

Topological dynamical systems

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Topological dynamical system

A **topological dynamical system** is a pair $(X; \varphi)$, with nonempty compact metrizable space X and $\varphi : X \rightarrow X$ continuous. It is called **invertible**, if φ is invertible i.e. a homeomorphism.

Finite state space

Finite set $X := \{0, \dots, d - 1\}$ with discrete topology.

Every map $\varphi : X \rightarrow X$ is continuous.

$(X; \varphi)$ invertible if and only if φ permutation.

Rotation on Compact Groups

A **topological group** is a group $(G, \cdot)$ with topology s.t. inversion and multiplication are continuous.

Compact topological group G with **left rotation** by $a \in G$

$$\varphi_a : G \rightarrow G, \varphi_a(g) := a \cdot g,$$

forms invertible topological system $(G; \varphi_a)$. i.e. $\mathbb{T}$ compact Abelian group.

Homomorphisms of topological dynamical systems

A **homomorphism** between topological systems $(X_1; \varphi_1), (X_2; \varphi_2)$ is continuous map $\psi : X_1 \rightarrow X_2$ such that $\psi \circ \varphi_1 = \varphi_2 \circ \psi$, i.e., the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{\varphi_1} & X_1 \\ \downarrow \psi & & \downarrow \psi \\ X_2 & \xrightarrow{\varphi_2} & X_2 \end{array}$$

is commutative. An **isomorphism** if ψ is bijective.

Topological Transitivity

If we have a look at topological dynamical systems we are interested in questions like:

How does φ mixes the points of X as it is applied over and over again?

Will a point return to its original position?

Will a point come arbitrarily close to any other point of X ?

Will a certain point x never leave a certain region?

Topological Transitivity

Orbits

Definition

For an invertible system the **orbit** of $x \in X$ is defined as

$$\text{orb}(x) := \{\varphi^n(x) : n \in \mathbb{Z}\}.$$

$x \in X$ is **transitive** if $\text{orb}(x)$ is dense in X .

An invertible system $(X; \varphi)$ is **transitive** if there exists one transitive point.

Topological Transitivity

Shift algebra

Consider $(W; \tau)$ with

$$W := \{0, 1\}^{\mathbb{Z}} \text{ with } \tau((x_n)_{n \in \mathbb{Z}}) = (x_{n+1})_{n \in \mathbb{Z}}$$

$$x = \dots |0|1|01|10|11|000|001|010| \dots$$

$\forall U \subset W$ open $\exists n \in \mathbb{Z}$ s.t. $\tau^n(x) \in U$ by definition of product topology.

Topological Transitivity

Proposition

Let $(X; \varphi)$ be an invertible topological system. Then the following assertions are equivalent:

- ▶ *$(X; \varphi)$ is topologically transitive, i.e., there is a point $x \in X$ with dense orbit.*
- ▶ *For all $\emptyset \neq U, V$ open sets in X there is $n \in \mathbb{Z}$ with $\varphi^n(U) \cap V \neq \emptyset$.*

Topological Transitivity

Proof.

- Suppose $x \in X$ has dense orbit. Let U, V be nonempty open subsets of X . Then $\varphi^m(x) \in V$ for some $m \in \mathbb{Z}$. There exists $k \in \mathbb{Z}$ s.t. $\varphi^{k+m}(x) \in U$. Hence

$$\varphi^m(x) \in \varphi^{-k}(U) \cap V.$$

- Countable base $\{U_n \mid n \in \mathbb{N}\}$. Consider

$$G_n := \bigcup_{k \in \mathbb{Z}} \varphi^k(U_n)$$

G_n is dense in X . By Baire Category Theorem $\bigcap_{n \in \mathbb{N}} G_n$ is nonempty, dense and every point has dense forward orbit.



Minimality

Definition

$(X; \varphi)$ is called **minimal** if there are no nontrivial closed Φ -bi-invariant sets in X .

i.e. $A \subset X$ closed and $\varphi^{-1}(A) = A \Rightarrow A = \emptyset$ or $A = X$.

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C*-dynamical system

Definition

A **C*-dynamical system** is a triple (A, G, α) with a group G , unital C*-algebra A and a group action $\alpha : G \rightarrow \text{Aut}(A)$.

We just use $G = \mathbb{Z}$ so write (A, T) for $T \in \text{Aut}(A)$, as the group action is completely determined by $\alpha(1)$.

From dynamical systems to C^* -dynamical systems

How to get from topological dynamical systems $(X; \varphi)$ to C^* -dynamical systems?

Obtain action on $A = C(X)$:

$$T_t(f)(x) = (t \cdot f)(x) = f(\varphi^{-t}(x))$$

C*-dynamical systems

Definition

(A, T) and (B, S) are **conjugate** if there exist group isomorphism $f : \mathbb{Z} \rightarrow \mathbb{Z}$ and *-isomorphism $\phi : A \rightarrow B$ s.t.

$$S^{f(t)}(\phi(a)) = \phi(T^t(a)).$$

(A, T) with commutative, unital C*-algebra A conjugate to $(C(X), \tilde{T})$ where $\tilde{T}(f) = f(\tilde{T}^{-1}(x))$ is induced action on $C(X)$.

