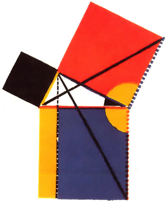
Quantum Families of Maps

Non-Commutative Mapping Spaces

$$\Bigg\{$$  $$\Bigg\}$$

$$\leftrightarrow$$

$$\left\{ \begin{aligned} & \|\boldsymbol{a}+\boldsymbol{b}\|^2 \\ &= \langle \boldsymbol{a}+\boldsymbol{b}, \boldsymbol{a}+\boldsymbol{b} \rangle \\ &= \langle \boldsymbol{a}, \boldsymbol{a} \rangle + 2\,\langle \boldsymbol{a}, \boldsymbol{b} \rangle + \langle \boldsymbol{b}, \boldsymbol{b} \rangle \\ &= \|\boldsymbol{a}\|^2 + \|\boldsymbol{b}\|^2 \end{aligned} \right\}$$

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The Duality of Geometry and Algebra

Geometric World		**Algebraic World**
$\{$Euclidean Geometry$\}$	$\longleftrightarrow$	$\{$vector spaces$\}$
$\{$smooth manifolds$\}$	$\longleftrightarrow$	$\{$commutative $\mathbb{R}$-algebras$\}$
$\{$measure spaces$\}$	$\longleftrightarrow$	$\{$commutative von Neumann algebras$\}$
$\boxed{\{\text{compact Hausdorff spaces}\}}$	$\longleftrightarrow$	$\boxed{\{\text{commutative unital } C^*\text{-algebras}\}}$
$\{$affine schemes$\}$	$\longleftrightarrow$	$\{$commutative rings$\}$

Non-Commutative Geometry in a Nutshell

- Find an association "$\leftrightarrow$" between a geometric space and a commutative algebraic structure (***classical setting***)
- Apply "$\leftrightarrow$" to the non-commutative version of said algebraic structure (***quantum*** or ***non-classical setting***)

quantum tools

generalize classical results / statements / theorems to the quantum setting

discover theories / concepts / methods that are "invisible" in the classical setting

Naïve Correspondence

Gelfand-Naimark

A is a commutative C*-algebra

$$\Updownarrow$$

$A \cong \mathrm{C}_0(X)$ for some locally compact Hausdorff space X,
which is unique up to homeomorphism.

Let's phrase this in terms of categories!

The Categories

The Category $\mathsf{C}^*_{\mathrm{com.}}$

- $\mathrm{Obj}(\mathsf{C}^*_{\mathrm{com.}}) := \{A \mid A$ is a commutative C^*-algebra$\}$
- $\mathrm{Mor}(A, B) := \{\varphi\colon A \to \mathrm{M}(B) \mid \varphi$ is a non-degen. $*$-hom.$\}$
  for any $A, B \in \mathrm{Obj}(\mathsf{C}^*_{\mathrm{com.}})$

The Category $\mathsf{Top}_{\mathrm{loc.cpt.}}$

- $\mathrm{Obj}(\mathsf{Top}_{\mathrm{loc.cpt.}}) := \{X \mid X$ is a locally compact Hausdorff space$\}$
- $\mathrm{Mor}(X, Y) := \{f\colon X \to Y \mid f$ is a continuous map$\}$
  for any $X, Y \in \mathrm{Obj}(\mathsf{Top}_{\mathrm{loc.cpt.}})$

Functorial Correspondence

Gelfand's Duality

The ***functors*** C_0 and Spec form an ***anti-equivalence of categories*** , i.e.

$$\mathsf{Top}_{\mathrm{loc.cpt.}} \underset{\mathrm{C}_0}{\overset{\mathrm{Spec}}{\rightleftarrows}} \mathsf{C}^*_{\mathrm{com.}}.$$

We also say that $\mathsf{Top}_{\mathrm{loc.cpt.}}$ and $\mathsf{C}^*_{\mathrm{com.}}$ are ***dual categories***.

We now have established "$\leftrightarrow$" for the classical setting.

Next we need to extend $\mathsf{C}^*_{\mathrm{com.}}$ to *arbitrary* C*-algebras.

Extending $\mathrm{C}^*_{\mathrm{com.}}$

The Category of C*-algebras

- $\mathrm{Obj}(\mathrm{C}^*) := \{A \mid A$ is a C*-algebra$\}$
- $\mathrm{Mor}(A, B) := \{\varphi\colon A \to \mathrm{M}(B) \mid \varphi$ is a non-degen. $*$-hom.$\}$ for any $A, B \in \mathrm{Obj}(\mathrm{C}^*)$

$\longrightarrow$ *Note:* With our definitions $\mathrm{C}^*_{\mathrm{com.}}$ is a so-called ***full subcategory*** of C^*.

Quantum spaces

We call an object in the ***dual category*** of C^* a ***quantum space***.

Notation: $\mathbb{X}, \mathbb{Y}, \ldots$ – quantum spaces
$\mathrm{C}_0(\mathbb{X}), \mathrm{C}_0(\mathbb{Y}), \ldots$ – corresponding C*-algebras

Quantum Space Dictionary

How do we define ***topological properties of quantum spaces***?

Idea: Use *Gelfandesque equivalences*, e.g.

$$X \text{ is compact} \quad \Leftrightarrow \quad \mathrm{C}_0(X) \text{ is unital}$$

to generalize classical concepts!

