C^*-uniqueness of group algebras

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Outline

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Algebraically C^*-unique group

Definition : A group *G* is said to be algebraically C^*-unique if the group algebra $\mathbb{C}[G]$ admits a unique C^*-norm.

Examples : Finite groups are algebraically C^*-unique. More generally, locally finite groups are also algebraically C^*-unique.

Every algebraically C^*-unique group is amenable.

Caspers, Skalski studied algebraically C^*-unique groups in the context of discrete quantum groups.

Non-example : For $G = \mathbb{Z}$, $\mathbb{C}[\mathbb{Z}]$ has more than one C^*-completion.

Let $\mathcal{A}(\mathbb{T})$ be the $*$-subalgebra of $\mathrm{C}(\mathbb{T})$ generated by the identity function on $\mathbb{T}$. $\mathbb{C}[\mathbb{Z}] \simeq \mathcal{A}(\mathbb{T})$.

Depending on the norm defined on $\mathcal{A}(\mathbb{T})$, we can have many C^*-completions on $\mathbb{C}[\mathbb{Z}]$.

Any infinite closed subset F of $\mathbb{T}$ gives rise to a distinct C^*-norm on $\mathcal{A}(\mathbb{T})$ defined by $\|P\| := \sup_{z \in F} |P(z)|$.

Outline

Necessary-sufficient condition for algebraically C^* unique group

Given a discrete group G, we write $\mathrm{C}^*(G)$ for the full group C^*-algebra of G.

Lemma [Alekseev, Kyed]: Let G be a discrete group. Then G is algebraically C^*-unique if and only if every non-trivial closed, two-sided ideal in $\mathrm{C}^*(G)$ intersects $\mathbb{C}[G]$ non-trivially.

Outline of the proof of $\Rightarrow$:

- Assume that there is a non-trivial two-sided closed ideal J intersecting $\mathbb{C}[G]$ trivially. So, there is a quotient map $q : \mathrm{C}^*(G) \to \mathrm{C}^*(G)/J$.
- We may restrict q to $\mathbb{C}[G]$ which yields a faithful $*$-homomorphism $\pi : \mathbb{C}[G] \to \mathrm{C}^*(G)/J$. This defines a C^*-norm on $\mathbb{C}[G]$ via $||x|| := ||\pi(x)||$. It is properly majorised by the universal C^*-norm of $\mathbb{C}[G]$. Contradiction!

Continue

Outline of the proof of $\Leftarrow$:

- Assume that G is not algebraically C^*-unique. So, there is a C^*-norm on $\mathbb{C}[G]$ which is properly majorised by the universal C^*-norm. Let A be the closure with respect to this C^*-norm. Then, there exists a $*$- homomorphism from $\mathrm{C}^*(G)$ onto A.
- The kernel of this surjective $*$-homomorphism intersects $\mathbb{C}[G]$ trivially, since restriction of this $*$-homomorphism to $\mathbb{C}[G]$ is an identity map-hence it is injective. Contradiction!

We will use this lemma to discuss a group which is torsion free and algebraically C^*-unique- $\mathbb{Z}\left[\frac{1}{pq}\right] \rtimes_\alpha \mathbb{Z}^2$ where p, q are multiplicatively independent integers.

A torsion-free algebraically C^*-unique group

Let $p, q \geq 2$ be two multiplicatively independent integers (i.e., there exist no $r, s \in \mathbb{N}$ such that $p^r = q^s$).
Let $\mathbb{Z}\left[\frac{1}{pq}\right]$ be the additive group $\left\{\frac{a}{(pq)^n} \in \mathbb{Q} \mid a \in \mathbb{Z},\ n \in \mathbb{N}\right\}$.
Let $\alpha : \mathbb{Z}^2 \curvearrowright \mathbb{Z}\left[\frac{1}{pq}\right]$ be given by $\alpha_{(m,n)}(x) = p^m q^n x$ for $m, n \in \mathbb{Z}$ and $x \in \mathbb{Z}\left[\frac{1}{pq}\right]$.
Our goal is to prove that the torsion-free semi-direct product group $\mathbb{Z}\left[\frac{1}{pq}\right] \rtimes_\alpha \mathbb{Z}^2$ is algebraically C^*-unique.

Outline

Faithful action

First we are going to prove that the action $\alpha : \mathbb{Z}^2 \curvearrowright \mathbb{Z}[1/pq]$ defined by $\alpha_{(n,m)}(x) = p^n q^m x$ is faithful.

Proof.

