$$\|p(x) - q(x)\| = \delta$$

Let v satisfies $v(x)p(x)v(x)^* = q(x)$,

$$v(x) = \begin{pmatrix} \lambda\sqrt{x} & -\mu\sqrt{1-x} \\ \lambda\sqrt{1-x} & \mu\sqrt{x} \end{pmatrix}$$

for $\lambda, \mu \in$ with $|\lambda| = |\mu| = 1$

Thus

$$\|1 - v\| \geq \sqrt{2}\sqrt{1-\sqrt{x}} = \sqrt{2}(1-(1-\delta^2)^{1/2})^{1/2}$$

Letting $\delta \to 1$ shows that $\sqrt{2}$ is the best possible value for arbitary P and Q.

Unitary equivalence of two **finite** projections

Corollary 2.4 (Raeburn/Sinclair)

Corollary 2.4 (Raeburn/Sinclair)

P, Q are two finite projections in a von Neumann algebra M.

P and Q are equivalent, i.e., there exists $T \in M$ such that

- $TT^* = P$
- $T^*T = Q$

if and only if there exists an element $W \in M$ such that

- WW^* is the projection onto $\ker P \cap \operatorname{ran} Q$
- W^*W is the projection onto $\ker Q \cap \operatorname{ran} P$

If so, there is a unitary $U \in M$ such that

a. $UPU^* = Q$

b. $U|P - Q| = |P - Q|U$

c. $|1 - U| = \sqrt{2}(1 - (1 - |P - Q|^2)^{1/2})^{1/2} \leq \sqrt{2}|P - Q|$

Proof

Let $\pi : C^*(p,q) \to B(H)$ be the representation such as $\pi(p) = P$ and $\pi(q) = Q$

$$\pi = \pi_c \oplus \pi_0$$

** As the unitary u satisfies $\pi_c(upu^*) = \pi_c(q)$, $\pi_c(p)$ is always equivalent to $\pi_c(q)$.

** Because P, Q are finite, P is equivalent to Q if and only if $\pi_0(p) = P - \pi_c(p)$ is equivalent to $\pi_0(q) = Q - \pi_c(q)$.

- $\operatorname{ran} \pi_0(p) = \ker Q \cap \operatorname{ran} P$
- $\operatorname{ran} \pi_0(q) = \ker P \cap \operatorname{ran} Q$

so $\pi_0(p)$ equivalent to $\pi_0(q)$ means precisely that $\exists W \in M$ as claimed, which ends the proof.

Unitary equivalence of two **finite** projections

Corollary 2.5 (Raeburn/Sinclair)

P, Q are two finite projections in a von Neumann algebra M.

P and Q are equivalent, i.e., there exists $T \in M$ and it exists an element $W \in M$ such that

- WW^* is the projection onto $\ker P \cap \operatorname{ran} Q$
- W^*W is the projection onto $\ker Q \cap \operatorname{ran} P$

Then there is an element V of M such that

a. $VV^* = Q$ and $V^*V = P$
b. $V|P - Q| = |P - Q|V$
c. $|P - V| \leq \sqrt{2}|P - Q|$ and $|Q - V| \leq \sqrt{2}|P - Q|$

Proof

First hypothesis implies the existence of partial isometry W.
Now let

$$\pi = \pi_c \oplus \pi_0$$

and consider

$$V(x) = \begin{pmatrix} \sqrt{x} & 0 \\ \sqrt{1-x} & 0 \end{pmatrix} \in M(J)$$

This has all the properties relative to $p, q \in M(J)$.

$$V = \pi_c(v) + W$$

is an element of the von Neumann algebra M satisfying a, b, and c.

Unitary equivalence of pairs of projections

Theorem 3.1. (Raeburn, Sinclair) Let H be a Hilbert space. Fix $\lambda > 1$ and two pairs of orthogonal projections P, Q and P', Q'. Then, the following assertions are equivalent.

