Topological dynamical systems

Niklas Handke
Lukas Rollier

June 2021

Table of Contents

Table of Contents

Topological dynamical system

A **topological dynamical system** is a pair $(X; \varphi)$, with nonempty compact metrizable space X and $\varphi : X \to X$ continuous. It is called **invertible**, if φ is invertible i.e. a homeomorphism.

Finite state space

Finite set $X := \{0, \ldots, d-1\}$ with discrete topology.
Every map $\varphi : X \to X$ is continuous.
$(X; \varphi)$ invertible if and only if φ permutation.

Rotation on Compact Groups

A **topological group** is a group $(G, \cdot)$ with topology s.t. inversion and multiplication are continuous.
Compact topological group G with **left rotation** by $a \in G$

$$\varphi_a : G \to G,\ \varphi_a(g) := a \cdot g,$$

forms invertible topological system $(G; \varphi_a)$. i.e. $\mathbb{T}$ compact Abelian group.

Homomorphisms of topological dynamical systems

A **homomorphism** between topological systems $(X_1; \varphi_1), (X_2; \varphi_2)$ is continuous map $\psi : X_1 \to X_2$ such that $\psi \circ \varphi_1 = \varphi_2 \circ \psi$, i.e., the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{\varphi_1} & X_1 \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi} \\ X_2 & \xrightarrow{\varphi_2} & X_2 \end{array}$$

is commutative. An **isomorphism** if ψ is bijective.

Topological Transitivity

If we have a look at topological dynamical systems we are interested in questions like:
How does φ mixes the points of X as it is applied over and over again?
Will a point return to its original position?
Will a point come arbitrarily close to any other point of X?
Will a certain point x never leave a certain region?

Definition

For an invertible system the **orbit** of $x \in X$ is defined as

$$orb(x) := \{\varphi^n(x) \ : \ n \in \mathbb{Z}\}.$$

$x \in X$ is **transitive** if $orb(x)$ is dense in X.
An invertible system $(X; \varphi)$ is **transitive** if there exists one transitive point.

Topological Transitivity

Shift algebra

Consider $(W; \tau)$ with

$$W := \left\{0, 1\right\}^{\mathbb{Z}} \text{ with } \tau((x_n)_{n\in\mathbb{Z}}) = (x_{n+1})_{n\in\mathbb{Z}}$$

$$x = \dots |0|1|01|10|11|000|001|010| \dots$$

$\forall U \subset W$ open $\exists n \in \mathbb{Z}$ s.t. $\tau^n(x) \in U$ by definition of product topology.

Topological Transitivity

Proposition

Let $(X;\varphi)$ be an invertible topological system. Then the following assertions are equivalent:

- *$(X;\varphi)$ is topologically transitive, i.e., there is a point $x \in X$ with dense orbit.*
- *For all $\emptyset \neq U,\ V$ open sets in X there is $n \in \mathbb{Z}$ with $\varphi^n(U) \cap V \neq \emptyset$.*

Topological Transitivity

Proof.

- Suppose $x \in X$ has dense orbit. Let U, V be nonempty open subsets of X. Then $\varphi^m(x) \in V$ for some $m \in \mathbb{Z}$. There exists $k \in \mathbb{Z}$ s.t. $\varphi^{k+m}(x) \in U$. Hence

$$\varphi^m(x) \in \varphi^{-k}(U) \cap V.$$

- Countable base $\left\{ U_n \middle| \ n \in \mathbb{N} \right\}$. Consider

$$G_n := \bigcup_{k \in \mathbb{Z}} \Phi^k(U_n)$$

G_n is dense in X. By Baire Category Theorem $\bigcap_{n \in \mathbb{N}} G_n$ is nonempty, dense and every point has dense forward orbit.

$\square$

Minimality

Definition

$(X;\varphi)$ is called **minimal** if there are no nontrivial closed Φ-bi-invariant sets in X.
i.e. $A \subset X$ closed and $\varphi^{-1}(A) = A \Rightarrow A = \emptyset$ or $A = X$.

Table of Contents

C*-dynamical system

Definition

A **C*-dynamical system** is a triple (A, G, α) with a group G, unital C*-algebra A and a group action $\alpha : G \to Aut(A)$.
We just use $G = \mathbb{Z}$ so write (A, T) for $T \in Aut(A)$, as the group action is completely determined by $\alpha(1)$.

From dynamical systems to C*-dynamical systems

How to get from topological dynamical systems $(X;\varphi)$ to C*-dynamical systems?
Obtain action on $A = C(X)$:

$$T_t(f)(x) = (t \cdot f)(x) = f(\varphi^{-t}(x))$$

C*-dynamical systems

Definition

(A, T) and (B, S) are **conjugate** if there exist group isomorphism $f : \mathbb{Z} \to \mathbb{Z}$ and *-isomorphism $\phi : A \to B$ s.t.

$$S^{f(t)}(\phi(a)) = \phi(T^t(a)).$$

(A, T) with commutative, unital C*-algebra A conjugate to $(C(X), \tilde{T})$ where $\tilde{T}(f) = f(\tilde{T}^{-1}(x))$ is induced action on C(X).