- p and q are multiplicatively independent
$$\implies p^n q^m = 1 \implies n = m = 0.$$
- Now $\alpha_{(n,m)} = Id_{\mathbb{Z}[1/pq]} \implies p^n q^m = 1$. So, the action is faithful. ∎

Topologically free action

Let G and $\mathcal{A}$ be discrete groups with $\mathcal{A}$ abelian. Write $\hat{\mathcal{A}}$ for the Pontryagin dual, which is compact Hausdorff. Recall that any action $\beta : G \curvearrowright \mathcal{A}$ induces a dual (continuous) action

$$\hat{\beta} : G \curvearrowright \hat{\mathcal{A}}; \quad \hat{\beta}_g(\chi)(x) = \chi(\beta_{g^{-1}}(x)).$$

Definition

An action of a group G on a locally compact Hausdorff space X is said to be topologically free if for each $g \in G \setminus \{e\}$, the set of points of X fixed by g has empty interior.

Lemma

If $\mathcal{A}$ is torsion-free, and $\beta : G \curvearrowright \mathcal{A}$ is a faithful action, then the dual action $\hat{\beta} : G \curvearrowright \widehat{\mathcal{A}}$ is topologically free.

Proof of lemma

- For $g \in G$ let $F_g = \{\chi \in \widehat{\mathcal{A}} : \hat{\beta}_g(\chi) = \chi\}$. We will show that if $(F_g)^\circ \neq \emptyset$ then $g = e$.
- Note that F_g is a subgroup of $\widehat{\mathcal{A}}$.
- F_g has nonempty interior $\implies$ F_g is open subgroup.
- $\hat{\beta}_g$ is continuous $\implies$ F_g is closed subgroup.

Proof...

- $\mathcal{A}$ is discrete, abelian and torsion-free $\implies \widehat{\mathcal{A}}$ is connected $\implies F_g = \widehat{\mathcal{A}}$.
- We have $\chi(g^{-1}p) = \chi(p)$, $\forall p \in \mathcal{A}$ and $\forall \chi \in \widehat{\mathcal{A}}$ $\implies g^{-1}p = p$, $\forall p$. As β is faithful, we have $g = e$. $\blacksquare$

Now $\alpha : \mathbb{Z}^2 \curvearrowright \mathbb{Z}[1/pq]$ is faithful $\implies$ from the previous lemma $\hat{\alpha} : \mathbb{Z} \curvearrowright \widehat{\mathbb{Z}[1/pq]}$ is a topologically free action.

Outline

Semidirect product of two groups

Let G and Γ be two discrete groups and suppose that $\beta : \Gamma \curvearrowright G$ is an action.
The **semidirect product** is defined as the set

$$\left\{(g, t) \mid g \in G, t \in \Gamma\right\}$$

equipped with the operation

$$(g, t) \cdot (h, s) := (g\beta_t(h), ts).$$

This gives the set a group structure (with the identity (e, e)).
We denote this group by $G \rtimes \Gamma$.

C^*-dynamical systems

Definition

By a C^***-dynamical system** we mean a triple (A, α, Γ) where A is a C^*-algebra, Γ is a discrete group, and $\alpha : \Gamma \mapsto \mathrm{Aut}(A)$ is a group homomorphism.

Given a C^*-dynamical system we may form the algebraic crossed product $*$-algebra

$$\mathrm{C}_c(\Gamma, A) = \left\{ \sum_{s \in F} a_s s \mid F \subseteq \Gamma \text{ finite },\ a_s \in A \right\}.$$

with operations

$$(a_s s)(b_t t) := a_s \alpha_s(b_t) st, \quad (a_s s)^* = \alpha_{s^{-1}}(a_s^*) s^{-1}.$$

Covariant representations

Definition

A **covariant representation** of a C^*-system (A, α, Γ) is a pair (π, u) consisting of a representation $\pi : A \mapsto \mathbb{B}(\mathcal{H})$ and a unitary representation $u : \Gamma \mapsto \mathcal{U}(\mathcal{H})$ satisfying:

$$\pi(\alpha_s(a)) = u_s \pi(a) u_s^*; \quad \forall s \in \Gamma, \ \forall a \in A.$$

If (π, u) is a covariant representation on $\mathcal{H}$, we get a $*$-representation

$$\pi \rtimes u : \mathrm{C}_c(\Gamma, A) \to \mathbb{B}(\mathcal{H}); \quad \pi \rtimes u(a_s s) = \pi(a_s) u(s).$$

We get a C^*-norm on $\mathrm{C}_c(\Gamma, A)$ as follows: for $x \in \mathrm{C}_c(\Gamma, A)$

$$\|x\|_{\max} := \sup \left\{ \|(\pi \rtimes u)(x)\| \mid (\pi, u) \text{ all covariant representations} \right\}.$$

$A \rtimes \Gamma$ and Universal properties

Completing with respect to this C^*-norm gives the **full crossed product** C^*-algebra $A \rtimes \Gamma := \overline{\mathrm{C}_c(\Gamma, A)}^{\|\cdot\|_{\max}}$.