Dictionary of the Quantum Space Language

$$\begin{aligned}
\mathbb{X} \text{ is a quantum space} \quad &:\Leftrightarrow \quad \mathrm{C}_0(\mathbb{X}) \text{ is a C}^*\text{-algebra} \\
\mathbb{X} \text{ is compact} \quad &:\Leftrightarrow \quad \mathrm{C}(\mathbb{X}) := \mathrm{C}_0(\mathbb{X}) \text{ is unital} \\
\mathbb{X} \text{ is finite} \quad &:\Leftrightarrow \quad \mathrm{C}_0(\mathbb{X}) \text{ is finite-dimensional} \\
\mathbb{X} \text{ is compact and metrizable} \quad &:\Leftrightarrow \quad \mathrm{C}(\mathbb{X}) \text{ is separable} \\
&\vdots
\end{aligned}$$

Our Classical Blueprint: Families of Maps and Jackson's Theorem

Setting: Let X, Y and P be topological spaces.
We call a continuous map

$$\psi : X \times P \longrightarrow Y$$

a ***(classical) family of maps*** parametrized or indexed over P.

The Problem

Find a (categorically) natural bijection

$$\mathrm{C}(X \times P, Y) \cong \mathrm{C}\left(P, Y^X\right),$$

where we call Y^X the *exponential space* with respect to X and Y.

Our Classical Blueprint: Jackson's Answer

$$\boxed{\mathrm{C}(X \times P, Y) \cong \mathrm{C}\left(P, Y^X\right)} \qquad (*)$$

$\longrightarrow$*Note:* If we ignore the topologies on X, Y and P,

$$Y^X = \{f \mid f\colon X \to Y \text{ is a map}\}\,.$$

Jackson's Theorem (1952)

If X is a locally compact Hausdorff space, then the bijection $(*)$ holds for

$$Y^X = \mathrm{C}(X, Y)$$

with the compact-open topology.

The Quantum Version

classical	$\longrightarrow$	**quantum**
topological spaces X, Y, P	$\longrightarrow$	quantum spaces $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{P}$ $\mathrm{C}_0(\mathbb{X}) \cong A, \mathrm{C}_0(\mathbb{Y}) \cong B, \mathrm{C}_0(\mathbb{P}) \cong C$
Cartesian product $\times$	$\longrightarrow$	topological tensor product $\otimes := \otimes_{\min}$
continuous maps $\mathrm{C}(\cdot, \cdot)$	$\longrightarrow$	morphisms $\mathrm{Mor}(\cdot, \cdot)$

New Problem

Find a natural bijection

$$\mathrm{Mor}(B, A \otimes C) \cong \mathrm{Mor}(\mathrm{C}_0(\mathbb{M}_{\mathbb{X},\mathbb{Y}}), C)$$

for some quantum space $\mathbb{M}_{\mathbb{X},\mathbb{Y}}$.

We call $\Psi \in \mathrm{Mor}(B, A \otimes C)$ a ***quantum family of maps*** indexed by $\mathbb{P}$.

A Universal Property for Quantum Families of Maps

We say that

$$\Phi_{\mathbb{X},\mathbb{Y}}\colon \mathrm{C}_0(\mathbb{Y}) \longrightarrow \mathrm{C}_0(\mathbb{X}) \otimes \mathrm{C}_0(\mathbb{M}_{\mathbb{X},\mathbb{Y}})$$

is ***universal*** if

- for all quantum spaces $\mathbb{P}$ and
- for all quantum families of maps $\Psi \in \mathrm{Mor}(\mathrm{C}_0(\mathbb{Y}), \mathrm{C}_0(\mathbb{X}) \otimes \mathrm{C}_0(\mathbb{P}))$ there exists a unique $\Lambda \in \mathrm{Mor}(\mathrm{C}_0(\mathbb{M}_{\mathbb{X},\mathbb{Y}}), \mathrm{C}_0(\mathbb{P}))$ such that

$$\begin{tikzcd} & \mathrm{C}_0(\mathbb{X}) \otimes \mathrm{C}_0(\mathbb{M}_{\mathbb{X},\mathbb{Y}}) \arrow[d, "\mathrm{id}\otimes\Lambda"] \\ \mathrm{C}_0(\mathbb{Y}) \arrow[ur, "\Phi_{\mathbb{X},\mathbb{Y}}"] \arrow[r, "\Psi"'] & \mathrm{C}_0(\mathbb{X}) \otimes \mathrm{C}_0(\mathbb{P}) \end{tikzcd}$$

commutes.

Simple Case

Existence of a Universal Family of Maps

If $\mathbb{X}$ and $\mathbb{Y}$ are compact quantum spaces such that

- $C(\mathbb{X}) = \mathrm{Mat}_n(\mathbb{C})$ is a simple matrix algebra and
- $C(\mathbb{Y}) = C^*(\mathbb{Z}) = C(\{z \in \mathbb{C} \mid |z| = 1\})$ is the algebra freely generated by a unitary δ,

the universal family of maps

$$\Phi_{\mathbb{X},\mathbb{Y}} \colon C(\mathbb{Y}) \to C(\mathbb{X}) \otimes C(\mathbb{M}_{\mathbb{X},\mathbb{Y}})$$

and the mapping space $\mathbb{M}_{\mathbb{X},\mathbb{Y}}$ do exist.

