Graph C*-Algebras

And Their Connection to Shift Spaces

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11. June 2021

Outline

Section 1

Graph Algebras

Graphs

Definition

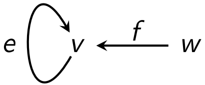
A *directed graph* $E = (E^0, E^1, s, r)$ consists of a countable set of vertices E^0, a countable set of edges E^1 and source/range functions $s, r : E^1 \to E^0$.

A *path of length n* is a sequence $\mu = \mu_1 \dots \mu_n$ of edges in E such that $s(\mu_i) = r(\mu_{i+1})$.

The *adjacency matrix* $A_E = (A_E(v, w))_{(v,w) \in E^0 x E^0}$ is defined as

$$A_E(v, w) = \#\{e \in E^1 : \; r(e) = v, s(e) = w\}$$

A graph is *row-finite* if each vertex receives at most finitely many edges.



$\{e, f, ee, ef, eef, eee, ...\}$

$$A_E = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

Dual Graph

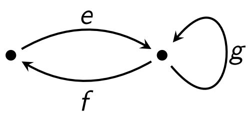
A vertex v is called a *source*, if it receives no edges.

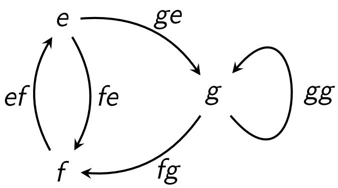
Definition

Let E be a row-finite directed graph without sources.

The *dual graph* is defined as

$$\hat{E}^0 := E^1 \qquad \hat{E}^1 := E^2 \qquad \text{and} \qquad s_{\hat{E}}(ef) = f \qquad r_{\hat{E}}(ef) = e.$$





The adjacency matrix of the dual graph is a 0-1–Matrix.

Cuntz-Krieger Family

Idea: Represent a graph by operators on a Hilbert space.

Definition (Cuntz–Krieger *E*-Family of Operators)

Let E be a row-finite graph. An *E-family $\{S, P\}$ on a Hilbert space $\mathcal{H}$* consists of

$\{P_v : v \in E^0\} \subset B(\mathcal{H})$ mutually orthogonal protections

$\{S_e : e \in E^1\} \subset B(\mathcal{H})$ partial isometries

such that

$$(CK1)\; S_e^* S_e = P_{s(e)} \qquad \text{for all edges } e \in E^1$$

$$(CK2)\; P_v = \sum_{e \in E^1 \,:\, r(e)=v} S_e S_e^* \qquad \forall v \in E^0 \text{ that are not a source}$$

Analogous for general C*-algebras due to representation on $\mathcal{H}$.

Definition (Graph Algebra of E)

$C^*(E) =$ Universal C*-algebra generated by a Cuntz-Krieger E-family.

Symbols: p_v and s_e for vertices $v \in E^0$ and edges $e \in E^1$

Relations: (CK1), (CK2)

p_v mutually orthogonal projections

s_e partial isometries

Proposition (Universal Property)

Let $\mathcal{A}$ be a C-algebra and $\{S, P\}$ a Cuntz–Krieger E-family in $\mathcal{A}$ for a row-finite graph E.*

*Then there exists a *-homomorphism $C^*(E) \to \mathcal{A}$.*

Graph Algebra

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Examples

Graph	CK-Relations	Graph C*-Algebra
v ●	$p_v^2 = p_v = p_v^*$	$\mathbb{C}$
	$s_e^* s_e = p_v = s_e s_e^*$	$C(\mathbb{T})$
	$s_{e_j}^* s_{e_j} = p_v$ $\sum_{j=0}^{n} s_{e_j} s_{e_j}^* = p_v$	Cuntz Algebra $\mathcal{O}_n$

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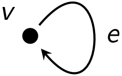	$s_e^* s_e = p_v = s_e s_e^*$	$C(\mathbb{T})$
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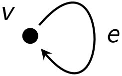
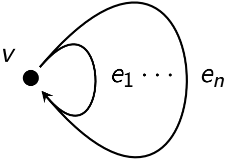
	$s_e^* s_e = p_v = s_e s_e^*$	$C(\mathbb{T})$
	$s_{e_j}^* s_{e_j} = p_v$ $\sum_{j=0}^n s_{e_j} s_{e_j}^* = p_v$	Cuntz Algebra $\mathcal{O}_n$

Example - Matrix Algebra

- $M_n(\mathbb{C})$ is a graph algebra

$$v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_2} \cdots \xrightarrow{e_{n-1}} v_n$$

$$\begin{aligned} s_{e_j}^* s_{e_j} &= p_{v_j} \\ s_{e_{j-1}} s_{e_{j-1}}^* &= p_{v_j} \end{aligned}$$

- $K(\mathcal{H})$ is a graph algebra

$$v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_2} v_3 \xrightarrow{e_3} \cdots$$

$$\begin{aligned} s_{e_j}^* s_{e_j} &= p_{v_j}, \quad j \in \mathbb{N} \\ s_{e_{j-1}} s_{e_{j-1}}^* &= p_{v_j}, \quad j \in \mathbb{N}_{\geqslant 1} \end{aligned}$$

- Different graphs can generate the same graph algebra

These graphs also generate $M_n(\mathbb{C})$.