- Recall that if $u : \Gamma \mapsto \mathcal{U}(\mathcal{H})$ is a unitary representation we get a $*$-homomorphism

$$\psi_u : \mathrm{C}^*(\Gamma) \to \mathbb{B}(\mathcal{H}); \quad \psi_u(\delta_t) = u_t.$$

- If (π, u) is a covariant representation of (A, Γ, α) on $\mathcal{H}$ the $*$-representation $\pi \rtimes u$ extends continuously to the full crossed product

$$\pi \rtimes u : A \rtimes \Gamma \mapsto \mathbb{B}(\mathcal{H}).$$

Crossed and semidirect products

Suppose Λ and Γ are discrete groups with an action $\beta : \Gamma \curvearrowright \Lambda$. The universal property of $\mathrm{C}^*(\Lambda)$ gives us an induced C^*-action

$$\tilde{\beta} : \Gamma \to \mathrm{Aut}(\mathrm{C}^*(\Lambda)); \quad \tilde{\beta}_t(\delta_x) = \delta_{\beta_t(x)}.$$

We then have

Fact

There is a C^*-isomorphism

$$\mathrm{C}^*(\Lambda \rtimes \Gamma) \cong \mathrm{C}^*(\Lambda) \rtimes \Gamma.$$

Proof

- $\phi : \Lambda \rtimes \Gamma \to \mathrm{C}^*(\Lambda) \rtimes \Gamma$, $(x, s) \mapsto \delta_x s$ is a unitary representation. By the universal property of $\mathrm{C}^*(\Lambda \rtimes \Gamma)$ we then get a $*$-homo $\Phi : \mathrm{C}^*(\Lambda \rtimes \Gamma) \to \mathrm{C}^*(\Lambda) \rtimes \Gamma$.
- Now $v : \Lambda \mapsto \mathrm{C}^*(\Lambda \rtimes \Gamma)$, $x \mapsto (x, e_\Gamma)$ is a unitary rep, so we get a $*$-homomorphism $\pi : \mathrm{C}^*(\Lambda) \mapsto \mathrm{C}^*(\Lambda \rtimes \Gamma)$ with $\pi|_\Lambda = v$.
- Also, $u : \Gamma \mapsto \mathrm{C}^*(\Lambda \rtimes \Gamma)$, $y \mapsto (e_\Lambda, y)$ is a unitary rep.
- $(\pi, u)-$covariant representation (by semi-direct product).
- Universal property of full crossed product C^*-algebra gives a $*$-homomorphism $\Psi : \mathrm{C}^*(\Lambda) \rtimes \Gamma \mapsto \mathrm{C}^*(\Lambda \rtimes \Gamma)$.
- Now we can prove that $\Psi \circ \Phi = Id$ and $\Phi \circ \Psi = Id$.

$\square$

Outline

Topologically Free Actions

Throughout, Γ is a discrete group with neutral element e, X is a compact Hausdorff space, and $\Gamma \curvearrowright X$ is a continuous action.

Given $t \in \Gamma$, we have the open set of elements displaced by t:

$$D_t := \{x \in X \mid t.x \neq x\}.$$

The action $\Gamma \curvearrowright X$ is called **free** if for all $t \neq e$, $D_t = X$.

The action $\Gamma \curvearrowright X$ is called **topologically free** if for all $t \neq e$, $\overline{D_t} = X$. Equivalently, $\Gamma \curvearrowright X$ is called **topologically free** if for all finite subsets $F \subseteq \Gamma$

$$\bigcap_{t \in F \setminus \{e\}} D_t \quad \text{is dense in } X .$$

$\Gamma \curvearrowright X$ topologically free $\rightsquigarrow$ ideal structure of $\mathrm{C}(X) \rtimes \Gamma$.

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Theorem

Let $\Gamma \curvearrowright X$ be a topologically free action. If $I \subseteq \mathrm{C}(X) \rtimes_{\mathrm{r}} \Gamma$ is any non-zero closed ideal, then $I \cap \mathrm{C}(X) \neq \{0\}$.

Our proof will use the idea of **definite states**.