Simple Case

Construction

- Generators: $u_{ij} \in A$ for $0 \le i, j < n$
- Matrix: $U = (u_{ij})_{ij} \in \mathrm{Mat}_n(A) = \mathrm{C}(\mathbb{X}) \otimes A$
- Choose A as the Brown algebra: universal such that U is unitary
- Take $\mathrm{C}(\mathbb{M}_{\mathbb{X},\mathbb{Y}}) = A$ and $\Phi_{\mathbb{X},\mathbb{Y}}(\delta) = U$

Reminder

- $\mathrm{C}(\mathbb{X}) = \mathrm{Mat}_n(\mathbb{C})$ is a simple matrix algebra
- $\mathrm{C}(\mathbb{Y})$ is the universal algebra with a unitary generator δ

Simple Case

Property

- Morphisms $\mathrm{C}(\mathbb{Y}) \to \mathrm{C}(\mathbb{X}) \otimes B$
- Unitary elements of $\mathrm{M}(\mathrm{C}(\mathbb{X}) \otimes B)$
- Unitary elements of $\mathrm{Mat}_n(\mathrm{M}(B))$
- Morphisms $\mathrm{C}(\mathbb{M}_{\mathbb{X},\mathbb{Y}}) \to B$

Reminder

- $\mathrm{C}(\mathbb{Y})$ is the universal algebra with a unitary generator δ
- $\mathrm{C}(\mathbb{X}) = \mathrm{Mat}_n(\mathbb{C})$ is a simple matrix algebra
- $\mathrm{Mat}_n(\mathrm{C}(\mathbb{M}_{\mathbb{X},\mathbb{Y}}))$ is universal with a unitary $n \times n$ matrix

Larger Spaces

$\mathbb{X}$ finite

- $\mathrm{C}(\mathbb{X}) = \bigoplus_{0 \le k < m} \mathrm{Mat}_{n_k}(\mathbb{C})$
- Generators: u_{ij}^k with $0 \le i, j < n_k$, $0 \le k < m$ $\quad (u_{ij}^k \in \mathrm{C}(\mathbb{M}_{\mathbb{X},\mathbb{Y}}))$

More generators

- $\mathrm{C}(\mathbb{Y}) = \mathrm{C}^*(\mathbb{F}_\ell)$ freely generated by unitary δ_p for $0 \le p < \ell$
- Generators: u_{ij}^p with $0 \le i, j < n$, $0 \le p < \ell$ $\quad (u_{ij}^p \in \mathrm{C}(\mathbb{M}_{\mathbb{X},\mathbb{Y}}))$

Subspaces

More relations

- $\mathrm{C}(\mathbb{Y})$ generated by unitary δ_p, subject to relations
- Impose relations on $U^p = (u^p_{ij})_{ij}$

Algebraic quotients

- $\mathrm{C}(\widehat{\mathbb{Y}}) = \mathrm{C}(\mathbb{Y})/K$ for some ideal K
- $I = \langle (\omega \otimes \mathrm{id})\Phi_{\mathbb{X},\mathbb{Y}}(k) \mid \omega \in \mathrm{C}(\mathbb{X})^*, k \in K \rangle$
- $\mathrm{C}(\mathbb{M}_{\mathbb{X},\widehat{\mathbb{Y}}}) = \mathrm{C}(\mathbb{M}_{\mathbb{X},\mathbb{Y}})/I$

General Case

Existence of a Universal Family of Maps

If $\mathbb{X}$ and $\mathbb{Y}$ are compact quantum spaces such that

- $\mathbb{X}$ is finite, i.e. $\mathrm{C}(\mathbb{X})$ is finite-dimensional, and
- $\mathrm{C}(\mathbb{Y})$ is finitely generated,

the universal family of maps

$$\Phi_{\mathbb{X},\mathbb{Y}} \colon \mathrm{C}(\mathbb{Y}) \to \mathrm{C}(\mathbb{X}) \otimes \mathrm{C}(\mathbb{M}_{\mathbb{X},\mathbb{Y}})$$

and the mapping space $\mathbb{M}_{\mathbb{X},\mathbb{Y}}$ do exist.

Universal Property

Isomorphic Functors

$$\mathrm{Mor}(\mathrm{C}(\mathbb{M}_{\mathbb{X},\mathbb{Y}}), -) \cong \mathrm{Mor}(\mathrm{C}(\mathbb{Y}), \mathrm{C}(\mathbb{X}) \otimes (-))$$

Diagram

$$\begin{tikzcd}
 & \mathrm{C}(\mathbb{X}) \otimes \mathrm{C}(\mathbb{M}_{\mathbb{X},\mathbb{Y}}) \arrow[d, "\mathrm{id}\otimes\Lambda"] & \mathrm{C}(\mathbb{M}_{\mathbb{X},\mathbb{Y}}) \arrow[d, dashed, "\Lambda"] \\
\mathrm{C}(\mathbb{Y}) \arrow[ur, "\Phi_{\mathbb{X},\mathbb{Y}}"] \arrow[r, "\Psi"'] & \mathrm{C}(\mathbb{X}) \otimes B & B
\end{tikzcd}$$

Definitions

Functors, covariant and contravariant

- $F\colon \mathsf{C} \to \mathsf{D}$
- $X, Y \in \mathrm{Obj}(C)$: $X \mapsto F(X)$
- $f \in \mathrm{Mor}(X,Y) \mapsto F(f) \in \mathrm{Mor}(F(X), F(Y))$ (covariant) or $f \in \mathrm{Mor}(X,Y) \mapsto F(f) \in \mathrm{Mor}(F(Y), F(X))$, s.t.
- 1. $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$, $X \in \mathrm{Obj}(C)$.
- 2a. $F(g \circ f) = F(g) \circ F(f)$, $f\colon X \to Y$, $g\colon Y \to Z$ (Covariant)
- 2b. $F(g \circ f) = F(f) \circ F(g)$, $f\colon X \to Y$, $g\colon Y \to Z$. (Contravariant)

