Project 11: W*-algebras

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Part I:
Strongly closed *-subalgebras of $\mathcal{B}(\mathcal{H})$

Titus Pinta, Aaron Kettner

The SOT

The strong operator topology (SOT):

- Locally convex topology on $\mathcal{B}(\mathcal{H})$ defined by the family of seminorms

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- Basis neighborhood of the origin:

$$\bigcap_{i=1}^{n} \mathcal{S}(0, x_i) = \{T \in \mathcal{B}(\mathcal{H}) : \|Tx_i\| < 1, 1 \leq i \leq n\}$$

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Some basic facts about the SOT:

- The adjoint map $T \mapsto T^*$ is not continuous

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The WOT

The weak operator topology (WOT):

- Locally convex topology on $\mathcal{B}(\mathcal{H})$ defined by the family of seminorms

$$p_{x,y}(T) = |(Tx, y)|$$

- A net converges WOT iff it converges pointwise weakly, i.e.

$$\lim_{\alpha} T_\alpha = T \iff \forall x, y \in \mathcal{H} : \lim_{\alpha}(T_\alpha x, y) = (Tx, y)$$

- Coarsest topology such that the sets $\mathcal{W}(T, x, y) = \{A \in \mathcal{B}(\mathcal{H}) : |((T - A)x, y)| < 1\}$ are open
- Basis neighborhood of the origin:

$$\bigcap_{i=1}^{n} \mathcal{W}(0, x_i, y_i) = \{T \in \mathcal{B}(\mathcal{H}) : |(Tx_i, y_i)| < 1, 1 \leq i \leq n\}$$

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- The adjoint map $T \mapsto T^*$ is continuous
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- The unit ball of $\mathcal{B}(\mathcal{H})$ is compact (adapted version of Banach-Alaoglu)

SOT and WOT

Theorem

The WOT-continuous linear functionals on $\mathcal{B}(\mathcal{H})$ and the SOT-continuous linear functionals coincide, and each functional has the form

$$f(T) = \sum_{i=1}^{n} (Tx_i, y_i)$$

for a finite set of vectors $x_1, ..., x_n, y_1, ..., y_n$ in $\mathcal{H}$.

Corollary

$\mathcal{B}(\mathcal{H})$ has the same closed convex sets in WOT and SOT.

Commutant of a W*-algebra

W*-Algebra: WOT-closed unital C^*-subalgebra of $\mathcal{B}(\mathcal{H})$
Commutant of a set $\mathcal{S} \subset \mathcal{B}(\mathcal{H})$:

$$\mathcal{S}' := \{T \in \mathcal{B}(\mathcal{H}) : ST = TS \text{ for all } S \in \mathcal{S}\}$$

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- $\mathcal{S}'$ is a WOT-closed unital algebra
- S selfadjoint $\Longrightarrow \mathcal{S}'$ is selfadjoint, hence a W*-Algebra

Von Neumann bicommutant theorem

Theorem (Von Neumann bicommutant theorem)

Suppose that $\mathcal{U}$ is a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ with trivial null space (i.e. $\mathcal{U}y = 0$ implies $y = 0$). Then

$$\mathcal{U}'' = \overline{\mathcal{U}}^{SOT} = \overline{\mathcal{U}}^{WOT}.$$

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Proof: $\overline{\mathcal{U}}^{\mathrm{SOT}} \subset \mathcal{U}''$ is clear. Let $T \in \mathcal{U}''$ and fix $x_1, \ldots x_n \in \mathcal{H}$.
To show:

$$\exists A \in \mathcal{U} : \sum_{i=1}^{n} \|(T - A)x_i\|^2 < 1$$

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First consider $n = 1$. We will show that $Tx_1 \in \overline{\mathcal{U}x_1}$.

- Let $P := \left[\overline{\mathcal{U}x_1}\right]$. $\implies P \in \mathcal{U}' \implies PT = TP$

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$\Longrightarrow$ Use the $n = 1$ case with $\boldsymbol{x} = (x_1, ..., x_n)$ to find an $A \in \mathcal{U}$ such that

$$1 > \left\|(T^{(n)} - A^{(n)})\boldsymbol{x}\right\|_{\mathcal{H}^{(n)}}^2 = \sum_{i=1}^{n} \|(T - A)x_i\|^2 .$$

Kaplansky's density theorem

Theorem (Kaplansky's density theorem)

If $\mathcal{U}$ is a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ with trivial null space, then the unit ball of $\mathcal{U}_{sa}$ is SOT-dense in the unit ball of $\mathcal{U}''_{sa}$ and the unit ball of $\mathcal{U}$ is SOT-dense in the unit ball of $\mathcal{U}''$. ...