The three worlds

$$\begin{tikzcd} & (X,\varphi) \arrow[dl, leftrightarrow] & \\ (C(X),T_\varphi) \arrow[rr, "\longrightarrow"{description, draw=none}, phantom] & & C(X) \rtimes_{T_\varphi} \mathbb{Z} \end{tikzcd}$$

Inner actions

Definition

A group action of $\mathbb{Z}$ on A is **inner** if there is a group homomorphism $U : \mathbb{Z} \to U(A)$ s.t.

$$\alpha^t(a) = u_t a u_t^* \ \forall t \in \mathbb{Z}, a \in A$$

.

Table of Contents

Crossed products

Covariant representation

Definition

A **covariant representation** of (A, T) consists of:

- unital *-representation $\pi : A \to B(H)$
- unitary representation $u : \mathbb{Z} \to U(H)$ s.t.

$$\pi(\alpha^t(a)) = u_t \pi(a) u_t^* \ \forall a \in A, \ t \in \mathbb{Z} \,.$$

Convolution algebra

$$C_c(\mathbb{Z}, A) = \left\{ f : \mathbb{Z} \to A | f(t) \neq 0 \text{ for finitely many } t \in \mathbb{Z} \right\}$$

$$(f * g)(t) = \sum_{s \in \mathbb{Z}} f(s) s \cdot g(s^{-1} t)$$

$$f^*(t) = t \cdot f(t^{-1})^*$$

becomes *-algebra.

Crossed product

(A, T) C*-dynamical system. The **reduced crossed product** $A \rtimes_r \mathbb{Z}$ is C*-algebra with dense subset $C_c(\mathbb{Z}, A)$ having universal property:
Given (π, U), there exists unique unital *-homomorphism $\pi \rtimes U : A \rtimes_r \mathbb{Z} \to B(H)$ s.t.

$$(\pi \rtimes U) \circ i_A = \pi, \quad (\pi \rtimes U) \circ i_{\mathbb{Z}} = U.$$

Table of Contents

Some intuition

An attractor of an action (X, φ) is some set of points in the phase space, X, to which the system tends as 'time' passes on.

- Thermal equilibrium in an isolated heat system.
- Stillness of surface in an isolated fluid system.
- $\{0\}$ in $\mathbb{N} \curvearrowright \mathbb{D} : \varphi(z) = \frac{z}{2}$.

Various notions of attractors

Let (X, φ) be a topological dynamical system. A closed invariant set $\emptyset \neq M \subset X$ is called

- *Nilpotent* if there is some $n \in \mathbb{N}$ such that $\varphi^n(X) \subset M$.
- *Uniformly attractive* if for any open $U \supset M$, there exists $n \in \mathbb{N}$ such that $\varphi^n(X) \subset U$.
- *Pointwise attractive* if for any $x \in X$ and any open $U \supset M$, there exists $n \in \mathbb{N}$ such that $\varphi^n(x) \in U$.

Clearly, implications flow top to bottom.

Translation to Koopmanism

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[dr, "f \circ \phi"'] & X \arrow[d, "f"] \\ & \mathbb{C} \end{tikzcd}$$

The map $\varphi : X \to X$ induces a map $T : C(X) \to C(X) : f \mapsto f \circ \varphi$.
Invariant closed sets in X correspond to invariant closed ideals in $C(X)$ via

$$M \mapsto I_M := \{f \in C(X) : f|_M \equiv 0\}$$

and

$$I \mapsto M_I := \bigcap_{f \in I} f^{-1}(\{0\})$$

.
We can characterize the various notions of attractivity by the behaviour of T on ideals.

Nilpotency

Theorem

Let (X, φ) be a topological dynamical system, and $\emptyset \neq M \subset X$ a closed subset. M is nilpotent if and only if the Koopman operator T restricted to the ideal I_M is nilpotent.

Proof.

- If M is nilpotent
  - Take $n \in \mathbb{N}$ such that $\varphi^n(X) \subset M$
  - Then $T^n(f)(x) = f(\varphi^n(x)) \in f(M) = \{0\}$
- If M is not nilpotent
  - Take $n \in \mathbb{N}$ arbitrarily
  - Get $x \in X$ such that $\varphi^n(x) \notin M$
  - By Urysohn's lemma, find $f \in I_M$ such that $1 = f(\varphi^n(x)) = T^n(f)(x)$

$\square$

Uniform attractivity

Theorem

Let (X, φ) be a topological dynamical system, and $\emptyset \neq M \subset X$ a closed subset. M is uniformly attractive if and only if for every $f \in I_M$ we have $\lim_{n\to\infty} \|T^n(f)\|_\infty = 0$.