Bifunctors

- $G\colon \mathsf{C} \times \mathsf{C}' \to \mathsf{D}$
- Functor (co- or contravariant) in both arguments

Set-up

Spaces and maps

- $\mathbb{X}_i$, $\mathbb{Y}_i$ are quantum spaces such that $\mathrm{C}(\mathbb{X}_i)$ is finite dimensional & $\mathrm{C}(\mathbb{Y}_i)$ is finitely generated and unital (for any index, including empty indices)
- Given any $\pi\colon \mathrm{C}(\mathbb{Y}_2) \to \mathrm{C}(\mathbb{Y}_1)$, there is a unique morphism $\Lambda\colon \mathrm{C}(\mathbb{M}_{\mathbb{X},\mathbb{Y}_2}) \to \mathrm{C}(\mathbb{M}_{\mathbb{X},\mathbb{Y}_1})$ making the upper diagram on the next slide commute.
- Given any $\rho\colon \mathrm{C}(\mathbb{X}_1) \to \mathrm{C}(\mathbb{X}_2)$, there is a unique morphism $\tilde{\Lambda}\colon \mathrm{C}(\mathbb{M}_{\mathbb{X}_2,\mathbb{Y}}) \to \mathrm{C}(\mathbb{M}_{\mathbb{X}_1,\mathbb{Y}})$ making the lower diagram on the next page commute.

Defining Λ and $\tilde{\Lambda}$

Diagrams

$$\begin{tikzcd} C(\mathbb{Y}_2) \arrow[r, "\Phi_{\mathbb{X},\mathbb{Y}_2}"] \arrow[d, "\pi"'] & C(\mathbb{X}) \otimes C(\mathbb{M}_{\mathbb{X},\mathbb{Y}_2}) \arrow[d, "\mathrm{id}\otimes\Lambda"] \\ C(\mathbb{Y}_1) \arrow[r, "\Phi_{\mathbb{X},\mathbb{Y}_1}"] & C(\mathbb{X}) \otimes C(\mathbb{M}_{\mathbb{X},\mathbb{Y}_1}) \end{tikzcd}$$

$$\begin{tikzcd} C(\mathbb{Y}) \arrow[r, "\Phi_{\mathbb{X}_2,\mathbb{Y}}"] \arrow[d, "\Phi_{\mathbb{X}_1,\mathbb{Y}}"'] & C(\mathbb{X}_2) \otimes C(\mathbb{M}_{\mathbb{X}_2,\mathbb{Y}}) \arrow[d, "\mathrm{id}\otimes\tilde{\Lambda}"] \\ C(\mathbb{X}_1) \otimes C(\mathbb{M}_{\mathbb{X}_1,\mathbb{Y}}) \arrow[r, "\rho\otimes\mathrm{id}"] & C(\mathbb{X}_2) \otimes C(\mathbb{M}_{\mathbb{X}_1,\mathbb{Y}}) \end{tikzcd}$$

Notation

- Given $\pi\colon C(\mathbb{Y}_2) \to C(\mathbb{Y}_1)$ & $\rho\colon C(\mathbb{X}_1) \to C(\mathbb{X}_2)$, consider

$$(\rho \otimes \mathrm{id}) \circ \Phi_{\mathbb{X}_1,\mathbb{Y}_1} \circ \pi\colon C(\mathbb{Y}_2) \to C(\mathbb{X}_2) \otimes C(\mathbb{M}_{\mathbb{X}_1,\mathbb{Y}_1})$$

- This morphism is also of the form $(\mathrm{id} \otimes \Lambda) \circ \Phi_{\mathbb{X}_2,\mathbb{Y}_2}$ for some $\Lambda\colon C(\mathbb{M}_{\mathbb{X}_2,\mathbb{Y}_2}) \to C(\mathbb{M}_{\mathbb{X}_1,\mathbb{Y}_1})$

The New Notation

Call this unique $\Lambda \stackrel{\text{def}}{=} \mathbb{M}_{\rho,\pi}$

Notation continued

- Hence $\mathbb{M}_{\rho,\pi}$ is the unique morphism making the following diagram commute:

$$\begin{tikzcd}
C(\mathbb{Y}_2) \arrow[rr, "\pi"] \arrow[d, "\Phi_{\mathbb{X}_2,\mathbb{Y}_2}"'] & & C(\mathbb{Y}_1) \arrow[d, "(\rho\otimes \mathrm{id})\circ(\Phi_{\mathbb{X}_1,\mathbb{Y}_1})"] \\
C(\mathbb{X}_2)\otimes C(\mathbb{M}_{\mathbb{X}_2,\mathbb{Y}_2}) \arrow[rr, "\mathrm{id}\otimes \mathbb{M}_{\rho,\pi}"'] & & C(\mathbb{X}_2)\otimes C(\mathbb{M}_{\mathbb{X}_1,\mathbb{Y}_1})
\end{tikzcd}$$

Functoriality of $\mathbb{M}_{\cdot,\cdot}$

Theorem

$\mathbb{M}_{\cdot,\cdot}$ is a bifunctor which assigns the object $\mathbb{M}_{\mathbb{X},\mathbb{Y}}$ to a pair of objects $(\mathbb{X},\mathbb{Y})$.