(i) There is a unitary operator U such that

$$UPU^* = P' \text{ and } UQU^* = Q'$$

(ii) There is a unitary operator U such that

$$U(\lambda P + Q)U^* = \lambda P' + Q',$$

i.e., $\lambda P + Q$ is unitarily equivalent to $\lambda P' + Q'$.

Remarks

1. By swapping P and Q also $\lambda \in (0,1)$ can be considered.
2. The theorem is a version of Dixmier's theorem:
   - **Question:** Let P, Q and P', Q' be two pairs of projections. When is there a unitary U such that
     $$UPU^* = P' \text{ and } UQU^* = Q'?$$
   - **Diximier.** Let P, Q be in generic position, i.e.,
     $$\ker(P) \cap \ker(1-Q) = \ker(Q) \cap \ker(1-P) = \{0\}$$
     $$\ker(P) \cap \ker(Q) = \ker(1-Q) \cap \ker(1-P) = \{0\}.$$
     Then, the self-adjoint operator $P+Q$ is a complete unitary invariant of the pair P, Q. That corresponds to the theorem with parameter $\lambda = 1$.

Proof (main idea)

- It is clear that Assertion (i) implies Assertion (ii). Hence, only the reverse implications needs a proof.
- Consider the C^*-algebra $C^*(p, q)$ and define

$$s := \lambda p + q.$$

- The key idea is that an irreducible representation of $C^*(p, q)$ is determined up to unitary equivalence by its restriction to $C^*(s)$.
- For Diximier's version of the theorem, the subalgebra $C^*(a)$, $a = p + q$ does not distinguish between the irreducible components of the representation $f \mapsto f(0)$. Thus, extra assumptions are needed, namely that P, Q are in generic position.

Proof (sketch)

- Assume $\lambda P + Q$ is unitary equivalent to $\lambda P' + Q'$.
- Let π, ρ be representations of $C^*(p,q)$ with

$$\pi(p) = P, \quad \pi(q) = Q \text{ and } \rho(p) = P', \quad \rho(q) = Q'.$$

- Write

$$s = \lambda p + q.$$

  Then $\pi_{|C^*(s)}$ and $\rho_{|C^*(s)}$ are unitarily equivalent. Without loss of generality assume $\pi_{|C^*(s)} = \rho_{|C^*(s)}$.

We know that there is an isomorphism of $C^*(p, q)$ generated by two projections onto

$$A := \{f : C([0,1]; M_2(\mathbb{C})) : f(0), f(1) \text{ diagonal}\}.$$

The elements p, q corresponds to

$$p(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } q(x) = \begin{pmatrix} x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & 1-x \end{pmatrix},$$

$p, q \in A$. Hence

$$s(x) = \begin{pmatrix} \lambda + x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & 1-x \end{pmatrix},$$

$s \in A$.

Computing $\sigma(s)$

- The Gelfand transform induces an isomporphism $C^*(s)$ onto $C(\sigma(s))$.
- An element $f \in A$ is invertible if and only if $f(x)$ is invertible for all x. Thus,

$$\sigma(s) = \bigcup_{x \in [0,1]} \sigma(s(x)) = \bigcup_{x \in [0,1]} \frac{1}{2}(1 + \lambda \pm \sqrt{(\lambda - 1)^2 + 4\lambda x})$$
$$= [0,1] \cup [\lambda, \lambda + 1].$$

Decomposing π and ρ

- Consider the ideal $I = \{f \in C^*(p,g) : f(0) = f(1) = 0\}$ as before and decompose

$$\pi = \pi_c \oplus \pi_0 \oplus \pi_1 \text{ and } \rho = \rho_c \oplus \rho_0 \oplus \rho_1.$$

- We consider the subspaces

$$\mathscr{H}(\pi_c), \mathscr{H}(\pi_0), \mathscr{H}(\pi_1) \text{ and } \mathscr{H}(\rho_c), \mathscr{H}(\rho_0), \mathscr{H}(\rho_1)$$

where $\mathscr{H}(\nu)$ is the so-called *essential space* of ν defined by

$$\mathscr{H}(\nu) := \overline{\mathrm{span}\{\nu(a)\xi : a \in C^*(p,q), \xi \in H\}}$$

for $\nu \in \{\pi_c, \pi_0, \pi_1, \rho_c, \rho_0, \rho_1\}$.