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$$p_x(T) = \sqrt{\sum_i \| Tx_i \|^2}, \text{ with } x \in \ell_2(\mathcal{H})$$

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Relationships between the Topologies

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\begin{array}{ccccc}
\sigma\text{-WOT} & \longleftarrow & \sigma\text{-SOT} & \longleftarrow & \text{Norm} \\
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On bounded sets σ-SOT coincides with SOT and σ-WOT with WOT.

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On bounded sets σ-SOT coincides with SOT and σ-WOT with WOT.
On convex sets σ-SOT closed is equivalent with σ-WOT closed and SOT closed with WOT closed.

Predual of a W*-algebra

Theorem

Suppose $\mathcal{U}$ *is a* W^**-subalgebra of* $\mathcal{B}(\mathcal{H})$*, then there exists a Banach space* $\mathcal{U}_*$ *with*

$$(\mathcal{U}_*)^* = \mathcal{U}$$

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Lemma

For $\varphi \in \mathcal{B}(\mathcal{H})^*$ the following statements are equivalent

1. $\exists x_1, \ldots, x_n, y_1, \ldots, y_n$ *such that* $\varphi(T) = \sum_{i=1}^{n} (Tx_i, y_i)$
2. φ *is WOT continuous*
3. φ *is SOT continuous*

Predual of a W*-algebra

Proof: (iii) $\Rightarrow$ (i)

Predual of a W*-algebra

Proof: (iii) $\Rightarrow$ (i) $\varphi^{-1}(\mathbb{D})$ is open in SOT, so it contains an SOT ball around 0

$$\exists x_1, \ldots x_n : \sum_{i=1}^{n} \|Tx_i\|^2 < 1 \Rightarrow |\varphi(T)| < 1$$

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$\forall T \in \mathcal{B}(\mathcal{H})$

$$\exists x_1, \dots x_n : \sum_{i=1}^{n} \left\| \frac{T}{\sqrt{4 \sum_{j=1}^{n} \| Tx_j \|^2}} x_i \right\|^2 = \frac{1}{2} \Rightarrow \left| \varphi \left(\frac{T}{\sqrt{4 \sum_{j=1}^{n} \| Tx_j \|^2}} \right) \right| < 1$$

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Consider $\psi : \overline{\{(Tx_1, \dots, Tx_n) : \forall T \in \mathcal{B}(\mathcal{H})\}} \subseteq \mathcal{H}^{(n)}$,
$\psi(Tx_1, \dots, Tx_n) = \varphi(T)$

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$\forall T \in \mathcal{B}(\mathcal{H})$

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$$\varphi(T) = \sum_{i=1}^{n} (Tx_i, y_i)$$

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Proof: (iii) $\Rightarrow$ (i) $\mathcal{B}(\mathcal{H})$ is included in $\mathcal{B}(\ell_2(\mathcal{H}))$ and φ is SOT continuous and by Hahn-Banach there exists an extension $\psi \in \mathcal{B}(\ell_2(\mathcal{H}))^*$

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$$\varphi(T) = \psi(T, \dots) = \sum_{i=1}^{n} \sum_{j} (Tx_{ij}, y_{ij})$$

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Proof of the Theorem: Let $\mathcal{U}_* = \{\phi \in \mathcal{B}(\mathcal{H})^* : \phi \text{ is } \sigma\text{-WOT continuous}\}$

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From the previous Lemma

$$\mathcal{U}_* \cong \ell_2(\mathcal{H})^{(2)}/{}^{\perp}\mathcal{U}$$

where ${}^{\perp}\mathcal{U} = \{x, y \in \ell_2(\mathcal{H}) : \sum_i (Tx_i, y_i) = 0, \forall T \in \mathcal{U}\}$

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$$({}^\perp\mathcal{U})^\perp = \overline{\mathcal{U}}^{\text{weak}^*}$$

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$$({}^{\perp}\mathcal{U})^{\perp} = \{T \in \mathcal{B}(\mathcal{H})^{**} : \sum_i (Tx_i, y_i) = 0, \forall x, y \in {}^{\perp}\mathcal{U}\}$$

$$({}^{\perp}\mathcal{U})^{\perp} = \overline{\mathcal{U}}^{\text{weak}^*}$$

$$(\mathcal{U}_*)^* = \mathcal{U}$$

Part II: Non-commutative W^*-algebras

Mino Nicola Kraft, Chaitanya J. Kulkarni

Reminder on C*-algebras

Theorem (Thm. 5.18)

$\mathcal{A}$ C-algebra. Then:*
*$\exists$ $\mathcal{H}$ Hilbert space, $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ injective (=faithful) *-homomorphism*

Proof.