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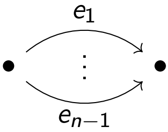

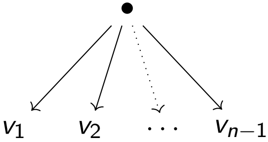

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Example - Toeplitz algebra

Want to investigate the following graph:

$$\begin{tikzcd} v \arrow[loop left, "e"] & w \arrow[l, "f"'] \end{tikzcd}$$

The associated Cuntz-Krieger relations are

$$\begin{aligned} (CK1) &: s_e^* s_e = p_v, \qquad s_f^* s_f = p_w \\ (CK2) &: s_e s_e^* + s_f s_f^* = p_v \end{aligned}$$

A representation on $\mathcal{H} = \ell^2(\mathbb{N}_0)$ is given by

$$P_v(x_0, x_1, \ldots) = (0, x_1, \ldots) \qquad P_w(x_0, x_1, \ldots) = (x_0, 0, \ldots)$$
$$S_e(x_0, x_1, \ldots) = (0, 0, x_1, x_2, \ldots) \qquad S_f(x_0, x_1, \ldots) = (0, x_0, 0, \ldots)$$

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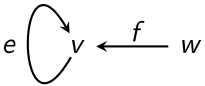

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Example -Toeplitz algebra

$S_e + S_f$ is an isometry for $C^*(S, P)$. This leads to the following:

$$\begin{aligned} P_v &= (S_e + S_f)(S_e + S_f)^* \\ P_w &= (S_e + S_f)^*(S_e + S_f) - P_v \\ S_e &= (S_e + S_f)P_v \\ S_f &= (S_e + S_f)P_w \end{aligned}$$

- $C^*(S, P)$ is generated by the isometry $S_e + S_f$.
- Coburn's Theorem: $C^*(S, P)$ is isomorphic to the Toeplitz algebra $\mathcal{T}$ generated by the unilateral shift.

Idea: If CK-E-families are nontrivial ($P_v \neq 0$)
$\Rightarrow$ CK-E-families generate isomorphic C*-algebras.

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Uniqueness Theorems

Let

- E row-finite directed graph
- $\{S, P\}$ CK E-family in C*-algebra $\mathcal{B}$ with $P_v \neq 0$

Proposition (Gauge-Invariant Uniqueness Theorem)

If there is a gauge action, i.e. $\beta : \mathbb{T} \to \mathrm{Aut}(\mathcal{B})$ *continuous with*

$$\beta_z(P_v) = P_v \quad \forall v \in E^0 \quad \textit{and} \quad \beta_z(S_e) = zS_e \quad \forall e \in E^1.$$

Then

$$\pi_{S,P} : C^*(E) \to \mathcal{B}$$

is an isomorphism of $C^*(E)$ *onto* $C^*(S, P)$.

$C^*(E)$ always has gauge action!

Let

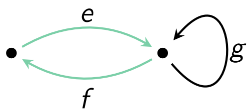
- E row-finite directed graph
- $\{S, P\}$ CK E-family in C*-algebra $\mathcal{B}$ with $P_v \neq 0$

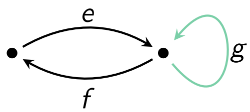
Proposition (Cuntz-Krieger Uniqueness Theorem)

If every cycle in E has an entry, i.e.

$$\forall \text{ cycle } \mu = \mu_1, \ldots \mu_n \quad \exists \text{ edge } e \notin \mu : \quad r(e) = s(\mu_i).$$

Then $C^*(E) \cong C^*(S, P)$.





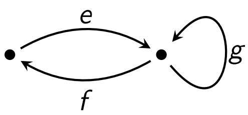
Application of Uniqueness Theorem

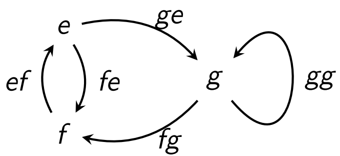
We consider again the graph of the Toeplitz algebra:

$$\begin{tikzcd} v \arrow[loop left, "e"] & w \arrow[l, "f"'] \end{tikzcd}$$

- Every cycle has an entry.
- CK-Uniqueness: $C^*(S, P)$ unique up to isomorphism
- The CK Uniqueness Theorem generalizes Coburn's Theorem.