The three worlds

$$\begin{array}{ccc} & (X, \varphi) & \\ \swarrow & & \nearrow \\ (C(X), T_\varphi) & \longrightarrow & C(X) \rtimes_{T_\varphi} \mathbb{Z} \end{array}$$

Inner actions

Definition

A group action of $\mathbb{Z}$ on A is **inner** if there is a group homomorphism $U : \mathbb{Z} \rightarrow U(A)$ s.t.

$$\alpha^t(a) = u_t a u_t^* \quad \forall t \in \mathbb{Z}, a \in A$$

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Covariant representation

Definition

A **covariant representation** of (A, T) consists of:

- ▶ unital $*$ -representation $\pi : A \rightarrow B(H)$
- ▶ unitary representation $u : \mathbb{Z} \rightarrow U(H)$ s.t.

$$\pi(\alpha^t(a)) = u_t \pi(a) u_t^* \quad \forall a \in A, \quad t \in \mathbb{Z}.$$

Convolution algebra

$$C_c(\mathbb{Z}, A) = \left\{ f : \mathbb{Z} \rightarrow A \mid f(t) \neq 0 \text{ for finitely many } t \in \mathbb{Z} \right\}$$

$$(f * g)(t) = \sum_{s \in \mathbb{Z}} f(s) s \cdot g(s^{-1}t)$$

$$f^*(t) = t \cdot f(t^{-1})^*$$

becomes $*$ -algebra.

Crossed product

(A, T) C^* -dynamical system. The **reduced crossed product** $A \rtimes_r \mathbb{Z}$ is C^* -algebra with dense subset $C_c(\mathbb{Z}, A)$ having universal property:

Given (π, U) , there exists unique unital $*$ -homomorphism $\pi \rtimes U : A \rtimes_r \mathbb{Z} \rightarrow B(H)$ s.t.

$$(\pi \rtimes U) \circ i_A = \pi, \quad (\pi \rtimes U) \circ i_{\mathbb{Z}} = U.$$

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Some intuition

An attractor of an action (X, φ) is some set of points in the phase space, X , to which the system tends as 'time' passes on.

- ▶ Thermal equilibrium in an isolated heat system.
- ▶ Stillness of surface in an isolated fluid system.
- ▶ $\{0\}$ in $\mathbb{N} \curvearrowright \mathbb{D} : \varphi(z) = \frac{z}{2}$.

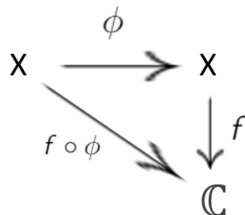
Various notions of attractors

Let (X, φ) be a topological dynamical system. A closed invariant set $\emptyset \neq M \subset X$ is called

- ▶ *Nilpotent* if there is some $n \in \mathbb{N}$ such that $\varphi^n(X) \subset M$.
- ▶ *Uniformly attractive* if for any open $U \supset M$, there exists $n \in \mathbb{N}$ such that $\varphi^n(X) \subset U$.
- ▶ *Pointwise attractive* if for any $x \in X$ and any open $U \supset M$, there exists $n \in \mathbb{N}$ such that $\varphi^n(x) \in U$.

Clearly, implications flow top to bottom.

Translation to Koopmanism



The map $\varphi : X \rightarrow X$ induces a map

$T : C(X) \rightarrow C(X) : f \mapsto f \circ \varphi$.

Invariant closed sets in X correspond to invariant closed ideals in $C(X)$ via

$$M \mapsto I_M := \{f \in C(X) : f|_M \equiv 0\}$$

and

$$I \mapsto M_I := \bigcap_{f \in I} f^{-1}(\{0\})$$

.

We can characterize the various notions of attractivity by the behaviour of T on ideals.

Nilpotency

Theorem

Let (X, φ) be a topological dynamical system, and $\emptyset \neq M \subset X$ a closed subset. M is nilpotent if and only if the Koopman operator T restricted to the ideal I_M is nilpotent.

Proof.

- ▶ If M is nilpotent
 - ▶ Take $n \in \mathbb{N}$ such that $\varphi^n(X) \subset M$
 - ▶ Then $T^n(f)(x) = f(\varphi^n(x)) \in f(M) = \{0\}$
- ▶ If M is not nilpotent
 - ▶ Take $n \in \mathbb{N}$ arbitrarily
 - ▶ Get $x \in X$ such that $\varphi^n(x) \notin M$
 - ▶ By Urysohn's lemma, find $f \in I_M$ such that $1 = f(\varphi^n(x)) = T^n(f)(x)$



Uniform attractivity

Theorem

Let (X, φ) be a topological dynamical system, and $\emptyset \neq M \subset X$ a closed subset. M is uniformly attractive if and only if for every $f \in I_M$ we have $\lim_{n \rightarrow \infty} \|T^n(f)\|_\infty = 0$.