Proof.

- If M is uniformly attractive
  - Take $f \in I_M$ and $\epsilon > 0$ arbitrarily
  - $[f < \epsilon] := \{x \in X : |f(x)| < \epsilon\}$ is a neighbourhood of M
  - Take $n \in \mathbb{N}$ such that $\varphi^n(X) \subset [f < \epsilon]$
  - $\|T^n(f)\|_\infty = \sup_{x\in X} |f(\varphi^n(x))| \leq \sup_{x\in[f<\epsilon]} |f(x)| \leq \epsilon$
- If $\lim_{n\to\infty} \|T^n(f)\|_\infty = 0$ for all $f \in I \lhd C(X)$
  - Take a neighbourhood $U \supset M_I$
  - By Urysohn's lemma, we find $f \in I$ such that $[f < 1] \subset U$
  - Take $n \in \mathbb{N}$ such that $\|T^n(f)\|_\infty < 1$
  - For all $x \in X$, we have $|f(\varphi^n(x))| < 1$, hence $\varphi^n(x) \in [f < 1] \subset U$

Pointwise attractivity

Theorem

Let (X, φ) be a topological dynamical system, and $\emptyset \neq M \subset X$ a closed subset. M is uniformly attractive if and only if for every $f \in I_M$ and $\psi \in C(X)'$ we have $\lim_{n\to\infty} \psi(T^n(f)) = 0$.

Proof.

Almost identical to uniform attractivity $\square$

Topological Halmos-Von-Neumann Theorem

ISem24 – Project phase – Classical dynamics versus C*-dynamics

Presenters: Patrick Hermle & Jens de Vries

Supervisors: Henrik Kreidler & Bálint Farkas

Goal & Overview

Completely classify minimal dynamical systems that are “spectrally discrete”.

- Representatives for isomorphism classes: “minimal group rotations”.
- Invariant $(\mathrm{X}, \phi) \mapsto \Gamma(X, \phi)$: “point spectrum of Koopman operator”.

Koopman Operator

X compact Hausdorff space, $\phi: X \to X$ homeomorphism

The Koopman operator of (X, ϕ) is:

$$T_\phi: C(X) \to C(X), \qquad T_\phi f := f \circ \phi.$$

Note: T_ϕ is a $*$-automorphism of $C(X)$.

Point Spectra

E Banach space, $T\colon E \to E$ bounded operator

$\lambda \in \mathbb{C}$ is an eigenvalue of T if:

$$\exists\, 0 \neq f \in E \text{ with } Tf = \lambda f.$$

The point spectrum of T is:

$$\sigma_{\mathrm{p}}(T) \coloneqq \{\text{eigenvalues of } T\} \subset \mathbb{C}.$$

Point Spectrum of Koopman Operator

<u>Lemma:</u>

If (X, ϕ) is minimal, then $\sigma_{\mathrm{p}}(T_\phi)$ is a subgroup of $\mathbb{T}$.

Sketch of proof:

T_ϕ unital $\Rightarrow$ $\sigma_{\mathrm{p}}(T_\phi)$ non-empty

$\lambda \in \sigma_{\mathrm{p}}(T_\phi)$ $\Rightarrow$ $\exists\, f \neq 0$ with $T_\phi f = \lambda f$

$\Rightarrow$ $|\lambda| \cdot ||f|| = ||\lambda f|| = ||T_\phi f|| = ||f||$

$\Rightarrow$ $\lambda \in \mathbb{T}$

(X, ϕ) minimal $\Rightarrow$ $\sigma_{\mathrm{p}}(T_\phi)$ closed under products and inverses [proof omitted]

Spectrally Discrete Systems

X compact Hausdorff space, $\phi: X \to X$ homeomorphism

$0 \neq f \in C(X)$ is an eigenfunction of T_ϕ if:

$$\exists\, \lambda \in \sigma_{\mathrm{p}}(T_\phi) \text{ with } T_\phi f = \lambda f.$$

(X, ϕ) is spectrally discrete if:

$$C(X) = \overline{\mathrm{span}}\,\{\text{eigenfunctions of } T_\phi\}.$$

Important Example: Group Rotations

G compact Hausdorff group, $a \in G$

The rotation by a on G is:

$$\rho_a : G \to G, \qquad \rho_a(g) \coloneqq ag.$$

<u>Lemma:</u>

The system (G, ρ_a) is minimal if and only if $G = \overline{\{a^k : k \in \mathbb{Z}\}}$.

<u>Corollary:</u>

If (G, ρ_a) is minimal, then G is abelian.