Graph Algebra of the Dual Graph





Proposition

Let E be a row finite graph with no sources: $C^*(\hat{E}) \cong C^*(E)$

Sketch of the proof:

- Let $\{s, p\}$ be a CK family generating $C^*(E)$
- Define $Q_e := s_e s_e^*, \quad T_{fe} := s_f s_e s_e^*$
- Can check that $\{T, Q\}$ is a CK $\hat{E}$-family in $C^*(E)$
- Universal property: $\exists$ *-hom. $\pi_{T,Q} : C^*(\hat{E}) \to C^*(E)$
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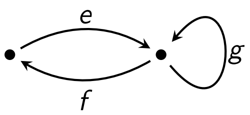

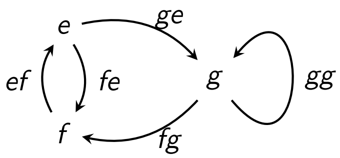

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- Preorder on E^0: $v \leqslant w$ if $\exists$ a path $\mu \in E^*$ from w to v
- $E^{\leqslant\infty} := E^\infty \cup \{\text{finite path beginning at sources}\}$

Definition

A directed graph E is *cofinal* if for every $\mu \in E^{\leqslant\infty}$ and $v \in E^0$ there exists a vertex w on μ such that $v \leqslant w$, i.e. there is a path from w to v.

Proposition

Suppose E is a row finite graph. Then

$$C^*(E) \text{ simple} \Leftrightarrow \text{Every cycle in } E \text{ has an entry and } E \text{ is cofinal.}$$

E strongly connected and every cycle has an entry $\Rightarrow C^*(E)$ simple

Simplicity of Graph Algebras

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Sketch of the proof of "$\Leftarrow$":

- Every ideal in a C*-algebra is a kernel of a representation.
- Aim: every nonzero representation $\pi_{S,P}$ of $C^*(E)$ is faithful.
- Let $\{S, P\}$ be a CK-E-family such that $\pi_{S,P} \neq 0$.
  $\Rightarrow P_v \neq 0$ for some $v \in E^0$.
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The graph is cofinal and every cycle has an entry $\Rightarrow \mathcal{O}_n$ is simple.

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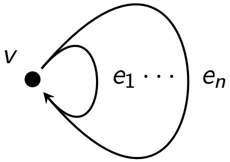

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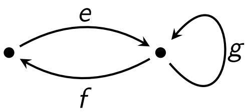
Section 2

Shift Spaces

Edge Shift



Definition (Edge Shift)

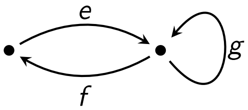
Let $E = (E^0, E^1)$ be a graph with no sinks or sources. The *edge shift* of E is the set of bi-infinite paths in E:

$$\mathsf{X}_E := \left\{ (\mu_i)_{i\in\mathbb{Z}} \in (E^1)^{\mathbb{Z}} \mid \forall i : r(\mu_{i+1}) = s(\mu_i) \right\}.$$

The dynamics of the system are described by the left shift

$$\begin{aligned} \sigma \colon \mathsf{X}_E &\to \mathsf{X}_E \\ (\mu_i)_{i\in\mathbb{Z}} &\mapsto (\mu_{i+1})_{i\in\mathbb{Z}} \end{aligned}$$

Edge Shift



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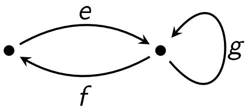
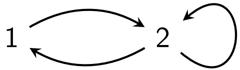
Edge Shift – Forbidden Blocks



- The requirement $r(\mu_{i+1}) = s(\mu_i)$ can be restated by saying that blocks of the form $\mu\nu$ are forbidden whenever $r(\nu) \neq s(\mu)$.
- If E is a finite graph, the set of forbidden blocks is finite.
- In the example graph, the forbidden blocks are

$$\mathcal{F} = \{ee, eg, ff, gf\}.$$



Definition (Vertex Shift)

Let $E = (E^0, E^1)$ be a graph with no sinks, sources or multiple edges. The *vertex shift* of E is the set of bi-infinite "vertex-paths" in E:

$$\hat{X}_E := \left\{ (v_i)_{i\in\mathbb{Z}} \in (E^0)^{\mathbb{Z}} \mid \forall i: \text{ there is an edge from } v_{i+1} \text{ to } v_i \right\}.$$