Proof.

- ▶ If M is uniformly attractive
 - ▶ Take $f \in I_M$ and $\epsilon > 0$ arbitrarily
 - ▶ $[f < \epsilon] := \{x \in X : |f(x)| < \epsilon\}$ is a neighbourhood of M
 - ▶ Take $n \in \mathbb{N}$ such that $\varphi^n(X) \subset [f < \epsilon]$
 - ▶ $\|T^n(f)\|_\infty = \sup_{x \in X} |f(\varphi^n(x))| \leq \sup_{x \in [f < \epsilon]} |f(x)| \leq \epsilon$
- ▶ If $\lim_{n \rightarrow \infty} \|T^n(f)\|_\infty = 0$ for all $f \in I \triangleleft C(X)$
 - ▶ Take a neighbourhood $U \supset M_I$
 - ▶ By Urysohn's lemma, we find $f \in I$ such that $[f < 1] \subset U$
 - ▶ Take $n \in \mathbb{N}$ such that $\|T^n(f)\|_\infty < 1$
 - ▶ For all $x \in X$, we have $|f(\varphi^n(x))| < 1$, hence $\varphi^n(x) \in [f < 1] \subset U$

Pointwise attractivity

Theorem

Let (X, φ) be a topological dynamical system, and $\emptyset \neq M \subset X$ a closed subset. M is uniformly attractive if and only if for every $f \in I_M$ and $\psi \in C(X)'$ we have $\lim_{n \rightarrow \infty} \psi(T^n(f)) = 0$.

Proof.

Almost identical to uniform attractivity



Topological Halmos-Von-Neumann Theorem

ISem24 — Project phase — Classical dynamics versus C^* -dynamics

Presenters: Patrick Hermle & Jens de Vries

Supervisors: Henrik Kreidler & Bálint Farkas

Goal & Overview

Completely classify minimal dynamical systems that are “spectrally discrete”.

- Representatives for isomorphism classes: “minimal group rotations”.
- Invariant $(X, \phi) \mapsto \Gamma(X, \phi)$: “point spectrum of Koopman operator”.

Koopman Operator

X compact Hausdorff space, $\phi: X \rightarrow X$ homeomorphism

The Koopman operator of (X, ϕ) is:

$$T_\phi: C(X) \rightarrow C(X), \quad T_\phi f := f \circ \phi.$$

Note: T_ϕ is a $*$ -automorphism of $C(X)$.

Point Spectra

E Banach space, $T: E \rightarrow E$ bounded operator

$\lambda \in \mathbb{C}$ is an **eigenvalue** of T if:

$$\exists 0 \neq f \in E \text{ with } Tf = \lambda f.$$

The **point spectrum** of T is:

$$\sigma_p(T) := \{\text{eigenvalues of } T\} \subset \mathbb{C}.$$

Point Spectrum of Koopman Operator

Lemma:

If (X, ϕ) is minimal, then $\sigma_p(T_\phi)$ is a subgroup of $\mathbb{T}$.

Sketch of proof:

T_ϕ unital $\Rightarrow \sigma_p(T_\phi)$ non-empty

$\lambda \in \sigma_p(T_\phi) \Rightarrow \exists f \neq 0$ with $T_\phi f = \lambda f$
 $\Rightarrow |\lambda| \cdot \|f\| = \|\lambda f\| = \|T_\phi f\| = \|f\|$
 $\Rightarrow \lambda \in \mathbb{T}$

(X, ϕ) minimal $\Rightarrow \sigma_p(T_\phi)$ closed under products and inverses [proof omitted]

Spectrally Discrete Systems

X compact Hausdorff space, $\phi: X \rightarrow X$ homeomorphism

$0 \neq f \in C(X)$ is an **eigenfunction** of T_ϕ if:

$$\exists \lambda \in \sigma_p(T_\phi) \text{ with } T_\phi f = \lambda f.$$

(X, ϕ) is **spectrally discrete** if:

$$C(X) = \overline{\text{span}} \{ \text{eigenfunctions of } T_\phi \}.$$

Important Example: Group Rotations

G compact Hausdorff group, $a \in G$

The rotation by a on G is:

$$\rho_a: G \rightarrow G, \quad \rho_a(g) := ag.$$

Lemma:

The system (G, ρ_a) is minimal if and only if $G = \overline{\{a^k : k \in \mathbb{Z}\}}$.

Corollary:

If (G, ρ_a) is minimal, then G is abelian.

Are Group Rotations Spectrally Discrete?

Lemma:

If (G, ρ_a) is minimal, then (G, ρ_a) is spectrally discrete.