Claim: $\mathscr{H}(\pi_j) = \mathscr{H}(\rho_j)$, $j = c, 0, 1$

- $j = c$. First, let $f_n \in C_0((0,1) \cup (\lambda, \lambda+1))$ that is equal to 1 on
$$\left\{\frac{1}{2}(1+\lambda \pm \sqrt{(\lambda-1)^2 + 4\lambda x} : \frac{1}{n} \leq x \leq 1 - \frac{1}{n}\right\} \subset \sigma(s).$$
Since $f_k(s)(x) = 1$ for $k \geq n$, $\pi(f_n(s))$ (and $\rho(f_n(s))$) converges strongly to the orthogonal projection onto $\mathscr{H}(\pi_c)$ (and $\mathscr{H}(\rho_c)$). Since $\pi_{|C^*(s)} = \rho_{|C^*(s)}$ we obtain that $\mathscr{H}(\pi_c) = \mathscr{H}(\rho_c)$.
- $j = 0, 1$. $s(j), 1$ generate the diagonal subalgebra of $M_2(\mathbb{C})$. Since $f \mapsto f(j)$ is surjective on $C^*(s)$, the representations π_j, ρ_j factor through these quotient maps and $\pi_{|C^*(s)} = \rho_{|C^*(s)}$ we obtain $\pi_j = \rho_j$.

In remains to show: π_c is unitarily equivalent to ρ_c

We diagonalise s by using the following lemma.
Lemma 3.2. (Raeburn, Sinclair)

- Let $f \in C([0,1], M_2(\mathbb{C})$ be self-adjoint.
- Let $v \in C([0,1], \mathbb{C}^2)$ such that $v(x)$ is a unit eigenvector for $f(x)$, $x \in [0,1]$.
- Let $p_1(x)$ be the orthogonal projection onto $\mathrm{span}(v(x))$.

Then there is a $w \in C([0,1], M_2(\mathbb{C}))$ such that

$$w(x)^* w(x) = p_1(x), w(x)w(x)^* = 1 - p_1(x).$$

Moreover, we can find for arbitrary $g \in C([0,1], M_2(\mathbb{C}))$ functions $a, b, c, d \in C([0,1])$ such that

$$g = ap_1 + bw^* + cw + d(1 - p_1)$$

- We apply the Lemma 3.2 to s. After messy computations, we find a function $v \in C([0,1], M_2(\mathbb{C}))$ such that $v(x)$ is a unit eigenvector for $s(x)$, $x \in [0,1]$.
- Note that $p_1 = \mathbb{1}_{[\lambda,\lambda+1]}(s) \in C^*(s)$. So we set $\pi_c(p_1) = \rho_c(p_1) =: P_1$. Let $V = \pi_c(w)$, $W = \rho_c(w)$. Defining $U = W^* V + (1 - P_1)$ yields an unitary operator satisfying

$$U\pi_c(g) = \rho_c(g)U$$

for arbitrary $g \in C^*(p,q)$ using the decomposition of the Lemma.
- Hence, π_c is unitary equivalent to ρ_c.

Summary

1. Two subspaces in generic position of a Hilbert space $\mathcal{H}$:
   $M \cap N = M \cap N^{\perp} = M^{\perp} \cap N = M^{\perp} \cap N^{\perp} = \{0\}$
2. Existence of a C^*-algebra $C^*(p,q)$, s.t. $\exists$ a representation π of $C^*(p,q)$ with $\pi(p) = P$ and $\pi(q) = Q$ for all projections $P, Q \in \mathcal{H}$
3. Unitary equivalence of projections P and Q in a van-Neumann-algebra M: Find some unitary $U \in M$ satisfying $UPU^* = Q$ and minimising $\|1 - U\|$
4. Unitary equivalence of pairs of projections $\{P, Q\}$ and $\{P', Q'\}$

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