Additionally, $\mathbb{M}_{\cdot,\cdot}$ is covariant in the first slot and contravariant in the second slot.

- In the proof of the above theorem, we will denote $\Phi_{\mathbb{X}_i,\mathbb{Y}_i}$ by Φ_i for notational simplicity.

Proof of the Functoriality of $\mathbb{M}_{\cdot,\cdot}$

Proof.

- Consider the maps

$$\mathrm{C}(\mathbb{Y}_3) \xrightarrow{\pi_2} \mathrm{C}(\mathbb{Y}_2) \xrightarrow{\pi_1} \mathrm{C}(\mathbb{Y}_1)$$

  and

$$\mathrm{C}(\mathbb{X}_1) \xrightarrow{\rho_1} \mathrm{C}(\mathbb{X}_2) \xrightarrow{\rho_2} \mathrm{C}(\mathbb{X}_3)$$

- $\mathbb{M}_{\rho_1,\pi_1}$ and $\mathbb{M}_{\rho_2,\pi_2}$ are respectively defined by

$$(\mathrm{id} \otimes \mathbb{M}_{\rho_1,\pi_1}) \circ \Phi_2 = (\rho_1 \otimes \mathrm{id}) \circ \Phi_1 \circ \pi_1,$$

$$(\mathrm{id} \otimes \mathbb{M}_{\rho_2,\pi_2}) \circ \Phi_3 = (\rho_2 \otimes \mathrm{id}) \circ \Phi_2 \circ \pi_2$$

Proof of the Functoriality of $\mathbb{M}_{\cdot,\cdot}$

Proof continued.

- This implies

$$\begin{aligned}
\mathrm{id} \otimes (\mathbb{M}_{\rho_1,\pi_1} \circ \mathbb{M}_{\rho_2,\pi_2}) \circ \Phi_3 &= (\mathrm{id} \otimes \mathbb{M}_{\rho_1,\pi_1}) \circ (\mathrm{id} \otimes \mathbb{M}_{\rho_2,\pi_2}) \circ \Phi_3) \\
&= (\mathrm{id} \otimes \mathbb{M}_{\rho_1,\pi_1}) \circ (\rho_2 \otimes \mathrm{id}) \circ \Phi_2 \circ \pi_2 \\
&= (\rho_2 \otimes \mathrm{id}) \circ ((\mathrm{id} \otimes \mathbb{M}_{\rho_1,\pi_1}) \circ \Phi_2) \circ \pi_2 \\
&= (\rho_2 \otimes \mathrm{id}) \circ (\rho_1 \otimes \mathrm{id}) \circ \Phi_1 \circ \pi_1 \circ \pi_2 \\
&= ((\rho_2 \circ \rho_1) \otimes \mathrm{id}) \circ \Phi_1 \circ (\pi_1 \circ \pi_2) \\
&= (\mathrm{id} \otimes \mathbb{M}_{\rho_2\circ\rho_1,\pi_1\circ\pi_2}) \circ \Phi_3
\end{aligned}$$

- It follows that $\mathbb{M}_{\rho_1,\pi_1} \circ \mathbb{M}_{\rho_2,\pi_2} = \mathbb{M}_{\rho_2\circ\rho_1,\pi_1\circ\pi_2}$ as desired. □

Theorem on surjectivity and injectivity

Full theorem

- if π is surjective, $\mathbb{M}_{\mathrm{id},\pi} : \mathrm{C}(\mathbb{M}_{\mathbb{X},\mathbb{Y}_2}) \to \mathrm{C}(\mathbb{M}_{\mathbb{X},\mathbb{Y}_1})$ is surjective
- if ρ is injective, $\mathbb{M}_{\rho,\mathrm{id}} : \mathrm{C}(\mathbb{M}_{\mathbb{X}_2,\mathbb{Y}}) \to \mathrm{C}(\mathbb{M}_{\mathbb{X}_1,\mathbb{Y}})$ is surjective
- if π is injective, $\mathbb{M}_{\mathrm{id},\pi} : \mathrm{C}(\mathbb{M}_{\mathbb{X},\mathbb{Y}_2}) \to \mathrm{C}(\mathbb{M}_{\mathbb{X},\mathbb{Y}_1})$ is injective
- if ρ is surjective, $\mathbb{M}_{\rho,\mathrm{id}} : \mathrm{C}(\mathbb{M}_{\mathbb{X}_2,\mathbb{Y}}) \to \mathrm{C}(\mathbb{M}_{\mathbb{X}_1,\mathbb{Y}})$ is injective

Proving the theorem

Proof of part 1

- if π is surjective, $\mathbb{M}_{\mathrm{id},\pi}\colon \mathrm{C}(\mathbb{M}_{\mathbb{X},\mathbb{Y}_2}) \to \mathrm{C}(\mathbb{M}_{\mathbb{X},\mathbb{Y}_1})$ is surjective
- $\{(\omega \otimes \mathrm{id})\Phi_{\mathbb{X},\mathbb{Y}_1}(y_1) | y_1 \in \mathrm{C}(\mathbb{Y}_1), \omega \in \mathrm{C}(\mathbb{X})^*\}$
$= \{(\omega \otimes \mathrm{id})\Phi_{\mathbb{X},\mathbb{Y}_1}(\pi(y_2)) \mid y_2 \in \mathrm{C}(\mathbb{Y}_2), \omega \in \mathrm{C}(\mathbb{X})^*\}$
$= \mathbb{M}_{\mathrm{id},\pi}(\{(\omega \otimes \mathrm{id})\Phi_{\mathbb{X},\mathbb{Y}_2}(y_2) \mid y_2 \in \mathrm{C}(\mathbb{Y}_2), \omega \in \mathrm{C}(\mathbb{X})^*\})$.