Sketch of proof:

Pontryagin dual: $G^* := \{\text{continuous homomorphisms } G \rightarrow \mathbb{T}\} \subset C(G)$

$$\gamma \in G^* \quad \Rightarrow \quad \forall g \in G \text{ one has } (T_{\rho_a} \gamma)(g) = (\gamma \circ \rho_a)(g) = \gamma(ag) = \gamma(a)\gamma(g)$$

$$\Rightarrow \quad T_{\rho_a} \gamma = \gamma(a)\gamma$$

$$\Rightarrow \quad G^* \subset \{\text{eigenfunctions of } T_{\rho_a}\}$$

$$G \text{ abelian} \quad \Rightarrow \quad C(G) = \overline{\text{span}} G^* \text{ by Pontryagin duality and Stone-Weierstrass [proof omitted]}$$

$$\Rightarrow \quad C(G) = \overline{\text{span}} G^* \subset \overline{\text{span}} \{\text{eigenfunctions of } T_{\rho_a}\} \subset C(G)$$

Halmos-Von-Neumann



MSD = Minimal &
Spectrally Discrete

Theorem:

- Every MSD system is isomorphic to a minimal group rotation.
- Two MSD systems (X_1, ϕ_1) and (X_2, ϕ_2) are isomorphic if and only if
$$\sigma_p(T_{\phi_1}) = \sigma_p(T_{\phi_2}).$$
- If Γ is a subgroup of $\mathbb{T}$, then there is an MSD system (X, ϕ) such that
$$\Gamma = \sigma_p(T_{\phi}).$$

The top. Halmos-von Neumann theorem

1. Sketch of the proof

1.1 constr.

(uniform enveloping Ellis semigroup)

Let $(X; \phi)$ be a TDS. Then

$$E(X, \phi) := \overline{\{\phi^n : n \in \mathbb{Z}\}}^u \subset C(X, X)$$

is a top. semigroup.

1.2 Prop.: Let $(X; \phi)$ be a TDS,

then t.f.a.e.:

(i) $(X; \phi)$ is spectrally discrete.

(ii) $E(X, \phi)$ is a compact top. group

$$(X, \phi) \leadsto E(X, \phi) \leadsto (E(X, \phi), S_\phi)$$

$\uparrow$
 $\sigma \mapsto \phi \sigma$

1.3 Thm. (HvN "Representation") :

Let $(X; \phi)$ be a min. spec. discrete

$$\Rightarrow (X; \phi) \cong (E(X, \phi), S_\phi).$$

Proof : Fix $x_0 \in X$. Then

$$\phi : E(X, \phi) \rightarrow X, \quad v \mapsto \phi(v)$$

is a homeomorphism.

Furthermore

$$\begin{array}{ccc} E(X, \phi) & \xrightarrow{v \mapsto \phi v} & E(X, \phi) \\ \phi \downarrow & \curvearrowright & \downarrow \phi \\ X & \xrightarrow{x \mapsto \phi(x)} & X \end{array}$$

$$\Rightarrow (X, \phi) \cong (E(X, \phi), S_\phi). \quad \square$$

1.3 Thm. (HvN „Uniqueness“) :

Let (X_1, ϕ_1) and (X_2, ϕ_2) be two min.
spec. discrete TDS's. Then :

$$(X_1, \phi_1) \cong (X_2, \phi_2) \Leftrightarrow \sigma_p(T_{\phi_1}) = \sigma_p(T_{\phi_2}).$$

Proof : " $\Rightarrow$ " : "clear".

" $\Leftarrow$ " : If $\sigma_p(T_{\phi_1}) = \sigma_p(T_{\phi_2}) =: \Gamma \subset \mathbb{T}$

$\Rightarrow \Gamma^*$ comp. top. group gen. by $\text{id}_p: p \mapsto p$.

$\sim_D (\Gamma^*, S_{\text{id}_p})$ min. + spectr. discrete

Enough to show : $\cong (X_1, \phi_1) \cong (X_2, \phi_2)$

$$(E(X_1, \phi_1), S_{\phi_1}) \stackrel{\sim}{\cong} (\Gamma^*, S_{\text{id}_{\Gamma^*}}) \stackrel{\sim}{\cong} (E(X_2, \phi_2), S_{\phi_2})$$

construct
main tool : $\text{Pontryagin duality}$

$$\psi \mapsto \phi_1 \psi$$

$$E(X_1, \phi_1) \longrightarrow E(X_2, \phi_2)$$

$$\downarrow i$$

$$\Gamma$$

$$\downarrow i$$

$$\Gamma^*$$

$$\longrightarrow$$

$$\Gamma^*$$

$$z' \mapsto \text{id}_{\Gamma^*} z'$$

2. Application of HVN to odometers

Recall: Odometer is a rotation system on the a -adic integers Δ_a :

• a -adic integers Δ_a : Let $a = (n_i)_{i \in \mathbb{N}} \subseteq \mathbb{N}, n_i > 1$

$$\Delta_a := \prod_{i=1}^{\infty} \{0, \dots, n_i - 1\} \text{ compact group}$$

w.r.p to product top. and the addition with carryover

even more: $x = (1, 0, 0, \dots, 0, \dots)$ gen. Δ_a as comp. group

$$\leadsto (\Delta_a, S_x^a)$$

$$\leadsto C(\Delta_a) \rtimes_{S_x^a} \mathbb{Z}$$

Lecture 11 ISEM: Let $a = (n_i)_{i \in \mathbb{N}} \subseteq \mathbb{N}$ and

$b = (m_i)_{i \in \mathbb{N}} \subseteq \mathbb{N}; n_i, m_i > 1$, then t.f.a.e:

$$(i) \quad C(\Delta_a) \rtimes_{S_x^a} \mathbb{Z} \cong C(\Delta_b) \rtimes_{S_x^b} \mathbb{Z}$$

(ii) $\forall$ prime powers p^r , we have:

p^r divides some $n_1 \dots n_k$ iff p^r divides some $m_1 \dots m_\ell$.