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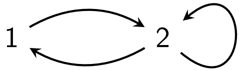
Vertex Shift



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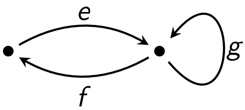
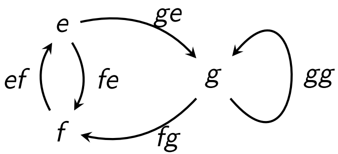
The dynamics of the system are described by the left shift

$$\sigma \colon \hat{\mathsf{X}}_E \to \hat{\mathsf{X}}_E$$
$$(v_i)_{i\in\mathbb{Z}} \mapsto (v_{i+1})_{i\in\mathbb{Z}}$$

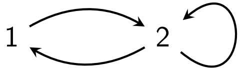
Remark

The edge shift of a graph E is the vertex shift of its dual graph $\hat{E}$.





Vertex Shift – Forbidden Blocks



- The requirement of having an edge from v_{i+1} to v_i can be restated by saying that blocks of the form vw are forbidden whenever there is no such edge.
- If E is a finite graph, the set of forbidden blocks is finite.
- In the example graph, the forbidden blocks are

$$\mathcal{F} = \{11\}.$$

Shift Spaces

Definition

The *full two-sided shift* over a finite alphabet $\mathcal{A}$ is the space $\mathcal{A}^{\mathbb{Z}}$ of bi-infinite $\mathcal{A}$-sequences.

The *shift map* $\sigma \colon \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is defined by $\sigma(x)_i = x_{i+1}$.

If $\mathcal{F}$ is a set of finite sequences in $\mathcal{A}$ called *forbidden blocks*, we define $X_{\mathcal{F}}$ to be the set of sequences in $\mathcal{A}^{\mathbb{Z}}$ that contain no block of $\mathcal{F}$.

A *shift space* is a subset $X \subseteq \mathcal{A}^{\mathbb{Z}}$ such that $X = X_{\mathcal{F}}$ for some $\mathcal{F}$.

One-sided shifts are defined analogously as subsets of $\mathcal{A}^{\mathbb{N}}$.

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Connections

Definition

If $\Phi\colon \mathcal{B}_{m+n+1}(X) \to \mathfrak{A}$ is a block map into another alphabet, the map $\phi\colon X \to \mathfrak{A}^{\mathbb{Z}}$ defined by

$$\phi(x)_i = \Phi(x_{i-m}x_{i-m+1}\dots x_{i+n}) = \Phi(x_{[i-m,i+n]})$$

is called *sliding block code*

A *conjugacy* between two shifts is a bijective sliding block code.

$$\begin{array}{rcl} x = & \dots x_{i-m-1} \boxed{x_{i-m}x_{i-m+1}\dots x_{i+n-1}x_{i+n}} \; x_{i+n+1}\dots \\ & \big\downarrow \Phi \\ \phi(x) = & \dots y_{i-1} \boxed{y_i} y_{i+1} \dots \end{array}$$

Higher Order Shifts

For a shift space X and $n \in \mathbb{N}$, take $\mathfrak{A} = \mathcal{B}_n(X)$ and $\Phi = \mathrm{id}_{\mathcal{B}_n(X)}$. We obtain the sliding block code $\beta_n \colon X \to (\mathcal{B}_n(X))^{\mathbb{Z}}$

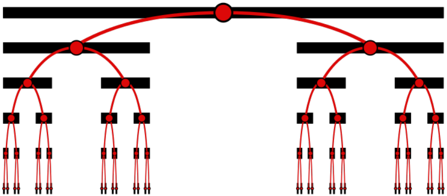
$$\beta_n(\dots x_{-1}x_0x_1\dots) = \dots \begin{bmatrix} x_{-1} \\ x_0 \\ \vdots \\ x_{n-3} \\ x_{n-2} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-2} \\ x_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} \dots$$

The image of β_n is called *higher order shift* $X^{[n]}$ and β_n is a conjugacy between X and $X^{[n]}$.

Topology

We equip $\mathcal{A}^{\mathbb{Z}}$ with the product topology of the discrete topology.

If $|\mathcal{A}| > 1$, $\mathcal{A}^{\mathbb{Z}}$ is a *Cantor set*, i.e. compact, totally disconnected, metrizable and has no isolated points.