Proof of part 2

- if ρ is injective, $\mathbb{M}_{\rho,\mathrm{id}}\colon \mathrm{C}(\mathbb{M}_{\mathbb{X}_2,\mathbb{Y}}) \to \mathrm{C}(\mathbb{M}_{\mathbb{X}_1,\mathbb{Y}})$ is surjective
- $\mathbb{M}_{\rho,\mathrm{id}}(\{(\omega_2 \otimes \mathrm{id})\Phi_{\mathbb{X}_2,\mathbb{Y}}(y) \mid y \in \mathrm{C}(\mathbb{Y}), \omega_2 \in \mathrm{C}(\mathbb{X}_2)^*\})$
$= \{((\omega_2 \circ \rho) \otimes \mathrm{id})\Phi_{\mathbb{X}_1,\mathbb{Y}}(y) \mid y \in \mathrm{C}(\mathbb{Y}), \omega_2 \in \mathrm{C}(\mathbb{X}_2)^*\}$
$= \{(\omega_1 \otimes \mathrm{id})\Phi_{\mathbb{X}_1,\mathbb{Y}}(y) \mid y \in \mathrm{C}(\mathbb{Y}), \omega_1 \in \mathrm{C}(\mathbb{X}_1)^*\}$.

Proving the theorem, continued

Proof of part 3

- if π is injective, $\mathbb{M}_{\mathrm{id},\pi}\colon \mathrm{C}(\mathbb{M}_{\mathbb{X},\mathbb{Y}_2}) \to \mathrm{C}(\mathbb{M}_{\mathbb{X},\mathbb{Y}_1})$ is injective
- See [Arkadiusz Bochniak, Paweł Kasprzak, Piotr M. Sołtan. *Quantum correlations on quantum spaces*, `https://arxiv.org/abs/2105.07820`, May 2021]

Proving the theorem, continued

Proof of part 4

- if ρ is surjective, $\mathbb{M}_{\rho,\mathrm{id}}\colon \mathrm{C}(\mathbb{M}_{\mathbb{X}_2,\mathbb{Y}}) \to \mathrm{C}(\mathbb{M}_{\mathbb{X}_1,\mathbb{Y}})$ is injective
- $\mathrm{C}(\mathbb{X}_1)$ and $\mathrm{C}(\mathbb{X}_2)$ finite dimensional, $\rho\colon \mathrm{C}(\mathbb{X}_1) \to \mathrm{C}(\mathbb{X}_2)$ surjective
- $\exists X', \sigma\colon \mathrm{C}(\mathbb{X}') \oplus \mathrm{C}(\mathbb{X}_2) \to \mathrm{C}(\mathbb{X}_1)$ isomorphism s. th.
  $\rho \circ \sigma\colon \mathrm{C}(\mathbb{X}') \oplus \mathrm{C}(\mathbb{X}_2) \to \mathrm{C}(\mathbb{X}_2)$ is the projection onto $\mathrm{C}(\mathbb{X}_2)$
- Fact: $\mathrm{C}(\mathbb{M}_{\mathbb{X}'\sqcup\mathbb{X}_2,\mathbb{Y}}) \cong \mathrm{C}(\mathbb{M}_{\mathbb{X}',\mathbb{Y}}) * \mathrm{C}(\mathbb{M}_{\mathbb{X}_2,\mathbb{Y}})$
- Inclusion mapping $\iota_2\colon \mathrm{C}(\mathbb{M}_{\mathbb{X}_2,\mathbb{Y}}) \to \mathrm{C}(\mathbb{M}_{\mathbb{X}'\sqcup\mathbb{X}_2,\mathbb{Y}})$
- Then $((\rho\circ\sigma)\otimes\mathrm{id})\circ\Phi_{\mathbb{X}_2\sqcup\mathbb{X}',\mathbb{Y}} = (\mathrm{id}\otimes\iota_2)\circ\Phi_{\mathbb{X}_2,\mathbb{Y}}$
- Find the associated $\mathbb{M}_{\sigma,\mathrm{id}}\colon \mathrm{C}(\mathbb{M}_{\mathbb{X}_1,\mathbb{Y}}) \to \mathrm{C}(\mathbb{M}_{\mathbb{X}_2\sqcup\mathbb{X}',\mathbb{Y}})$,
  which satisfies $(\sigma\otimes\mathrm{id})\circ\Phi_{\mathbb{X}_2\sqcup\mathbb{X}',\mathbb{Y}} = (\mathrm{id}\otimes\mathbb{M}_{\sigma,\mathrm{id}})\circ\Phi_{\mathbb{X}_1,\mathbb{Y}}$
- Equivalently $(\sigma\otimes\mathbb{M}_{\sigma,\mathrm{id}}^{-1})\circ\Phi_{\mathbb{X}_2\sqcup\mathbb{X}',\mathbb{Y}} = \Phi_{\mathbb{X}_1,\mathbb{Y}}$.