2.1. Corollary of HVN: (ii) $\Leftrightarrow$ (i).

main idea : calculate $(\Delta_a)^*$

2.2 Def. : a-Prüfer group

Let $a := (n_i)_{i \in \mathbb{N}} \subseteq \mathbb{N}; n_i > 1$.

$$C(a^\infty) := \left\{ e^{2\pi i \frac{\ell}{n_1 \cdots n_k}} : \ell \in \mathbb{Z}, k \in \mathbb{N}, 0 \leq \ell < n_1 \cdots n_k \right\}$$

2.3 Prop. : $C(a^\infty) \stackrel{j}{\cong} (\Delta_a)^*$ via a canonical iso.

Proof of corollary 2.1:

$$\pi \supseteq C(a^\infty) \stackrel{j}{\cong} (\Delta_a)^* \stackrel{x \mapsto x \circ q}{\cong} E(\Delta_a, S_x^a) \stackrel{\psi \mapsto \psi(S_x^a)}{\cong} \sigma(TS_x^a) \subseteq \pi$$

$$q: E(\Delta_a, S_x^a) \rightarrow \Delta_a$$

$$\vartheta \mapsto \vartheta(x_0)$$

$$z \mapsto z$$

$$\Rightarrow C(a^\infty) = \sigma(TS_x^a)$$

(D) $\rightarrow \parallel$

$$C(b^\infty) = \sigma(TS_x^b)$$

HVN

$$\Rightarrow (\Delta_a, S_x^a) \cong (\Delta_b, S_x^b)$$

$\Rightarrow$ assertion .



Topological dynamical systems and related crossed product C^* -algebras

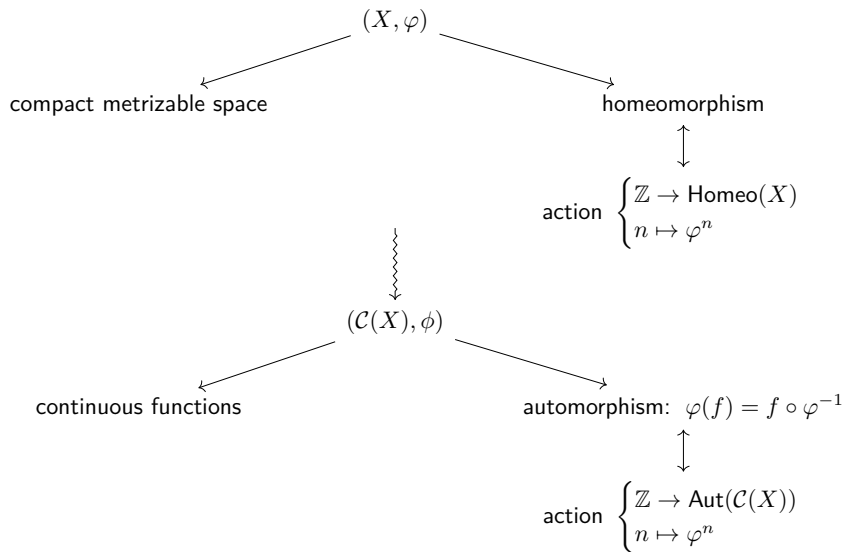
24th Internet Seminar, C^* -algebras and dynamics

Emiel Lanckriet and Matteo Pagliero

KU Leuven

June 7, 2021

Topological dynamical system



Crossed product C^* -algebra

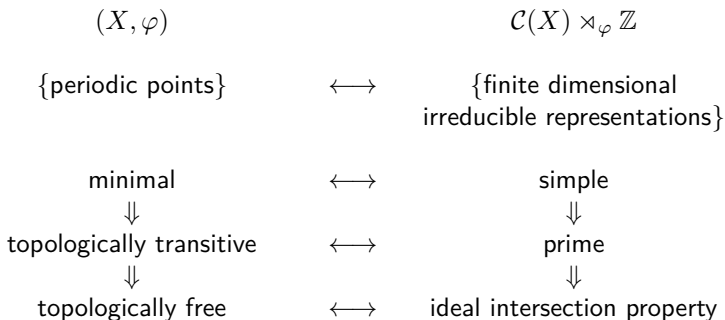
Given (X, φ) ,

$$\begin{array}{ccc} \mathcal{C}(X) & \xrightarrow{\quad} & \mathcal{C}(X) \rtimes_{\varphi} \mathbb{Z} \\ & \nearrow \text{dashed} & \\ \mathbb{Z} & & \end{array}$$

$\mathbb{Z}$ is amenable $\Rightarrow \mathcal{C}(X) \rtimes_{\varphi, f} \mathbb{Z} \cong \mathcal{C}(X) \rtimes_{\varphi, r} \mathbb{Z}$,

so we write the crossed product: $\mathcal{C}(X) \rtimes_{\varphi} \mathbb{Z}$.