Proposition

- *A subset $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is a shift space if and only if it is closed and σ-invariant.*
- *A map $\phi \colon X \to Y \subseteq \mathfrak{A}^{\mathbb{Z}}$ is a sliding block code if and only if ϕ is continuous and $\phi \circ \sigma_X = \sigma_Y \circ \phi$.*

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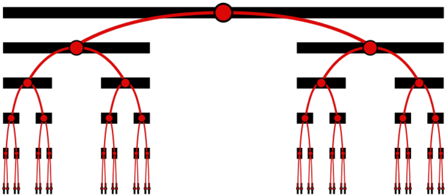

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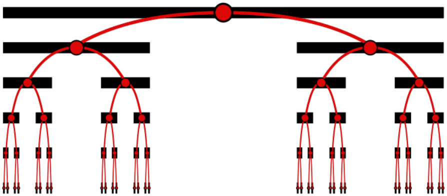

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Group 4

Graph Algebras

Shift Spaces

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A *(sub-)shift of finite type* is a shift space X which can be described by a finite set of forbidden blocks $\mathcal{F}$.

If the maximum length of a block in $\mathcal{F}$ is $k+1$, we say that X is a *k-step* SFT.

Remark

- 1-step SFT are exactly the vertex shifts of finite graphs.
- If X is a k-step SFT, then the higher order shift $X^{[n]}$ is $\max\{k-n, 1\}$-step.

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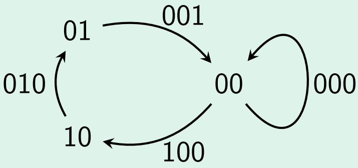
Shifts of finite type

Corollary

Every SFT is conjugate to a vertex shift and an edge shift. In particular, if X is a k-step SFT, then there is a graph E such that $X^{[k]} = \hat{\mathsf{X}}_E$ and $X^{[k+1]} = \mathsf{X}_E$.

Example

$$\mathcal{A} = \{0, 1\}, \quad \mathcal{F} = \{11, 101\}$$



(Stationary) Markov Chains

Here: family $(X_t)_{t \geqslant \mathbb{N}_0}$ of random variables $X_t : \Omega \to \Sigma$ where Σ is finite and Ω is ambient probability space s.t.

- Markov property (1-memory):
  $\forall n \in \mathbb{N}, t_0 < \cdots < t_n \in \mathbb{N}_0, i_0, \ldots, i_n \in \Sigma$:

$$\Pr[X_{t_n} = i_n \mid X_{t_{n-1}} = i_{n-1}, \ldots, X_{t_0} = i_0] = \Pr[X_{t_n} = i_n \mid X_{t_{n-1}} = i_{n-1}]$$

  whenever conditional prob.'s are well-defined
- stationary: $\forall t, t' \in \mathbb{N}_0,\ i, j \in \Sigma$:

$$\Pr[X_{t+1} = j \mid X_t = i] = \Pr[X_{t'+1} = j \mid X_{t'} = i] =: T_{ij}$$

$\to$ **we don't really care about Ω, only distributions $(p(t))_{t \in \mathbb{N}_0}$,**

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Markov Chains in C*-Language

Stationary Markov Chain on Finite Set Σ: Given by state $p(0) \in (\mathbb{C}^n)^*$ and (completely) positive unital operator T on C*-algebra $\mathbb{C}^n$: ($\mathbb{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$)

$$p(t) = p(0)\, T^t \qquad \text{at time } t \in \mathbb{N}_0$$

$$p(0) \in \mathfrak{S}(\mathbb{C}^n) \Longleftrightarrow p_i(0) \geqslant 0 \wedge p(0)\mathbb{1} = \sum_i p_i(0) = 1$$

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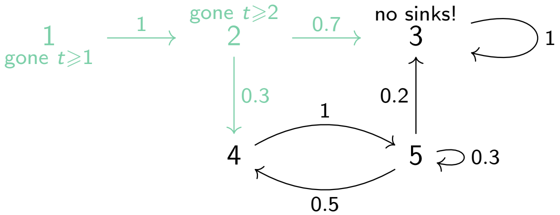

- forget probabilities and wait a finite time $\rightsquigarrow$ end up with graph without sinks or sources, no mutliple edges

$$E = \begin{tikzcd} & 3 \arrow[loop right] \\ 4 \arrow[r, bend left] & 5 \arrow[l, bend left] \arrow[u] \arrow[loop right] \end{tikzcd} \implies \begin{array}{c} A_E \in \{0,1\}^{\Sigma \times \Sigma} \\ \text{no zero columns} \\ \text{no zero rows} \\ \\ \text{new } \Sigma = \{3,4,5\} \end{array}$$

- want to be able to measure probabilities of cylinder sets

$$Z = \prod_{t \in \mathbb{N}_0} Z_t,\ Z_t = \Sigma \text{ for almost all } t \in \mathbb{N}_0$$

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- cylinder sets as basis of *topology* $\rightsquigarrow$ product topology $\prod_{t \in \mathbb{N}_0} \Sigma$

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endowed with subspace topology of the product topology of $(E^0)^{\mathbb{N}}$, is called *topological Markov chain*.