Are Group Rotations Spectrally Discrete?

Lemma:

If (G, ρ_a) is minimal, then (G, ρ_a) is spectrally discrete.

Sketch of proof:

Pontryagin dual: $G^* \coloneqq \{\text{continuous homomorphisms } G \to \mathbb{T}\} \subset C(G)$

$\gamma \in G^*$ $\Rightarrow$ $\forall\, g \in G$ one has $(T_{\rho_a}\gamma)(g) = (\gamma \circ \rho_a)(g) = \gamma(ag) = \gamma(a)\gamma(g)$

$\Rightarrow$ $T_{\rho_a}\gamma = \gamma(a)\gamma$

$\Rightarrow$ $G^* \subset \{\text{eigenfunctions of } T_{\rho_a}\}$

G abelian $\Rightarrow$ $C(G) = \overline{\operatorname{span}}\, G^*$ by Pontryagin duality and Stone-Weierstrass [proof omitted]

$\Rightarrow$ $C(G) = \overline{\operatorname{span}}\; G^* \subset \overline{\operatorname{span}}\,\{\text{eigenfunctions of } T_{\rho_a}\} \subset C(G)$

Halmos-Von-Neumann

MSD = Minimal &
Spectrally Discrete

Theorem:

- Every MSD system is isomorphic to a minimal group rotation.

- Two MSD systems (X_1, ϕ_1) and (X_2, ϕ_2) are isomorphic if and only if
$$\sigma_{\mathrm{p}}(T_{\phi_1}) = \sigma_{\mathrm{p}}(T_{\phi_2}).$$

- If Γ is a subgroup of $\mathbb{T}$, then there is an MSD system (X, ϕ) such that
$$\Gamma = \sigma_{\mathrm{p}}(T_\phi).$$

The top. Halmos-von Neumann theorem

1. Sketch of the proof

1.1 Constr.

(Uniform enveloping Ellis semigroup)

Let $(X;\phi)$ be a TDS. Then

$$E(X,\phi) := \overline{\{\phi^n : n \in \mathbb{Z}\}}^u \subset C(X,X)$$

is a top. semigroup.

1.2 Prop: Let $(X;\phi)$ be a TDS, then t.f.a.e.:

(i) $(X;\phi)$ is spectrally discrete.

(ii) $E(X,\phi)$ is a compact top. group.

$$(X,\phi) \rightsquigarrow E(X,\phi) \rightsquigarrow (E(X,\phi), S_\phi)$$

$$\vartheta \mapsto \phi\vartheta$$

1.3 Thm. (HvN „Representation"):

Let $(X;\phi)$ be a min. eqc. discrete

$\Longrightarrow$ $$(X;\phi) \cong (E(X,\phi), S_\phi).$$

Proof: Fix $x_0 \in X$. Then

$$\phi: E(X,\phi) \to X, \quad \vartheta \mapsto \vartheta(x_0)$$

is a homoemorphism.

Furthermore

$$\begin{tikzcd} E(X,\phi) \arrow[r, "\vartheta \mapsto \phi\vartheta"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\circlearrowleft"] & E(X,\phi) \arrow[d, "\phi"] \\ X \arrow[r, "x \mapsto \phi(x)"'] & X \end{tikzcd}$$

$\Longrightarrow$ $$(X,\phi) \cong (E(X,\phi), S_\phi). \quad \boxed{}$$

1.3 Thm. (HvN „Uniqueness")

Let $(X_1;\phi_1)$ and $(X_2;\phi_2)$ be two min. spec. discrete TDS's. Then:

$$(X_1;\phi_1) \cong (X_2;\phi_2) \Leftrightarrow \sigma_p(T_{\phi_1}) = \sigma_p(T_{\phi_2}).$$

Proof: „$\Longrightarrow$:" clear.

„$\Longleftarrow$:" If $\sigma_p(T_{\phi_1}) = \sigma_p(T_{\phi_2}) =: \Gamma \subset \mathbb{T}$

$\Rightarrow$ Γ^* comp. top. group gen. by $id_\Gamma : \Gamma \to \Gamma$.