- **(single) Hilbert space**
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Reminder on C*-algebras

Theorem (Thm. 5.18)

$\mathcal{A}$ C-algebra. Then:*
*$\exists\ \mathcal{H}$ Hilbert space, $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ injective (=faithful) *-homomorphism*

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New for weak topology

Definition

$\mathcal{A}$ C*-algebra, $\exists \mathcal{X}$ Banach space: $(\mathcal{X})^* \cong \mathcal{A}$. Then:
$\mathcal{A}$ is called W^*-algebra

New requirement

- π weakly continuous (called: W*-*representation*)

Consequences

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  - Trace class operators $N(\mathcal{H})$:

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Weak continuity of multiplication

Theorem

$\mathcal{A}$ W^*-algebra, $y \in \mathcal{A}$. Then:
$x \mapsto yx$, $x \mapsto xy$ are weakly continuous

Proof.

- $(x_\alpha) \subseteq \mathcal{A}$ converges weakly to 0
- Aim: $\forall f \in \mathcal{A}_* : \; f(x_\alpha y) \to 0$
- Show:
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Let $\mathcal{A}_1$, $\mathcal{A}_2$ be W*-algebras and $\phi\colon \mathcal{A}_1 \to \mathcal{A}_2$ be a *-homomorphism. Then ϕ is *W*-homomorphism.* $:\iff$ ϕ is $\sigma(\mathcal{A}_1, \mathcal{A}_{1*})$-$\sigma(\mathcal{A}_2, \mathcal{A}_{2*})$-continuous.

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6. Faithfulness :
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   - $\pi(a) = 0 \implies \pi_\varphi(a) = 0\ \forall \varphi \in \mathcal{S}_n \implies \varphi(a) = 0 \implies a = 0$

$\square$

Part III:
Commutative W*-algebras

Made Benny Prasetya, Mahesh Krishna Krishnanagara,
& Juan Galvis

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Later we are going to show that $L^\infty(\Gamma, \nu)$ is the only commutative von Neumann algebra.

Definition

Let $\mathcal{X}$ be a Banach space and $\mathcal{X}^*$ be its dual. Then $x_\alpha \to 0$ in $\mathcal{X}$ w.r.t. $\sigma(\mathcal{X}, \mathcal{X}^*)$ iff $\forall f \in X^* : f(x_\alpha) \to 0$. Similarly, $f_\alpha \to 0$ in $\mathcal{X}^*$ w.r.t. $\sigma(\mathcal{X}^*, \mathcal{X})$ iff $\forall x \in \mathcal{X} : f_\alpha(x) \to 0$.

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$\left(\mathcal{X}^*, \sigma(\mathcal{X}^*, \mathcal{X})\right)^* = \mathcal{X}$ *and* $\left(\mathcal{X}, \sigma(\mathcal{X}, \mathcal{X}^*)\right)^* = \mathcal{X}^*$.

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A positive linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ is said to be **normal** if $\varphi(\sup_\alpha(x_\alpha)) = \sup_\alpha(\varphi(x_\alpha))$ for every uniformly bounded increasing direct net (x_α) of positive elements in $\mathcal{A}$.

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Remark: A positive linear functional is normal if and only if it is $\sigma(\mathcal{A}, \mathcal{A}_*)$-continuous.

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Example

Let $\mathcal{A} = L^\infty(\Gamma, \nu)$. Given $f \in L^1(\Gamma, \nu)$, $f \geq 0$, define $\phi(g) \coloneqq \int fg \, d\nu, \forall g \in L^\infty(\Gamma, \nu)$. Then ϕ is normal on $L^\infty(\Gamma, \nu)$. If $\int f \, d\nu = 1$, then ϕ is a normal state on $L^\infty(\Gamma, \nu)$.

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Theorem

Let $\mathcal{A}$, $\mathcal{B}$ be W-algebras. Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a W*-homomorphism. Then $\Phi(\mathcal{A})$ is closed in $\mathcal{B}$ in the $\sigma(\mathcal{B}, \mathcal{B}_*)$-topology.*

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Theorem

(Polar decomposition for functional) Let $\mathcal{A}$ be W-algebra. Then every weakly continuous linear functional $f \in \mathcal{A}_*$ can be written as $f(\cdot) = |f|(\cdot v)$, where $v \in \mathcal{A}$ is a partial isometry and $|f| \in \mathcal{A}_*$ is a normal functional.*

Definition

Let $\mathcal{A}$ be W*-algebra and let $a \in \mathcal{A}$. Let $L = \{xa : xa = 0, x \in \mathcal{A}$. Then L is a $\sigma(\mathcal{A}, \mathcal{A}_*)$-closed left ideal. Hence $L = \mathcal{A}e$ for a unique projection e in $\mathcal{A}$. Then $1 - e$ is the least projection of all the projections q in $\mathcal{A}$ such that $qa = a$. Projection $1 - e$ is called the left support of a and is denoted by $l(a)$. Similarly we can define right support $r(a)$ of a. If a is self-adjoint, then $l(a) = r(a)$ and is called the support of a and is denoted by $s(a)$.