⚠ Wikipedia calls arbitrary shifts of finite type "topological Markov chain"

Recall: Arbitrary k-step shift X is conjugate (=isomorphic) to $X^{[k]} \cong \hat{X}_E$ for some graph E.

Similar: stochastic process $(X_t)_{t\in\mathbb{N}_0}$ is k-memory

$$\implies ((X_{t+k-1}, X_{t+k-2}, \dots, X_t))_{t\in\mathbb{N}_0} \text{ is Markov}$$

Group 4

Graph Algebras

Shift Spaces

Connec-tions

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Section 3

Connections

Definition (Cuntz-Krieger Algebra)

Let $A \in M_n(\{0,1\})$ be a matrix with no zero rows and columns. The *Cuntz-Krieger-algebra* $\mathcal{O}_A$ is defined as universal C*-algebra generated by partial isometries s_i satisfying

$$s_i^* s_i = \sum_{j=1}^{n} a_{ij} s_j s_j^* \qquad \text{and} \qquad \sum_{i=1}^{n} s_i s_i^* = 1.$$

We will call a family $S = \{S_i \mid i = 1, \ldots, n\}$ of partial isometries in a C*-algebra $\mathcal{A}$ satisfying these relations a *CK-A-family*.

Corresponding graph:

$$E_A^0 := \{1, \ldots, n\}, \quad E_A^1 := \{ij : \ a_{ij} = 1\}, \quad s(ij) = j, \quad r(ij) = i$$

A is the incidence matrix of E_A.

Proposition

$\mathcal{O}_A$ and $C^*(E_A)$ are isomorphic, where the Cuntz-Krieger E_A-family $\{t, q\}$ is given by

$$q_i = s_i s_i^*, \qquad t_{ij} = s_i s_j s_j^*$$

In particular, the projections q_i are mutually orthogonal.

How to get s_i back: $s_i = s_i \cdot \sum_{j=1}^{n} s_j s_j^* = \sum_{j\,:\,ij \in E_A^1} t_{ij}$.

$\Longrightarrow$ CK-Algebras are the C*-algebras of finite graphs with no sinks, sources ~~or multiple edges~~.

$\uparrow$

use $C^*(E) \cong C^*(\widehat{E})$ for graphs E without sources, dual graph $\widehat{E}$ never has multiple edges.

CK-Algebras are Graph Algebras

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Define

$$\Phi_A : \mathcal{A} \to \mathcal{A}, \quad x \mapsto \sum_i S_i x S_i^*$$

$\implies$ Φ_A is Quantum Operation (= completely positive unital operator on $\mathcal{A}$) with so-called Kraus operators S_i

Quantum Markov Chain $(\phi_0 \circ \Phi_A^t)_{t \in \mathbb{N}_0}$ for initial state ϕ_0 on $\mathcal{A}$

$\hookrightarrow$ compare with Markov chain $p(t) = p(0)T^t$

$$\mathcal{D}_A := C^*\left(\left\{\Phi_A^k(S_i S_i^*) \,\middle|\, i \in \Sigma, k \in \mathbb{N}_0\right\}\right) = \overline{\operatorname{span}}_{\mathbb{C}} \left\{S_\mu S_\mu^* \mid |\mu| \geqslant 1\right\}$$

where $S_\mu := S_{\mu_1} S_{\mu_2} \cdots S_{\mu_{|\mu|}}$

$\implies$ $\mathcal{D}_A$ is commutative AF-subalgebra of $\mathcal{A}$, invariant under Φ_A

Compare: $C^*(S) = \overline{\operatorname{span}}_{\mathbb{C}} \{S_\mu S_\nu^* \mid |\mu|, |\nu| \geqslant 1\} \subseteq \mathcal{A}$

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Theorem

Let $\mathcal{A}$ be C-algebra with a CK-A-family such that $S_i^* S_i \neq 0$ for all i, where $A \in \{0,1\}^{\Sigma \times \Sigma}$ has no sinks (i.e. no zero columns). There is an isomorphism $\omega : \mathcal{D}_A \to C(\widehat{X}_A)$ of commutative unital C*-algebras such that*

1. $$\begin{array}{ccc} \mathcal{D}_A & \xrightarrow{\omega} & C(\widehat{X}_A) \\ \downarrow{\scriptstyle \Phi_A} & & \downarrow{\scriptstyle \sigma_A^*} \\ \mathcal{D}_A & \xrightarrow{\omega} & C(\widehat{X}_A) \end{array}$$ *commutes, where* $\sigma_A^* f := f \circ \sigma_A$.
2. $\forall i : \ \omega(S_i S_i^*) = \chi_i$, *where* χ_i *is the characteristic function of the cylinder set* $Z(i) = \{x \in \widehat{X}_A : x_1 = i\}$

In other words, the Quantum Markov Chain generated by Φ_A is isomorphic to the dual dynamical system of the one-sided shift $\widehat{X}_A$, given corresponding initial states $\phi_0 = \mu_0 \circ \omega$.