$\rightsquigarrow$ $(\Gamma^*, S_{id_\Gamma})$ min. + spectr. discrete

Enough to show:

$$\begin{tikzcd} (X_1,\phi_1) \arrow[d, phantom, "\cong" rotate=90] \arrow[r, phantom, "\cong"] & (X_2,\phi_2) \arrow[d, phantom, "\cong" rotate=90] \\ (E(X_1,\phi_1), S_{\phi_1}) \arrow[r, phantom, "\cong"] & (\Gamma^*, S_{id_{\Gamma^*}}) \cong (E(X_2,\phi_2), S_{\phi_2}) \end{tikzcd}$$

$(E(X_1,\phi_1), S_{\phi_1}) \cong (\Gamma^*, S_{id_{\Gamma^*}}) \cong (E(X_2,\phi_2), S_{\phi_2})$

construct: i

main tool: Pontryagin duality

$$\begin{tikzcd} E(X_1,\phi_1) \arrow[r, "\vartheta \mapsto \phi_1 \vartheta"] \arrow[d, "i"'] & E(X_1,\phi_1) \arrow[d, "i"] \\ \Gamma^* \arrow[r, "z' \mapsto id_{\Gamma^*} z'"'] & \Gamma^* \end{tikzcd}$$

$\boxed{\not L}$

2. Application of HVN to odometers

Recall: Odometer is a rotation system on the a-adic integers Δ_a:

- a-adic integers Δ_a: Let $a = (n_i)_{i \in \mathbb{N}} \subseteq \mathbb{N}, n_i > 1$

$$\Delta_a := \prod_{i=1}^{\infty} \{0, \dots, n_i - 1\} \text{ compact group}$$

w.r.p to product top. and the addition with carryover

even more: $x = (1, 0, 0, \dots, 0, \dots)$ gen. Δ_a as comp. group

$\rightsquigarrow (\Delta_a, S_x^a)$

$\rightsquigarrow C(\Delta_a) \rtimes_{S_x^a} \mathbb{Z}$

Lecture 11 ISEM: Let $a = (n_i)_{i \in \mathbb{N}} \subseteq \mathbb{N}$ and $b := (m_i)_{i \in \mathbb{N}} \subseteq \mathbb{N}$; $n_i, m_i > 1$, then t.f.a.e:

(i) $C(\Delta_a) \rtimes_{S_x^a} \mathbb{Z} \cong C(\Delta_b) \rtimes_{S_x^b} \mathbb{Z}$

(ii) $\forall$ prime powers p^r, we have:
p^r divides some $n_1 \cdots n_k$ iff p^r divides some $m_1 \cdots m_l$.

2.1. Corollary of HvN: (ii) $\Rightarrow$ (i).

main idea : calculate $(\Delta_a)^*$

2.2 <u>Def.</u> : a - Prüfer group

Let $a := (n_i)_{i \in \mathbb{N}} \subseteq \mathbb{N}$; $n_i > 1$.

$$C(a^\infty) := \left\{ e^{2\pi i \frac{\ell}{n_1 \cdots n_k}} : \ell \in \mathbb{Z}, k \in \mathbb{N}, 0 \leq \ell \leq n_1 \cdots n_k \right\}$$

2.3 <u>Prop</u>:

$$C(a^\infty) \underset{j}{\cong} (\Delta_a)^* \text{ via a canonical iso.}$$

Proof of corollary 2.1:

$$\mathbb{T} \supseteq C(a^\infty) \underset{j}{\cong} (\Delta_a)^* \underset{x \mapsto x \circ q}{\cong} E(\Delta_a, \rho_x^a) \underset{\psi \mapsto \psi(\rho_x^a)}{\cong} \sigma(T_{\rho_x^a}) \subseteq \mathbb{T}$$

$$q : E(\Delta_a, \rho_x^a) \to \Delta_a$$
$$\vartheta \mapsto \vartheta(x_0)$$

$$z \mapsto z$$

$$\Rightarrow \quad C(a^\infty) = \sigma(T_{\rho_x^a})$$

(D) $\longrightarrow \; \|$

$$C(b^\infty) = \sigma(T_{\rho_x^b})$$

$$\overset{HvN}{\Longrightarrow} \quad (\Delta_a, \rho_x^a) \cong (\Delta_b, \rho_x^b)$$

$\Rightarrow$ assertion. $\boxed{}$

Topological dynamical systems and related crossed product C^*-algebras

24th Internet Seminar, C^*-algebras and dynamics

Emiel Lanckriet and Matteo Pagliero
KU Leuven
June 7, 2021

Topological dynamical system

(X, φ)

compact metrizable space

homeomorphism

$\updownarrow$

action $\begin{cases} \mathbb{Z} \to \mathsf{Homeo}(X) \\ n \mapsto \varphi^n \end{cases}$

$\rightsquigarrow$

$(\mathcal{C}(X), \phi)$

continuous functions

automorphism: $\varphi(f) = f \circ \varphi^{-1}$

$\updownarrow$

action $\begin{cases} \mathbb{Z} \to \mathsf{Aut}(\mathcal{C}(X)) \\ n \mapsto \varphi^n \end{cases}$

Crossed product C^*-algebra

Given (X, φ),

$$
\begin{tikzcd}
\mathcal{C}(X) \arrow[dr, hook] & \\
& \mathcal{C}(X) \rtimes_{\varphi} \mathbb{Z} \\
\mathbb{Z} \arrow[ur, hook, dashed] &
\end{tikzcd}
$$

$$\mathbb{Z} \text{ is amenable } \Rightarrow \mathcal{C}(X) \rtimes_{\varphi,f} \mathbb{Z} \cong \mathcal{C}(X) \rtimes_{\varphi,r} \mathbb{Z},$$

so we write the crossed product: $\mathcal{C}(X) \rtimes_{\varphi} \mathbb{Z}$.