Goal



Minimality, transitivity and freeness

Recall:

- (X, φ) is minimal if **for every** $x \in X$ the orbit of x is dense:

$$\overline{\{\varphi^n(x) : n \in \mathbb{Z}\}} = X.$$

- (X, φ) is topologically transitive if **there exists** $x \in X$ the orbit of x is dense.
- (X, φ) is topologically transitive if and only if for any two open nonempty sets U and V there is a $n \in \mathbb{Z}$, such that $\varphi^n(U) \cap V \neq \emptyset$.

Minimality, transitivity and freeness

Definition

(X, φ) is topologically free if the set of aperiodic points is dense in X :

$$\overline{\{x \in X : \varphi^n(x) \neq x\}} = X.$$

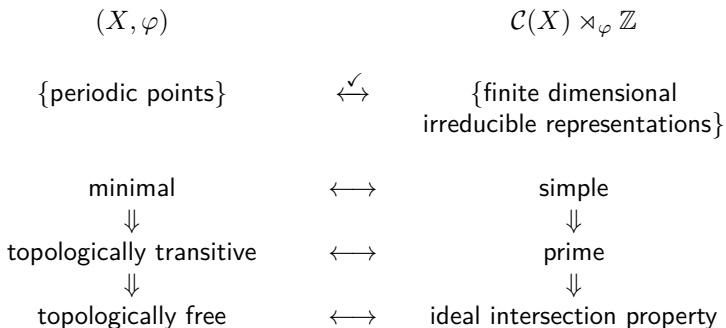
It is clear that a topologically transitive system (with infinite X) is topologically free.

Periodic points actually determine finite representations on the crossed product:

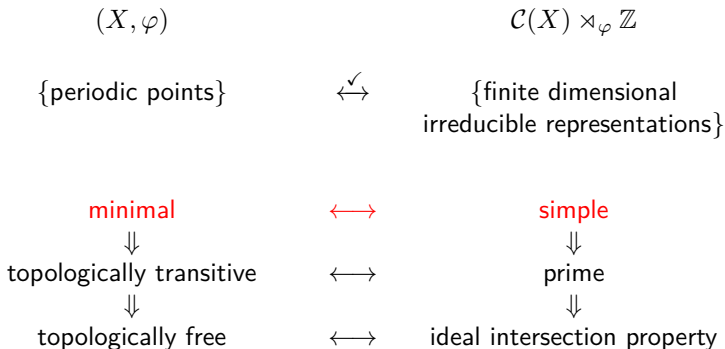
Theorem

Each periodic point in (X, φ) induces a finite dimensional irreducible representation of $\mathcal{C}(X) \rtimes_{\varphi} \mathbb{Z}$ that is unique up to unitary equivalence.

Goal



Goal



Minimal systems vs simple crossed products

Example (Irrational rotation on the circle)

Let

- $X = \mathbb{T}$,
- $\varphi^n(z) = e^{2\pi i n \vartheta} z$ with irrational ϑ .

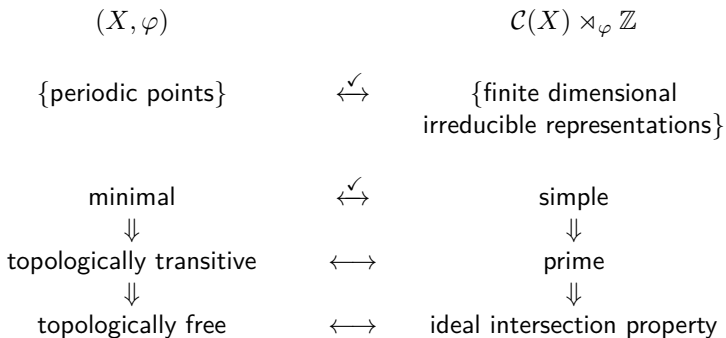
As we have shown during the online lectures:

Theorem

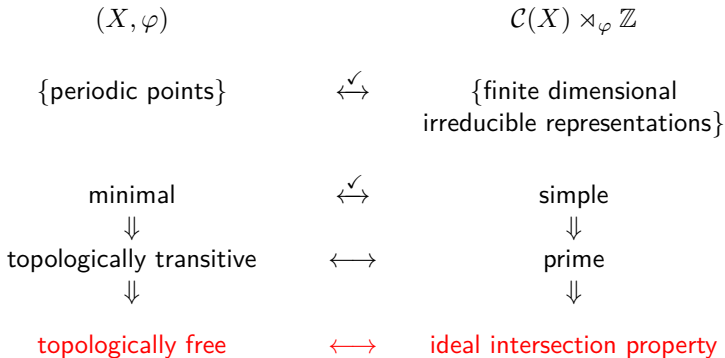
If X is infinite, (X, φ) is minimal if and only if $\mathcal{C}(X) \rtimes_{\varphi} \mathbb{Z}$ is simple.