Proving the theorem, continued

Proof of part 4, continued

- Combine $((\rho \circ \sigma) \otimes \mathrm{id}) \circ \Phi_{\mathbb{X}_2 \sqcup \mathbb{X}', \mathbb{Y}} = (\mathrm{id} \otimes \iota_2) \circ \Phi_{\mathbb{X}_2, \mathbb{Y}}$ and $\Phi_{\mathbb{X}_1, \mathbb{Y}} = (\sigma \otimes \mathbb{M}_{\sigma,\mathrm{id}}^{-1}) \circ \Phi_{\mathbb{X}_2 \sqcup \mathbb{X}', \mathbb{Y}}$
- $(\rho \otimes \mathrm{id}) \circ \Phi_{\mathbb{X}_1, \mathbb{Y}} = (\rho \otimes \mathrm{id}) \circ (\sigma \otimes \mathbb{M}_{\sigma,\mathrm{id}}^{-1}) \circ \Phi_{\mathbb{X}_2 \sqcup \mathbb{X}', \mathbb{Y}} = ((\rho \circ \sigma) \otimes \mathbb{M}_{\sigma,\mathrm{id}}^{-1}) \circ \Phi_{\mathbb{X}_2 \sqcup \mathbb{X}', \mathbb{Y}} = (\mathrm{id} \otimes (\mathbb{M}_{\sigma,\mathrm{id}}^{-1} \circ \iota_2)) \circ \Phi_{\mathbb{X}_2, \mathbb{Y}}$
- By definition, $(\rho \otimes \mathrm{id}) \circ \Phi_{\mathbb{X}_1, \mathbb{Y}} = (\mathrm{id} \otimes \mathbb{M}_{\rho,\mathrm{id}}) \circ \Phi_{\mathbb{X}_2, \mathbb{Y}}$
- Conclusion: $\mathbb{M}_{\rho,\mathrm{id}} = \mathbb{M}_{\sigma,\mathrm{id}}^{-1} \circ \iota_2$

Maps from a set to a compact quantum space

Quantum family of maps from X to $\mathbb{A}$

- A quantum family Ψ of maps indexed by $\mathbb{B}$ (with $A := \mathrm{C}(\mathbb{A})$, $B := \mathrm{C}_0(\mathbb{B})$):

$$\begin{aligned}\Psi \in &\mathrm{Mor}(A, \mathrm{C}_0(X) \otimes B) \simeq \mathrm{Mor}(A, \mathrm{C_b}(X, \mathrm{M}(B))) \\ &\simeq \mathrm{Mor}(A, \ell^\infty_X(\mathrm{M}(B))) \simeq \{\mathrm{Mor}(A, \mathrm{M}(B))\}^X\end{aligned} \tag{$**$}$$

- By $(**)$, to give such a Ψ is the same as to give a family of unital C^*-morphisms from A to $\mathrm{M}(B)$. Universal way to do this is
- **Universal quantum family of maps**: $\mathrm{C}(\mathbb{M}_{X,\mathbb{A}}) = \circledast_X A$ (free power of A over X), Φ determined by $(\iota_x : A \to \circledast_X A)_{x \in X}$ on r.h.s. of $(**)$ with $B = \mathrm{C}(\mathbb{M}_{X,\mathbb{A}})$.

Maps from a two point set to itself

- Special case of the previous case with $\mathbb{A} = X = \mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$, $A = \mathrm{C}_0(X) = \mathrm{C}(X) = \mathbb{C}^2$.
- **Universal quantum family of maps**: $\mathrm{C}(\mathbb{M}_{X,X}) = \mathbb{C}^2 * \mathbb{C}^2$ (free product of $\mathbb{C}^2$ with itself), $\Phi\colon \mathbb{C}^2 \to \mathbb{C}^2 \otimes (\mathbb{C}^2 * \mathbb{C}^2)$ the unital C*-morphism sending $v \in \mathbb{C}^2$ to $e_1 \otimes \iota_1(v) + e_2 \otimes \iota_2(v)$.
- $\mathrm{C}(\mathbb{M}_{X,X})$ as a group algebra: by the universal properties of free products of C*-algebras and groups, $\mathbb{C}^2 * \mathbb{C}^2 \simeq \mathrm{C}^*(\mathbb{Z}_2 * \mathbb{Z}_2)$.
- $\mathbb{Z}_2 * \mathbb{Z}_2 \simeq \mathbb{Z} \rtimes \mathbb{Z}_2$. *Proof*: A presentation for $\mathbb{Z}_2 * \mathbb{Z}_2$ is $\langle a, b \mid a^2 = b^2 = 1 \rangle$. Then $\mathbb{Z}_2 * \mathbb{Z}_2 = AN = NA = N \rtimes A$, where $N = \langle ab \rangle \simeq \mathbb{Z}$, $A = \langle a \rangle \simeq \mathbb{Z}_2$.
- $\mathrm{C}(\mathbb{M}_{X,X}) = \mathrm{C}^*(\mathbb{Z}_2 * \mathbb{Z}_2) = \mathrm{C}^*(\mathbb{Z} \rtimes \mathbb{Z}_2) = \mathrm{C}^*(\mathbb{Z}) \rtimes \mathbb{Z}_2$, where the semidirect product and the cross product are the unique non-trivial ones.
- By sharp contrast, in the classical case, $|\{X \to X\}| = 4$.