Goal

(X, φ)		$\mathcal{C}(X) \rtimes_\varphi \mathbb{Z}$
{periodic points}	$\longleftrightarrow$	{finite dimensional irreducible representations}
minimal	$\longleftrightarrow$	simple
$\Downarrow$		$\Downarrow$
topologically transitive	$\longleftrightarrow$	prime
$\Downarrow$		$\Downarrow$
topologically free	$\longleftrightarrow$	ideal intersection property

Minimality, transitivity and freeness

Recall:

- (X, φ) is minimal if **for every** $x \in X$ the orbit of x is dense:

$$\overline{\{\varphi^n(x) : n \in \mathbb{Z}\}} = X.$$

- (X, φ) is topologically transitive if **there exists** $x \in X$ the orbit of x is dense.
- (X, φ) is topologically transitive if and only if for any two open nonempty sets U and V there is a $n \in \mathbb{Z}$, such that $\varphi^n(U) \cap V \neq \emptyset$.

Minimality, transitivity and freeness

Definition

(X, φ) is topologically free if the set of aperiodic points is dense in X:

$$\overline{\{x \in X : \varphi^n(x) \neq x\}} = X.$$

It is clear that a topologically transitive system (with infinite X) is topologically free.

Periodic points actually determine finite representations on the crossed product:

Theorem

Each periodic point in (X, φ) induces a finite dimensional irreducible representation of $\mathcal{C}(X) \rtimes_\varphi \mathbb{Z}$ that is unique up to unitary equivalence.

Goal

$$
\begin{array}{ccc}
(X,\varphi) & & \mathcal{C}(X)\rtimes_\varphi \mathbb{Z} \\
\\
\{\text{periodic points}\} & \overset{\checkmark}{\longleftrightarrow} & \{\text{finite dimensional} \\
& & \text{irreducible representations}\} \\
\\
\text{minimal} & \longleftrightarrow & \text{simple} \\
\Downarrow & & \Downarrow \\
\text{topologically transitive} & \longleftrightarrow & \text{prime} \\
\Downarrow & & \Downarrow \\
\text{topologically free} & \longleftrightarrow & \text{ideal intersection property}
\end{array}
$$

Goal

$$
\begin{array}{ccc}
(X,\varphi) & & \mathcal{C}(X) \rtimes_\varphi \mathbb{Z} \\
\\
\{\text{periodic points}\} & \overset{\checkmark}{\longleftrightarrow} & \{\text{finite dimensional irreducible representations}\} \\
\\
\color{red}{\text{minimal}} & \color{red}{\longleftrightarrow} & \color{red}{\text{simple}} \\
\Downarrow & & \Downarrow \\
\text{topologically transitive} & \longleftrightarrow & \text{prime} \\
\Downarrow & & \Downarrow \\
\text{topologically free} & \longleftrightarrow & \text{ideal intersection property}
\end{array}
$$

Minimal systems vs simple crossed products

Example (Irrational rotation on the circle)

Let

- $X = \mathbb{T}$,
- $\varphi^n(z) = e^{2\pi i n \vartheta} z$ with irrational ϑ.

As we have shown during the online lectures:

Theorem

If X is infinite, (X, φ) is minimal if and only if $\mathcal{C}(X) \rtimes_\varphi \mathbb{Z}$ is simple.

Then since we know that $(\mathbb{T}, \varphi)$ is minimal, it follows that $\mathcal{C}(\mathbb{T}) \rtimes_\varphi \mathbb{Z} = A_\vartheta$ is simple for every irrational ϑ.

Goal

$$
\begin{array}{ccc}
(X,\varphi) & & C(X) \rtimes_\varphi \mathbb{Z} \\
\\
\{\text{periodic points}\} & \overset{\checkmark}{\longleftrightarrow} & \{\text{finite dimensional irreducible representations}\} \\
\\
\text{minimal} & \overset{\checkmark}{\longleftrightarrow} & \text{simple} \\
\Downarrow & & \Downarrow \\
\text{topologically transitive} & \longleftrightarrow & \text{prime} \\
\Downarrow & & \Downarrow \\
\text{topologically free} & \longleftrightarrow & \text{ideal intersection property}
\end{array}
$$