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Theorem

(Polar decomposition for elements) Let $\mathcal{A}$ be W-algebra and let $a \in \mathcal{A}$. Then a can be decomposed as $a = u|a|$ where $|a| = (a^*a)^{\frac{1}{2}}$ and u is a partial isometry in $\mathcal{A}$ such that $u^*u = s(|a|)$. Such a decomposition is unique.*

Proof.

For each natural number n, define $h_n = \left(a^*a + \frac{1}{n}\right)^{\frac{1}{2}}$ and $a_n = a\left(a^*a + \frac{1}{n}\right)^{\frac{-1}{2}}$. Then $\|a_n\| \leq 1$ for all n and $a_n\left(a^*a + \frac{1}{n}\right)^{\frac{1}{2}} = a$. Since h_n converges to $(a^*a)^{\frac{1}{2}}$, given $\epsilon > 0$, there exists n_0 such that

$$\|h_n - (a^*a)^{\frac{1}{2}}\| < \epsilon, \forall n \geq n_0.$$

Since the unit sphere S of $\mathcal{A}$ is compact in the weak*-topology, there exists a limit point, say b of $\{a_n\}$. We then have

$$a_n(a^*a)^{\frac{1}{2}} \in a + \epsilon S, \forall n \geq n_0 \quad \text{and} \quad b(a^*a)^{\frac{1}{2}} \in a + \epsilon S$$

Since ϵ was arbitrary $a = b(a^*a)^{\frac{1}{2}}$. Let p be the support of $(a^*a)^{\frac{1}{2}}$ and q be the support of $(aa^*)^{\frac{1}{2}}$. A little calculation says that $p = pb^*qbp$. Define $u = qbp$. Then u becomes partial isometry and $a = u|a|$. $\square$

Proof.

Suppose $a = u|a| = u'|a|$ is another polar decomposition of a. Then $(p - (u')^*u)|a| = 0$. Let

$$R = \{x|(p - (u')^*u)x = 0, x \in \mathcal{A}\}.$$

Then R is a σ-closed left ideal. Hence $R = e\mathcal{A}$ for some projection e. Hence $s(|a|) = p \leq e$. Therefore $(p - (u')^*u)p = 0$. We also have $p = (u')^*u$.Therefore $u' = u$. $\square$

Proposition

Let $\mathcal{A}$ be a W^*-algebra and S be its unit sphere. Then S has an extreme point iff $\mathcal{A}$ has an identity.

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- This implies $\mathcal{A}$ has an identity.

Theorem

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Let $\Lambda : \mathcal{M} \to \mathcal{N}$ be a linear map between W^*-algebra. Then Λ is σ-continuous iff for any σ-continuous linear functional φ on $\mathcal{N}$, $\varphi \circ \Lambda$ is a σ-continuous linear functional on $\mathcal{M}$.

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- Therefore, $f_n \xrightarrow{\sigma} f$.

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4. Let $g \in L^1(\Omega, \mu_\varphi)$. We define $\overline{g} := g \circ \Lambda_\varphi : \mathcal{A} \to \mathbb{C}$ by

$$(g \circ \Lambda_\varphi)(x) = \int_\Omega \hat{x} g \, d\mu_\varphi.$$

   Indeed, $\overline{g}$ is linear functional on $\mathcal{A}$.

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10. Finally, we have for all $x \in \mathcal{A}$,

$$z(x) = \lim_{n\to\infty} \overline{g_n}(x) = \overline{g}(x)$$

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$$\Lambda_\varphi(\mathcal{A}) = \overline{C(\Omega)}^{\sigma(L^\infty, L^1)} = L^\infty(\Omega, \mu_\varphi).$$

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- For $f \in L^1(\Gamma, \nu)$ normal state on $L^\infty(\Gamma, \nu)$, $s(f)$ corresponds to the characteristic function of its support.

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- The topological spaces Ω such that $C(\Omega)$ is a W^*-algebra are called hyper-Stonean.

Many thanks for your attention!