Fact

*Let B be a C^*-algebra and suppose $\varphi \in S(B)$ is a state. If φ is **definite** with respect to a self-adjoint element $k \in B$; that is $\varphi(k^2) = \varphi(k)^2$, then for every $b \in B$ we have*

$$\varphi(bk) = \varphi(kb) = \varphi(k)\varphi(b).$$

Proof. It is easily checked that $k - \varphi(k)1_B$ belongs to the left ideal

$$L_\varphi := \left\{ b \in B \mid \varphi(b^*b) = 0 \right\} \subseteq \ker(\varphi).$$

Therefore, if $b \in B$ then $bk - \varphi(k)b \in \ker(\varphi)$. Now compute...

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Proof of Theorem

Let $I \subseteq \mathrm{C}(X) \rtimes_{\mathrm{r}} \Gamma$ be a closed ideal with $I \cap \mathrm{C}(X) = \{0\}$. We will show that $I = \{0\}$ by showing that $I_+ = \{0\}$.

Let $a \in I_+$. It suffices to show that $f := \mathbb{E}(a) = 0$, since the expectation $\mathbb{E}$ is faithful. If $\varepsilon > 0$ we will arrive at $\|f\| \leq \varepsilon$.

Fix $x \in X$. Since f is continuous there is an open neighborhood U of x with

$$z \in U \implies |f(z) - f(x)| \leq \varepsilon/3.$$

Now find a $b \in \mathrm{C}_c(\Gamma, \mathrm{C}(X))$ with $\|a - b\| \leq \varepsilon/3$, and say

$$b = \sum_{t \in F} b_t u_t, \quad b_t \in \mathrm{C}(X).$$

Then

$$\|f - b_e\| = \|\mathbb{E}(a) - \mathbb{E}(b)\| = \|\mathbb{E}(a - b)\| \leq \|\mathbb{E}\|\|a - b\| \leq \varepsilon/3.$$

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Proof

For any $y \in U$ we have

$$|f(x)| \leq |f(x) - f(y)| + |f(y) - b_e(y)| + |b_e(y)| \\ \leq \varepsilon/3 + \|f - b_e\|_u + |b_e(y)| \leq 2\varepsilon/3 + |b_e(y)|.$$

Using topological freeness we will choose a certain $y \in U$ making $|b_e(y)|$ small.
In fact we pick

$$y \in \left(\bigcap_{t \in F_0} D_t \right) \cap U \neq \emptyset,$$

where $F_0 := F \setminus \{e\}$.
We see that $y \neq t^{-1}.y$ for any $t \in F_0$.

Proof

For any $y \in U$ we have

$$|f(x)| \leq |f(x) - f(y)| + |f(y) - b_e(y)| + |b_e(y)|$$
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Using topological freeness we will choose a certain $y \in U$ making $|b_e(y)|$ small.
In fact we pick

$$y \in \left(\bigcap_{t \in F_0} D_t \right) \cap U \neq \emptyset,$$

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The evaluation map $\mathrm{ev}_y : \mathrm{C}(X) \to \mathbb{C},\ f \mapsto f(y)$ is a state on $\mathrm{C}(X)$, and since $\mathrm{C}(X) \cap I = \{0\}$ there is a well-defined state

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Note that φ kills I and is definite with respect to self-adjoint elements of $\mathrm{C}(X)$; indeed, if $k \in \mathrm{C}(X)_{s.a.}$ then

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Claim. For every $t \in F_0$ we have $\varphi(u_t) = 0$.

For $t \in F_0$ we know $t^{-1}.y \neq y$. Urysohn's Lemma gives a $k \in \mathrm{C}_c(X, [0,1])$ with $k(y) \neq k(t^{-1}.y)$. By our above fact we see that

$$k(t^{-1}.y)\varphi(u_t) = \alpha_t(k)(y)\varphi(u_t) = \varphi(\alpha_t(k))\varphi(u_t)$$
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Thus $(k(t^{-1}.y) - k(y))\varphi(u_t) = 0$ which means $\varphi(u_t) = 0$. Now for each $t \in F_0$ we write $b_t = h_t + ik_t$ with h_t, k_t self-adjoint, and using our fact again we get

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Outline

1. Part 1, by Arnab Bhattacherjee
   - Algebraically C^*-unique groups: definition, (non)-example
   - Conditions for C^*-uniqueness
2. Part 2, by Malay Mandal
   - A Bit on Actions . . .
   - C^*-algebra of semidirect product vs crossed product
3. Part 3, by Timothy Rainone
   - Topologically free actions and ideal structure
   - A useful description of $C^*(\mathbb{Z}[\frac{1}{d}])$
4. Part 4, by Marco Roschkowski
   - Statement of the Theorem (Furstenberg's Approximation)
   - Proof
5. Part 5, by Ujan Chakraborty
   - Actions on the (Abelian) C^* algebra vs the Spectrum
   - Proof of C^* uniqueness: The Main Theorem