Goal

$$
\begin{array}{ccc}
(X, \varphi) & & \mathcal{C}(X) \rtimes_\varphi \mathbb{Z} \\
\\
\{\text{periodic points}\} & \overset{\checkmark}{\longleftrightarrow} & \{\text{finite dimensional} \\
 & & \text{irreducible representations}\} \\
\\
\text{minimal} & \overset{\checkmark}{\longleftrightarrow} & \text{simple} \\
\Downarrow & & \Downarrow \\
\text{topologically transitive} & \longleftrightarrow & \text{prime} \\
\Downarrow & & \Downarrow \\
\\
\text{topologically free} & \longleftrightarrow & \text{ideal intersection property}
\end{array}
$$

Topologically free systems vs ideal intersection property

Example (Irrational rotation on the disk)

- $X = \overline{\mathbb{D}}$,
- $\varphi^n(z) = e^{2\pi i n \vartheta} z$ for an irrational ϑ.

$(\overline{\mathbb{D}}, \varphi)$ is topologically free since $\overline{\mathbb{D}} = \bigcup_{r \in [0,1]} r\,\mathbb{T}$.

Definition

We say that $\mathcal{C}(X) \rtimes_\varphi \mathbb{Z}$ has the ideal intersection property if for each closed ideal $I \subseteq \mathcal{C}(X) \rtimes_\varphi \mathbb{Z}$,

$$I \neq \{0\} \;\Leftrightarrow\; I \cap \mathcal{C}(X) \neq \{0\}.$$

Note that "$\Leftarrow$" is always true!

Theorem

(X, φ) is topologically free if and only if $\mathcal{C}(X) \rtimes_\varphi \mathbb{Z}$ has the ideal intersection property.

$$\text{“}\Rightarrow\text{”}$$

Towards a contradiction, suppose $I \cap \mathcal{C}(X) = \{0\}$.

$$E : \mathcal{C}(X) \rtimes_\varphi \mathbb{Z} \longrightarrow \mathcal{C}(X) \text{ faithful}$$

Since $I \neq \{0\}$, there is $a \in I$ with $E(a) \neq 0 \;\Rightarrow\; E(a)(x) \neq 0$ for some $x \in X$. By topological freeness we can choose x to be aperiodic.
Result: if x is aperiodic, $\mathrm{ev}_x : \mathcal{C}(X) \to \mathbb{C},\; f \mapsto f(x)$ has a unique pure state extension to $\mathcal{C}(X) \rtimes_\varphi \mathbb{Z}$.

$$\begin{tikzcd}
\mathcal{C}(X) \rtimes_\varphi \mathbb{Z} \arrow[r, "q"] \arrow[d, "E"'] & \frac{\mathcal{C}(X) \rtimes_\varphi \mathbb{Z}}{I} \arrow[d, "\exists\psi"] \\
\mathcal{C}(X) \arrow[r, "\mathrm{ev}_x"'] & \mathbb{C}
\end{tikzcd}$$

but $\psi \circ q(a) = 0 \neq \mathrm{ev}_x \circ E(a)$, a contradiction.

Goal

$$
\begin{array}{ccc}
(X, \varphi) & & \mathcal{C}(X) \rtimes_\varphi \mathbb{Z} \\[1em]
\{\text{periodic points}\} & \overset{\checkmark}{\longleftrightarrow} & \begin{array}{c}\{\text{finite dimensional} \\ \text{irreducible representations}\}\end{array} \\[1em]
\text{minimal} & \overset{\checkmark}{\longleftrightarrow} & \text{simple} \\
\Downarrow & & \Downarrow \\
\text{topologically transitive} & \longleftrightarrow & \text{prime} \\
\Downarrow & & \Downarrow \\
\text{topologically free} & \overset{\checkmark}{\longleftrightarrow} & \text{ideal intersection property}
\end{array}
$$

Goal

(X, φ)		$\mathcal{C}(X) \rtimes_\varphi \mathbb{Z}$
{periodic points}	$\overset{\checkmark}{\longleftrightarrow}$	{finite dimensional irreducible representations}
minimal	$\overset{\checkmark}{\longleftrightarrow}$	simple
$\Downarrow$		$\Downarrow$
topologically transitive	$\longleftrightarrow$	prime
$\Downarrow$		$\Downarrow$
topologically free	$\overset{\checkmark}{\longleftrightarrow}$	ideal intersection property

Topological transitive systems vs prime crossed products

Example (Bernoulli shift)

- $X = \{0,1\}^{\mathbb{Z}}$,
- $\varphi((x_k)_{k\in\mathbb{Z}}) = (x_{k+1})_{k\in\mathbb{Z}}$.

The sequence $\ldots \mid 0 \mid 1 \mid 0\ 0 \mid 0\ 1 \mid 1\ 0 \mid 1\ 1 \mid 0\ 0\ 0 \mid 0\ 0\ 1 \mid \ldots$ has dense orbit and $(\{0,1\}^{\mathbb{Z}}, \varphi)$ is topologically transitive.