Example 1: $M_2(C(\mathbb{T}))$

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad E_A = \begin{tikzcd} 0 \arrow[r, bend left, "10"] & 1 \arrow[l, bend left, "01"] \end{tikzcd} = \begin{tikzcd} v \arrow[r, bend left, "e"] & w \arrow[l, bend left, "f"] \end{tikzcd}$$

CK A-family and CK E_A-family, generating $M_2(C(\mathbb{T}))$:

$$\begin{cases} s_0 = \begin{pmatrix} 0 & t\,\mathrm{id}_{\mathbb{T}} \\ 0 & 0 \end{pmatrix} = s_0 s_1 s_1^* =: t_f & s_1 = \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix} = s_1 s_0 s_0^* =: t_e \\ s_0 s_0^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} =: q_v & s_1 s_1^* = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} =: q_w \end{cases}$$

where $t \in \mathbb{T}$ is arbitrary.

Gauge Uniqueness Theorem $\implies$ $M_2(C(\mathbb{T})) \cong C^*(E)$

$$\Phi_A \;: x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \longmapsto s_0 x s_0^* + s_1 x s_1^* = \begin{pmatrix} x_{22} & 0 \\ 0 & x_{11} \end{pmatrix}$$

$$\mathcal{D}_A = \left\{ \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} \,\middle|\, z_1, z_2 \in \mathbb{C} \right\} =: D_2(\mathbb{C}),$$

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Example 2: $\mathcal{O}_2$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad E_A = \begin{tikzcd} 0 \arrow[loop left, "00"] \arrow[r, bend left, "10"] & 1 \arrow[l, bend left, "01"] \arrow[loop right, "11"] \end{tikzcd}$$

$$\text{dual graph of } E = \begin{tikzcd} \bullet \arrow[loop left, "0"] \arrow[loop right, "1"] \end{tikzcd}$$

Cuntz-Krieger Uniqueness Thm. (every cycle has an entry!) + dual graph (no sources!) $\Longrightarrow$ every C*-algebra generated by CK-A-family or C.K.-E_A-family is isomorphic to $C^*(E) = \mathcal{O}_2$.

iso.:	CK A-family s:	s_i	
	CK E_A-family $\{t, q\}$:	$t_{ij} = s_i s_j s_j^*$,	$q_i = s_i s_i^*$
	CK E-family $\{s, p\}$:	$s_i = s_i$,	$p_\bullet = 1$

one-sided vertex shift $\widehat{X}_A$ = one-sided edge shift $X_E = \{0, 1\}^{\mathbb{N}}$

X_E is Cantor set $\rightsquigarrow$ What's $C(X_E)$?

Example 2: $\mathcal{O}_2$

$$C(X_E) = \overline{\operatorname{span}}_{\mathbb{C}} \left\{ \chi_{Z(\mu)} \mid \mu \in \{0,1\}^n,\ n \geqslant 1 \right\} \quad Z(\mu) = \{\mu \cdots\}$$

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$\to$ fits to ω and such sums are dense!

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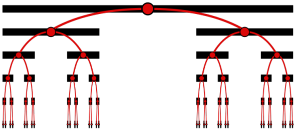
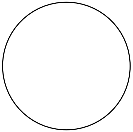
More Cantor Set Stuff

Theorem (The Cantor set X_E is very big)

Let Y be an arbitrary compact metrizable space, then there exists a continuous surjection $F : X_E \to Y$.

Example: $X_E =$  $\overset{\exists}{\twoheadrightarrow} \mathbb{T} =$ 

Corollary (The Cuntz algebra $\mathcal{O}_2$ is very big)

For every compact metrizable space Y, the C^*-algebra $C(Y)$ can be injectively embedded into $\mathcal{O}_2$. This embedding is given by

$$\omega^{-1} \circ F^* : C(Y) \xrightarrow{\circ F} C(X_E) \cong \mathcal{D}_A \subset \mathcal{O}_2$$

Either Φ_A acts periodically on image or there exist ∞ many embeddings: $\Phi_A^n \circ \omega^{-1} \circ F^*$, $n \in \mathbb{N}_0$.