Given $d \geq 1$, we are considering the (discrete) additive subgroup of $\mathbb{Q}$:

$$\Lambda := \mathbb{Z}\left[\frac{1}{d}\right] = \left\{\frac{a}{d^n} \mid a \in \mathbb{Z}, n \geq 0\right\}.$$

Λ can be realized as the inductive limit of the system

$$\mathbb{Z} \xrightarrow{\cdot d} \mathbb{Z} \xrightarrow{\cdot d} \mathbb{Z} \xrightarrow{\cdot d} \mathbb{Z} \cdots \longrightarrow \varinjlim_{n}(\mathbb{Z}, \cdot d) \cong \Lambda.$$

Identifying the circle group $\mathbb{T}$ with the Pontryagin dual $\hat{\mathbb{Z}}$ and dualizing this system gives the topological projective system:

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and is equipped with the relative product topology $X \subseteq \prod_k \mathbb{T}$.

Useful fact: if $(z_k)_k \in X$ and $m \geq n$, then

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Our goal is to show that X is homeomorphic to $\hat{\Lambda}$; the Pontryagin dual of Λ. We do this by establishing:

Lemma

There is an isomorphism of C^*-algebras

$$\psi : \mathrm{C}^*(\Lambda) \to \mathrm{C}(X); \quad \psi(\delta_{\frac{a}{d^n}})\big((z_k)_k\big) = z_n^a.$$

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The evaluation isomorphism

First we consider any discrete abelian group Λ with its Pontryagin dual $\hat{\Lambda}$, and $\Omega_\Lambda := \Omega(\mathrm{C}^*(\Lambda))$; the character space of $\mathrm{C}^*(\Lambda)$.

Each character $\chi \in \hat{\Lambda}$ gives rise to a character h_χ on $\mathrm{C}^*(\Lambda)$ satisfying $h_\chi(\delta_t) = \chi(t)$, and the map

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Building ψ via a unitary representation

Given $\frac{a}{d^n} \in \Lambda$, the map

$$U_{\frac{a}{d^n}} : X \to \mathbb{C}; \quad U_{\frac{a}{d^n}}(z) = \pi_n(z)^a = z_n^a$$

is clearly continuous and $\mathbb{T}$-valued, so $U_{\frac{a}{d^n}}$ is a unitary in $\mathrm{C}(X)$.
We claim that

$$U : \Lambda \to \mathrm{C}(X); \quad U\left(\frac{a}{d^n}\right) := U_{\frac{a}{d^n}}$$

is a well-defined unitary representation of Λ.
well-defined: Suppose $\frac{a}{d^n} = \frac{b}{d^m}$ with $m \geq n$. In that case $z_n = z_m^{d^{m-n}}$ so

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$$\begin{aligned} U\left(\frac{a}{d^n}+\frac{b}{d^m}\right)(z) &= U\left(\frac{d^{m-n}a+b}{d^m}\right) = z_m^{ad^{m-n}+b} = z_m^{ad^{m-n}} z_m^b \\ &= (z_m^{d^{m-n}})^a z_m^b = z_n^a z_m^b = U\left(\frac{a}{d^n}\right)(z)\cdot U\left(\frac{b}{d^m}\right)(z) \\ &= \left(U\left(\frac{a}{d^n}\right)\cdot U\left(\frac{b}{d^m}\right)\right)(z). \end{aligned}$$

By the universal property of $\mathrm{C}^*(\Lambda)$, there is a unital $*$-homomorphism $\psi : \mathrm{C}^*(\Lambda) \to \mathrm{C}(X)$ satisfying

$$\psi(\delta_{\frac{a}{d^n}}) = U_{\frac{a}{d^n}}.$$

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We therefore have our desired unital $*$-homomorphism

$$\boxed{\psi : \mathrm{C}^*(\Lambda) \to \mathrm{C}(X); \quad \psi(\delta_{\frac{a}{d^n}})(z) = z_n^a.}$$

The range of ψ is a unital C^*-subalgebra of $\mathrm{C}(X)$.
Also, the range $\psi(\mathrm{C}^*(\Lambda))$ separates points; indeed, if $(z_k)_k = z \neq w = (w_k)_k$ in X, then $z_n \neq w_n$ for some n, so

$$\psi(\delta_{d^{-n}})(z) = z_n \neq w_n = \psi(\delta_{d^{-n}})(w).$$

By the Stone Weierstrass Theorem, ψ is surjective.
To complete the proof we need only show that ψ is injective, and we do this by constructing a commutative diagram.