$\mathbb{M}_{\mathbb{X},\mathbb{X}}$ as a compact quantum semi-group

- $\mathbb{X}$ a finite quantum space, so the **universal quantum family of maps** $\Phi \in \mathrm{Mor}(\mathrm{C}(\mathbb{X}), \mathrm{C}(\mathbb{X}) \otimes \mathrm{C}(\mathbb{M}_{\mathbb{X},\mathbb{X}}))$ exists and is unique up to isomorphism.
- Consider

$$\Psi\colon \mathrm{C}(\mathbb{X}) \xrightarrow{\Phi} \mathrm{C}(\mathbb{X}) \otimes \mathrm{C}(\mathbb{M}_{\mathbb{X},\mathbb{X}}) \xrightarrow{\Phi\otimes\mathrm{id}} \mathrm{C}(\mathbb{X}) \otimes \mathrm{C}(\mathbb{M}_{\mathbb{X},\mathbb{X}}) \otimes \mathrm{C}(\mathbb{M}_{\mathbb{X},\mathbb{X}}).$$

- (Sołtan) Uniqueness part of the universal property of Φ yields a unique unital C*-morphism $\Delta\colon \mathrm{C}(\mathbb{M}_{\mathbb{X},\mathbb{X}}) \to \mathrm{C}(\mathbb{M}_{\mathbb{X},\mathbb{X}}) \otimes \mathrm{C}(\mathbb{M}_{\mathbb{X},\mathbb{X}})$, such that $\Psi = (\mathrm{id} \otimes \Delta)\Phi$. The morphism Δ (called comultiplication) is coassociative, i.e. $(\mathrm{id} \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id})\Delta$ (uniqueness part of the universal property of Φ again).
- (Sołtan) Similarly, the morphism $\mathrm{id}\colon \mathrm{C}(\mathbb{X}) \to \mathrm{C}(\mathbb{X}) = \mathrm{C}(\mathbb{X}) \otimes \mathbb{C}$ yields a counit $\epsilon\colon \mathrm{C}(\mathbb{M}_{\mathbb{X},\mathbb{X}}) \to \mathbb{C}$ for Δ. So $(\mathrm{C}(\mathbb{M}_{\mathbb{X},\mathbb{X}}), \Delta, \epsilon)$ is a counital coalgebra.

$\mathbb{M}_{\mathrm{Mat}_2,\mathbb{Z}_2}$ is not a compact quantum group

- (Woronowicz) **Definition.** A **compact quantum group** is given by a pair $\mathbb{G} = (A, \Delta)$, where A is a *unital* C*-algebra (we often write $A = \mathrm{C}(\mathbb{G})$), $\Delta\colon A \to A \otimes A$ a unital C*-morphism that is coassociative, such that Δ is bi-simplifiable in the sense that the linear spans of both $\{(1 \otimes a)\Delta(b) \mid a, b \in A\}$ and $\{(a \otimes 1)\Delta(b) \mid a, b \in A\}$ are dense in $A \otimes A$.
- $\mathbb{G} = (A, \Delta)$ a compact quantum group implies that $\chi(\mathbb{G}) := \{$nonzero multiplicative functionals on $A\}$ is a compact Hausdorff group, where the multiplication is the convolution $f * g := (f \otimes g)\Delta$, and the underlying topology is the weak-$*$ topology.
- (Sołtan) $\mathrm{C}(\mathbb{M}_{\mathrm{Mat}_2,\mathbb{Z}_2})$ is the unital universal C*-algebra generated by p, q, z with the relations

$$p = p^2 + z^* z,\ q = q^2 + zz^*,\ pz = (1 - z)q,$$
$$p = p^*,\ q = q^*.$$

$\mathbb{M}_{\mathrm{Mat}_2,\mathbb{Z}_2}$ is not a compact quantum group (continued)

Theorem (Sołtan)

$\chi\Big(\mathrm{C}(\mathbb{M}_{\mathrm{Mat}_2,\mathbb{Z}_2})\Big)$ is homeomorphic to the topological sum the two sphere S^2 and two isolated points, thus does not carry a topological group structure. Consequently, $\mathbb{M}_{\mathrm{Mat}_2,\mathbb{Z}_2}$ can not be a compact quantum group.

Remark

- (S. Wang) If $A = \mathrm{C}(\mathbb{G})$ for some compact quantum group, $n \in \mathbb{N}$, then the free power A^{*n} carries a compact quantum group structure.
- $A^{*n} = \mathrm{C}(\mathbb{M}_{X,\mathbb{G}})$ where X is a set of n points.
- The analogue result of S. Wang fails even for $\mathbb{G} = \mathbb{Z}_2$ if one replaces X with Mat_2 (new phenomenon of the quantum mapping space).

Further results

Suppose $\mathbb{X}$ is a finite quantum space, $\mathbb{Y}$ a compact quantum space.
Recall

- A C*-algebra A is said to be **RFD** (residually finite dimensional), if for all $0 \neq a \in A$, there is a finite dimensional representation $\pi\colon A \to \mathrm{Mat}_n$ such that $\pi(a) \neq 0$.
- A C*-algebra B is said to have the **lifting property**, if whenever J is a closed ideal of B, $u\colon C \to B/J$ is a c.c.p map, then u lifts to a c.c.p map $\tilde{u}\colon C \to B$.

Theorem (Bochniak, Kasprzak and Sołtan)

- If $\mathrm{C}(\mathbb{Y})$ is RFD, then $\mathrm{C}(\mathbb{M}_{\mathbb{X},\mathbb{Y}})$ is RFD.
- If $\mathrm{C}(\mathbb{Y})$ is *separable* and has the lifting property, then $\mathrm{C}(\mathbb{M}_{\mathbb{X},\mathbb{Y}})$ has the lifting property.