Then since we know that $(\mathbb{T}, \varphi)$ is minimal, it follows that $\mathcal{C}(\mathbb{T}) \rtimes_{\varphi} \mathbb{Z} = A_{\vartheta}$ is simple for every irrational ϑ .

Goal



Goal



Topologically free systems vs ideal intersection property

Example (Irrational rotation on the disk)

- $X = \overline{\mathbb{D}}$,
- $\varphi^n(z) = e^{2\pi i n \vartheta} z$ for an irrational ϑ .

$(\overline{\mathbb{D}}, \varphi)$ is topologically free since $\overline{\mathbb{D}} = \bigcup_{r \in [0,1]} r \mathbb{T}$.

Definition

We say that $\mathcal{C}(X) \rtimes_{\varphi} \mathbb{Z}$ has the ideal intersection property if for each closed ideal $I \subseteq \mathcal{C}(X) \rtimes_{\varphi} \mathbb{Z}$,

$$I \neq \{0\} \Leftrightarrow I \cap \mathcal{C}(X) \neq \{0\}.$$

Note that “ $\Leftarrow$ ” is always true!

Theorem

(X, φ) is topologically free if and only if $\mathcal{C}(X) \rtimes_{\varphi} \mathbb{Z}$ has the ideal intersection property.

“ $\Rightarrow$ ”

Towards a contradiction, suppose $I \cap \mathcal{C}(X) = \{0\}$.

$$E : \mathcal{C}(X) \rtimes_{\varphi} \mathbb{Z} \longrightarrow \mathcal{C}(X) \text{ faithful}$$

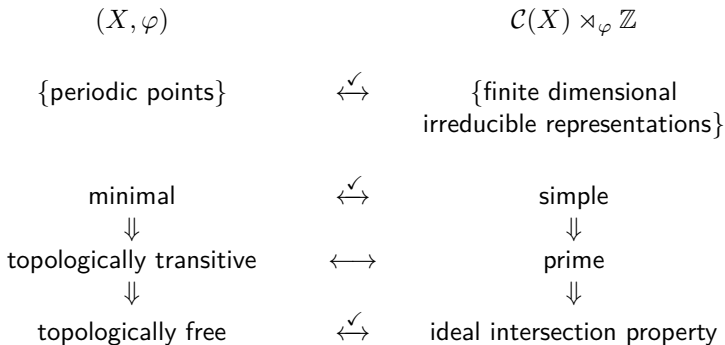
Since $I \neq \{0\}$, there is $a \in I$ with $E(a) \neq 0 \Rightarrow E(a)(x) \neq 0$ for some $x \in X$. By topological freeness we can choose x to be aperiodic.

Result: if x is aperiodic, $\text{ev}_x : \mathcal{C}(X) \rightarrow \mathbb{C}$, $f \mapsto f(x)$ has a unique pure state extension to $\mathcal{C}(X) \rtimes_{\varphi} \mathbb{Z}$.

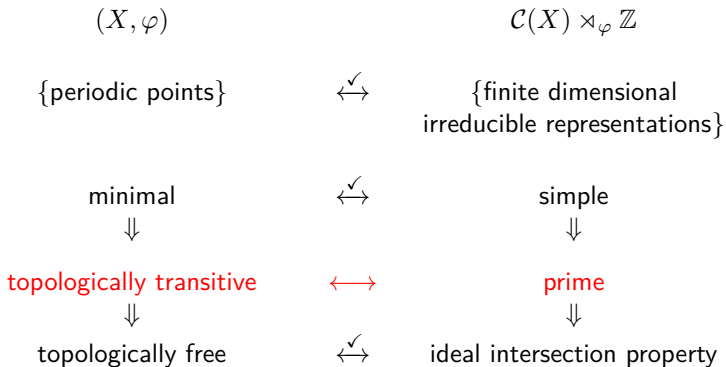
$$\begin{array}{ccc} \mathcal{C}(X) \rtimes_{\varphi} \mathbb{Z} & \xrightarrow{q} & \frac{\mathcal{C}(X) \rtimes_{\varphi} \mathbb{Z}}{I} \\ E \downarrow & & \downarrow \exists \psi \\ \mathcal{C}(X) & \xrightarrow{\text{ev}_x} & \mathbb{C} \end{array}$$

but $\psi \circ q(a) = 0 \neq \text{ev}_x \circ E(a)$, a contradiction.

Goal



Goal



Topological transitive systems vs prime crossed products

Example (Bernoulli shift)

- $X = \{0, 1\}^{\mathbb{Z}}$,
- $\varphi((x_k)_{k \in \mathbb{Z}}) = (x_{k+1})_{k \in \mathbb{Z}}$.