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Consider the sequence of continuous maps

$$p_n : \hat{\Lambda} \to \mathbb{T}; \quad p_n(\chi) = \chi(d^{-n}).$$

These satisfy $\rho \circ p_{n+1} = p_n$. Indeed,

$$\begin{aligned}\rho \circ p_{n+1}(\chi) &= \rho(\chi(d^{-(n+1)})) = \chi(d^{-(n+1)})^d \\ &= \chi(dd^{-(n+1)}) = \chi(d^{-n}) = p_n(\chi).\end{aligned}$$

By the universal property of X there is a unique continuous map $h : \hat{\Lambda} \to X$ satisfying $\pi_n \circ h = p_n$.
We dualize h and obtain a unital $*$-homomorphism

$$\boxed{T_h : \mathrm{C}(X) \to \mathrm{C}(\hat{\Lambda}); \quad T_h(f) = f \circ h.}$$

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We claim that $T_h \circ \psi = \mathrm{ev}$.

$$\begin{tikzcd}
C^*(\Lambda) \arrow[rr, "\mathrm{ev}"] \arrow[dr, "\psi"'] & & C(\hat{\Lambda}) \\
& C(X) \arrow[ur, "T_h"'] &
\end{tikzcd}$$

For every $\frac{a}{d^n} \in \Lambda$ and $\chi \in \hat{\Lambda}$

$$T_h \circ \psi(\delta_{\frac{a}{d^n}})(\chi) = T_h \circ U\left(\frac{a}{d^n}\right)(\chi) = U\left(\frac{a}{d^n}\right) \circ h(\chi)$$
$$= \left(\pi_n(h(\chi))\right)^a = p_n(\chi)^a = \chi(d^{-n})^a = \chi\left(\frac{a}{d^n}\right) = \mathrm{ev}_{\frac{a}{d^n}}(\chi).$$

By linearity the $*$-homomorphisms $T_h \circ \psi$ and ev agree on $\mathbb{C}[\Lambda]$, and by continuity they agree on $C^*(\Lambda)$.
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Outline

Furstenberg's Theorem

Theorem (Furstenberg)

If $B \subset \mathbb{T}$ is an infinite closed subset such that

$$B = \{z^{p^r q^s} \mid z \in B \text{ and } r, s \in \mathbb{N}\}, \tag{1}$$

then $B = \mathbb{T}$.

Why do we need this?

Furstenberg's Theorem

- $C^*(\mathbb{Z}[\frac{1}{pq}] \rtimes \mathbb{Z}^2) \cong C(X) \rtimes \mathbb{Z}^2$
- Need to show: $I \lhd C(X)$ is trivial
- $I = C_0(B^c)$ with $B \subseteq X$ closed
  $(B = \{x \in X \mid f(x) = 0 \;\; \forall f \in I\})$
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Definition

A subset $B \subseteq \mathbb{T}$ is said to be $\times p-$ *invariant* if $p \cdot B = B$.
$(p \cdot B = \{p \cdot z \mid z \in B\})$

Theorem (Furstenberg)

If $B \subseteq \mathbb{T}$ is an infinite, closed subset, which is $\times p-$ and $\times q-$ invariant, then $B = \mathbb{T}$.

Fix a set B with these properties

outline of the proof

For $S \subseteq \mathbb{T}$ let S' denote the set of **limit points of S.**
$(S' = \{z \mid z \in \overline{S} \cap \overline{S \setminus \{z\}})$

Lemma (Lemma 1)

If $B' \cap \mathbb{Q} \neq \varnothing$, then $B = \mathbb{T}$.

Lemma (Lemma 2)

If $\varnothing \neq S \subseteq \mathbb{T}$ is closed and $\times p-$, $\times q-$ invariant .
Then $S \cap \mathbb{Q} \neq \varnothing$.

Outline

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proof of Furstenberg's Theorem.

- B infinite $\Rightarrow B' \neq \varnothing$
- $B' \times p-, \times q - invariant \Rightarrow B' \cap \mathbb{Q} \neq \varnothing$
- $\Rightarrow B = \mathbb{T}$

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Lemma

Write $\{p^r q^s \mid r, s \in \mathbb{N}_0\} = \{s_1, s_2, ...\}$ with $s_i < s_{i+1}$. Then $s_{i+1}/s_i \to 1 \ (i \to \infty)$.