Definition

A C^*-algebra is prime if, for any two ideals I and J,

$$I \cap J = \{0\} \;\Rightarrow\; I = \{0\} \vee J = \{0\}.$$

Theorem

(X, φ) is an infinite topologically transitive dynamical system if and only if $\mathcal{C}(X) \rtimes_\varphi \mathbb{Z}$ is a prime C^*-algebra.

"$\Rightarrow$"

Recall that topological transitivity is equivalent to: For all open $U, V \subset X$,

$$\text{there is } n \in \mathbb{Z},\ \varphi^n(U) \cap V \neq \emptyset \quad \xrightarrow{\text{if inv.}} \quad U \cap V \neq \emptyset.$$

Let I, J be ideals in $\mathcal{C}(X) \rtimes_\varphi \mathbb{Z}$ with $I \cap J = \{0\}$.
Towards a contradiction, assume $I \neq \{0\}$ and $J \neq \{0\}$.Since topological transitivity implies topological freeness,

$$\begin{cases} \mathcal{C}(X) \cap I \neq \{0\} \\ \mathcal{C}(X) \cap J \neq \{0\} \end{cases} \qquad \begin{cases} \mathcal{C}(X) \cap I = \{f \in \mathcal{C}(X) : f|_E = 0\} \\ \mathcal{C}(X) \cap J = \{f \in \mathcal{C}(X) : f|_F = 0\} \end{cases}$$

E^c and F^c are invariant open subsets of X, so

$$E^c \cap F^c \neq \emptyset \ \Rightarrow\ E \cup F \neq X \ \Rightarrow\ I \cap J \neq \{0\}, \text{ a contradiction.}$$

Goal

$$
\begin{array}{ccc}
(X,\varphi) & & C(X)\rtimes_\varphi \mathbb{Z} \\
\{\text{periodic points}\} & \overset{\checkmark}{\longleftrightarrow} & \{\text{finite dimensional irreducible representations}\} \\
\text{minimal} & \overset{\checkmark}{\longleftrightarrow} & \text{simple} \\
\Downarrow & & \Downarrow \\
\text{topologically transitive} & \overset{\checkmark}{\longleftrightarrow} & \text{prime} \\
\Downarrow & & \Downarrow \\
\text{topologically free} & \overset{\checkmark}{\longleftrightarrow} & \text{ideal intersection property}
\end{array}
$$

Thank you for your attention!

Questions?

Minimal C^*-dynamical systems

Let A be a generic C^*-algebra, G a discrete group, $\alpha : G \to \mathrm{Aut}(A)$ an action:

$$(A, \alpha) \text{ is called } \mathrm{C}^*\text{-dynamical system.}$$

Definition

(A, α) is minimal if A has no non-trivial invariant ideals.

Theorem (Archbold–Spielberg 1994)

If (A, α) is minimal and topologically free (on the spectrum $\hat{A}$) then $A \rtimes_{\alpha,r} G$ is simple.

Topological freeness

α is an action of a discrete group G on $\mathcal{C}(X)$,

Theorem (Kawamura–Tomiyama 1990)

If G is amenable,

$$\begin{array}{ccc} (\mathcal{C}(X),\alpha) \text{ is} & \iff & \mathcal{C}(X) \rtimes_\alpha G \text{ has the ideal} \\ \text{topologically free} & & \text{intersection property.} \end{array}$$

Theorem (Archbold–Spielberg 1993)

Let $q : \mathcal{C}(X) \rtimes_{\sigma,f} G \to \mathcal{C}(X) \rtimes_{\sigma,r} G$ be the canonical surjection.

$$\begin{array}{ccc} (\mathcal{C}(X),\alpha) \text{ is} & \iff & I \triangleleft \mathcal{C}(X) \rtimes_{\sigma,f} G,\ I \cap \mathcal{C}(X) = \{0\} \\ \text{topologically free} & & \Rightarrow\ q(I) = \{0\}. \end{array}$$

For non-abelian algebras does not hold:
For $\mathcal{K}$ and $G = \mathbb{Z} \oplus \mathbb{Z}$ there is an action for which "$\not\Leftarrow$"

Invariant ideals

Let G be a discrete group. The map

$$\{\text{ideals in } A \rtimes_{\alpha,r} G\} \to \{\text{ invariant ideals in A}\}$$
$$I \mapsto I \cap A$$

is always surjective. When it is also injective we say that A separates ideals in $A \rtimes_{\alpha,r} G$.

Topological freeness is not enough to ensure that A separates ideals in $A \rtimes_{\alpha,r} G$.

It is believed that essential freeness is enough (Renault), with some more assumptions it is (Sierakovski).