The sequence $\dots | 0 | 1 | 0 0 | 0 1 | 1 0 | 1 1 | 0 0 0 | 0 0 1 | \dots$ has dense orbit and $(\{0, 1\}^{\mathbb{Z}}, \varphi)$ is topologically transitive.

Definition

A C^* -algebra is prime if, for any two ideals I and J ,

$$I \cap J = \{0\} \Rightarrow I = \{0\} \vee J = \{0\}.$$

Theorem

(X, φ) is an infinite topologically transitive dynamical system if and only if $\mathcal{C}(X) \rtimes_{\varphi} \mathbb{Z}$ is a prime C^* -algebra.

“ $\Rightarrow$ ”

Recall that topological transitivity is equivalent to: For all open $U, V \subset X$,

there is $n \in \mathbb{Z}$, $\varphi^n(U) \cap V \neq \emptyset \xrightarrow{\text{if inv.}} U \cap V \neq \emptyset$.

Let I, J be ideals in $\mathcal{C}(X) \rtimes_{\varphi} \mathbb{Z}$ with $I \cap J = \{0\}$.

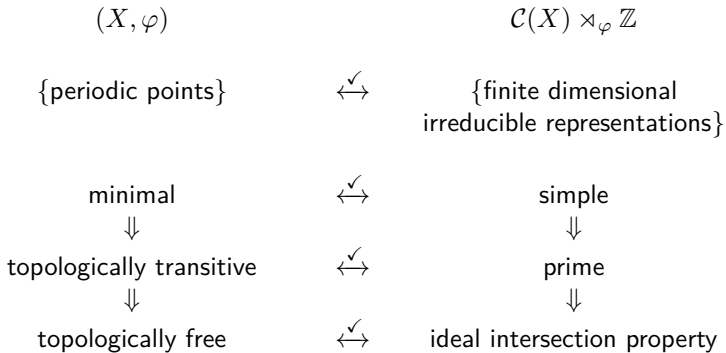
Towards a contradiction, assume $I \neq \{0\}$ and $J \neq \{0\}$. Since topological transitivity implies topological freeness,

$$\begin{cases} \mathcal{C}(X) \cap I \neq \{0\} \\ \mathcal{C}(X) \cap J \neq \{0\} \end{cases} \quad \begin{cases} \mathcal{C}(X) \cap I = \{f \in \mathcal{C}(X) : f|_E = 0\} \\ \mathcal{C}(X) \cap J = \{f \in \mathcal{C}(X) : f|_F = 0\} \end{cases}$$

E^c and F^c are invariant open subsets of X , so

$$E^c \cap F^c \neq \emptyset \Rightarrow E \cup F \neq X \Rightarrow I \cap J \neq \{0\}, \text{ a contradiction.}$$

Goal



Thank you for your attention!

Questions?

Minimal C^* -dynamical systems

Let A be a generic C^* -algebra, G a discrete group, $\alpha : G \rightarrow \text{Aut}(A)$ an action:

(A, α) is called C^* -dynamical system.

Definition

(A, α) is minimal if A has no non-trivial invariant ideals.

Theorem (Archbold–Spielberg 1994)

If (A, α) is minimal and topologically free (on the spectrum $\hat{A}$) then $A \rtimes_{\alpha, r} G$ is simple.

Topological freeness

α is an action of a discrete group G on $\mathcal{C}(X)$,

Theorem (Kawamura–Tomiya 1990)

If G is amenable,

$$(\mathcal{C}(X), \alpha) \text{ is topologically free} \iff \mathcal{C}(X) \rtimes_{\alpha} G \text{ has the ideal intersection property.}$$

Theorem (Archbold–Spielberg 1993)

Let $q : \mathcal{C}(X) \rtimes_{\sigma, f} G \rightarrow \mathcal{C}(X) \rtimes_{\sigma, r} G$ be the canonical surjection.

$$(\mathcal{C}(X), \alpha) \text{ is topologically free} \iff \begin{aligned} &I \triangleleft \mathcal{C}(X) \rtimes_{\sigma, f} G, I \cap \mathcal{C}(X) = \{0\} \\ &\Rightarrow q(I) = \{0\}. \end{aligned}$$

For non-abelian algebras does not hold:

For $\mathcal{K}$ and $G = \mathbb{Z} \oplus \mathbb{Z}$ there is an action for which “ $\neq$ ”

Invariant ideals

Let G be a discrete group. The map

$$\begin{aligned} \{\text{ideals in } A \rtimes_{\alpha,r} G\} &\rightarrow \{\text{invariant ideals in } A\} \\ I &\mapsto I \cap A \end{aligned}$$

is always surjective. When it is also injective we say that A separates ideals in $A \rtimes_{\alpha,r} G$.

Topological freeness is not enough to ensure that A separates ideals in $A \rtimes_{\alpha,r} G$.

It is believed that essential freeness is enough (Renault), with some more assumptions it is (Sierakowski).