Lemma 1

- First, suppose $0 \in B'$, fix small ϵ
- Take $n \in \mathbb{N}$ with $1 < s_{i+1}/s_i < 1 + \epsilon$
- Choose $z \in B$ with $0 < |z| < \epsilon/s_n$
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- Now, $r = n/t \in \mathbb{Q} \cap B'$
- $(n, t) = (t, p) = (t, q) = 1$ $[(\cdot, \cdot)$ *greatest common divisor*$]$
- p, q invertible (mod t)
- $\Rightarrow \exists u : p^u, q^u = 1 \ \ (mod \ \ t)$
- $B - r, B' - r \times p^u -, \times q^u - invariant$ and $0 \in B'$
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$$\begin{aligned} p^u \cdot (B - r) &= p^u \cdot B - p^u \cdot r \\ &= p^u \cdot B - (mt + 1)n/t \\ &= B - r \ \ (mod \ \ 1) \end{aligned} \tag{2}$$
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- Now, $r = n/t \in \mathbb{Q} \cap B'$
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If $\varnothing \neq S \subseteq \mathbb{T}$ is closed and $\times p-, \times q-$ invariant .
Then $S \cap \mathbb{Q} \neq \varnothing$.

Proof.

- Assume $S \cap \mathbb{Q} = \varnothing$
- Fix $\epsilon > 0$, $t > 1/\epsilon$
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Proof.

1) $S_i \times p^u-, \times q^u - invariant$

2) S_i closed

3) $S_i \subseteq \mathbb{R} \setminus \mathbb{Q}$ infinite

- Now S_{i+1}:

3) $K = S_i - S_i = \{z - z' \mid z, z' \in S_i\}$

- K closed, infinite, $\times p^u-, \times q^u - invariant$
- $0 \in K'$
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- $s_0 \in S_{t-1}$
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Outline

The Action on the C^* Algebra ...

- Recall, the additive group $\mathbb{Z}\left[\frac{1}{pq}\right] = \left\{\frac{a}{(pq)^n} \in \mathbb{Q} \mid a \in \mathbb{Z},\ n \in \mathbb{N}\right\}$ with the action $\alpha : \mathbb{Z}^2 \curvearrowright \mathbb{Z}\left[\frac{1}{pq}\right]$, $\alpha_{(r,s)}(x) = p^r q^s x$ for $r, s \in \mathbb{Z}$ and $x \in \mathbb{Z}\left[\frac{1}{pq}\right]$.
- Consider the induced action

$$\tilde{\alpha} : \mathbb{Z}^2 \curvearrowright \mathrm{C}^*\left(\mathbb{Z}\left[\frac{1}{pq}\right]\right)$$

(Consider $\hat{\alpha} : \mathbb{Z}^2 \curvearrowright \widehat{\mathbb{Z}\left[\frac{1}{pq}\right]}$, and correspondingly $\tilde{\hat{\alpha}} : \mathbb{Z}^2 \curvearrowright C_0\left(\widehat{\mathbb{Z}\left[\frac{1}{pq}\right]}\right)$, and recall $\mathrm{C}^*\left(\mathbb{Z}\left[\frac{1}{pq}\right]\right) \cong C_0\left(\widehat{\mathbb{Z}\left[\frac{1}{pq}\right]}\right)$.)

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... and the Spectrum

- Recall that for a group G and a (compact) topological space Y, an action $\xi : G \curvearrowright Y$ induces an action $\tilde{\xi} : G \curvearrowright C(Y)$ in the following fashion: for all $g \in G$ and $f \in C(Y)$, $\tilde{\xi}_g(f) = f \circ \xi_{g^{-1}}$, and vice versa.
- There exists an action $\beta : \mathbb{Z}^2 \curvearrowright X$ such that for $f \in C(X)$ and $r, s \in \mathbb{Z}$,

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What does the action look like?

- Recall, $X = \left\{ (x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathbb{T} \mid x_n = x_{n+1}^{pq} \right\}$
- Functions of the form $f_m^a \in C(X)$, with $a \in \mathbb{Z}$, $m \in \mathbb{N}$, $f(x) = (x_m)^a$, form a dense subset of $C(X)$, by the Stone-Weierstraß theorem.
- Recall the isomorphism $\psi : \mathrm{C}^*\left(\mathbb{Z}\left[\frac{1}{pq}\right]\right) \to C(X)$, $\delta_{\left(\frac{a}{(pq)^m}\right)} \mapsto f_m^a$.
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- $$\beta_{(1,1)} = T$$

- The case(s) for $rs < 0$ follow analogously.

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Outline

The Final Theorem

Theorem

The group $\mathbb{Z}\left[\frac{1}{pq}\right] \rtimes_\alpha \mathbb{Z}^2$ *is algebraically* C^* *unique.*

The Proof . . .

- Let $I \lhd \mathrm{C}^*\left(\mathbb{Z}\left[\frac{1}{pq}\right] \rtimes \mathbb{Z}^2\right)$ be an ideal such that $I \cap \mathbb{C}\left[\mathbb{Z}\left[\frac{1}{pq}\right] \rtimes \mathbb{Z}^2\right] = \{0\}$. We need to prove that $I = \{0\}$
- With the topological identification of the spectrum, $\mathrm{C}^*\left(\mathbb{Z}\left[\frac{1}{pq}\right] \rtimes \mathbb{Z}^2\right) \cong \mathrm{C}^*\left(\mathbb{Z}\left[\frac{1}{pq}\right]\right) \rtimes \mathbb{Z}^2 \cong C(X) \rtimes \mathbb{Z}^2$
- It is sufficient to prove that $I \cap C(X) = \{0\}$, because then it follows that $I = \{0\}$.
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$I \cap C(X)$ is a $\mathbb{Z}^2$ - invariant ideal of $C(X)$.

(This is a general fact for any group G acting on any $C(X)$.)

- Ideal, because $C(X)u_e$ is a subalgebra.
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The Proof . . .

There exists a $\mathbb{Z}^2$ - invariant closed subset $F \subseteq X$ such that $I \cap C(X) = C_0(F^C)$.

- $F = \{x \in X \mid f(x) = 0 \; \forall \; f \in I \cap C_0(X)\}$
- Since the action on $C(X)$ follows from the action β on X, F^C is a $\mathbb{Z}^2$-invariant subset of X, and so is F.

The Proof . . .

There exists a $\mathbb{Z}^2$ - invariant closed subset $F \subseteq X$ such that $I \cap C(X) = C_0(F^C)$.

- $F = \{x \in X \mid f(x) = 0 \ \forall \ f \in I \cap C_0(X)\}$
- Since the action on $C(X)$ follows from the action β on X, F^C is a $\mathbb{Z}^2$-invariant subset of X, and so is F.

The Proof . . .

- For $n \in \mathbb{N}$, let $\pi_n : X \to \mathbb{T}$ be the n^{th} canonical projection. Let $B := \pi_1(F)$.
- Recall, $X = \left\{(x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathbb{T} \mid x_n = x_{n+1}^{pq}\right\}$, and $\beta_{(1,1)}$ is the left shift on X.
- F is $\mathbb{Z}^2$ invariant.
- Hence, $\pi_n(F) = B \ \forall n \in \mathbb{N}$.
- By action of $\beta_{(1,0)}^{-1}$ and $\beta_{(0,1)}^{-1}$, we see that $\forall z \in B$, $z^p \in B$ and $z^q \in B$.

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The Proof . . .

B has infinitely many points.

- Suppose not: let B have only finitely many points.
  $\implies \exists\, p \in C(\mathbb{T})$, with $p(z) = \sum_{j=0}^{n} \lambda_j z^j$ and $p(B) = \{0\}$
  $\implies (p \circ \pi_m) \in C_0(F^C) = I \cap C(X) \ \ \forall m \in \mathbb{N}$
  But, recall the isomorphism $\psi : \mathrm{C}^*\left(\mathbb{Z}\left[\frac{1}{pq}\right]\right) \to C(X)$, $\delta_{\left(\frac{a}{(pq)^m}\right)} \mapsto f_m^a$, and hence

$$(p \circ \pi_m) = \psi\left(\sum_{j=1}^{n} \lambda_j \, \delta_{\left(\frac{j}{(pq)^m}\right)}\right)$$

which implies

$$I \cap \mathbb{C}\left[\mathbb{Z}\left[\frac{1}{pq}\right]\right] \neq \{0\}$$

A contradiction!

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A review . . .

- Finite $\implies$ C^* unique,
  but Infinite $\not\Longrightarrow$ C^* non-unique
- Non-amenable $\implies$ not C^* unique,
  but Amenable $\not\Longrightarrow$ C^* unique[1]
- Locally finite (finitely generated subgroups finite) $\Longrightarrow$ C^* unique[2],
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[2] R. Grigorchuk, M. Musat, and M. Rørdam, “Just-infinite C^*-algebras”, Comment. Math. Helv. 93:1 (2018), 157-201

[3] N. Ozawa showed that the Lamplighter group is C^* unique, V. Alekseev, “(Non)-uniqueness of C^*-norms on group rings of amenable groups”, pp. 2292-2293 in C^*-algebras, Oberwolfach Report 37/2019 16, European Mathematical Society Publishing House, 2019

[4] E. Scarparo, “A torsion-free algebraically C^*-unique group”, Rocky Mountain J. Math. 50:5 (2020), 1813-1815

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