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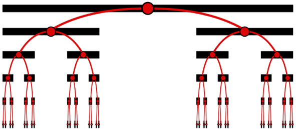
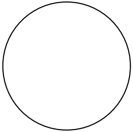
Example: $X_E =$  $\overset{\exists}{\twoheadrightarrow} \mathbb{T} =$ 

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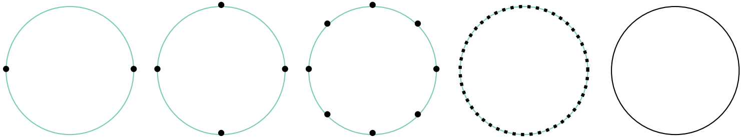
Embedding $\mathcal{O}_1 = C(\mathbb{T}) \hookrightarrow \mathcal{O}_2$

Can extract construction from the proof of the Theorem above, which leads to embedding given by:

$$(\omega^{-1} \circ F^*)(\mathrm{id}_{\mathbb{T}}) = \lim_{n\to\infty} \sum_{|\mu|=n} \underbrace{\exp\left(2\pi\mathrm{i} \cdot \sum_{k=1}^{n} \mu_k 2^{-k}\right)}_{\text{approx. every } t \in \mathbb{T}} \underbrace{s_\mu s_\mu^*}_{\text{mutually orth. proj.}}$$



Isomorphic Graph Algebras

Question: How to show that two graph algebras are isomorphic, given their graphs?

Already saw: Dual graph gives same C*-algebra, if no sources.

Also saw: $M_n(\mathbb{C})$ is isomorphic to graph algebra of

$$\bullet \longrightarrow \cdots \longrightarrow \bullet$$

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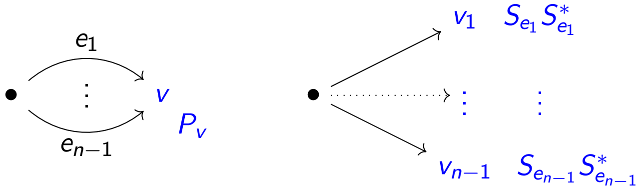

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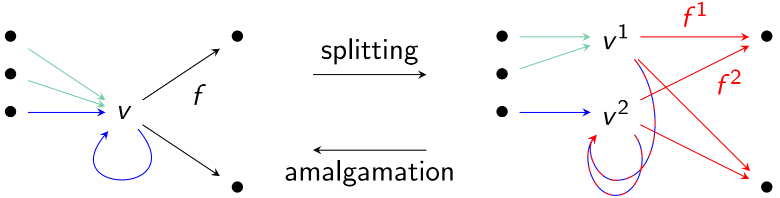
In-Splitting/In-Amalgamation



$$P_{v^1} := \sum_{e \text{ green}} S_e S_e^*, \quad P_{v^2} := \sum_{e \text{ blue}} S_e S_e^*, \quad S_{f^i} := S_f P_{v^i},$$

every other S_e, P_v unchanged. Now use

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$\Longrightarrow$ Cuntz-Krieger relations hold!

$\Longrightarrow$ get isom. of graph C*-algebras

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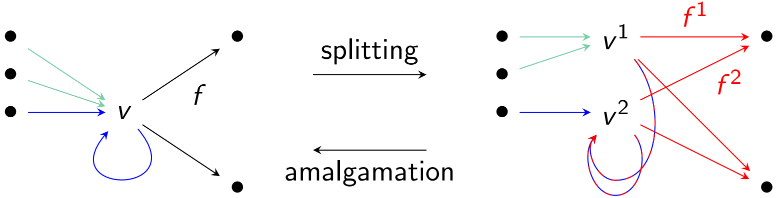

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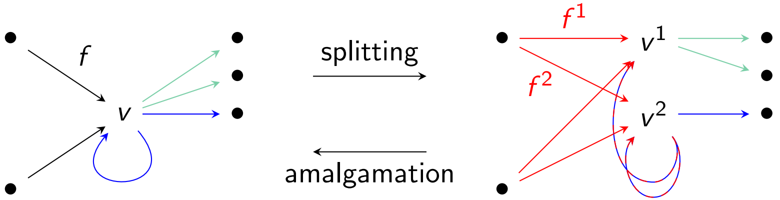

We could set $P_{v^1} = P_{v^2} = P_v$ and $S_{f^1} = S_{f^2} = S_f$ to satisfy Cuntz-Krieger relations, but P_{v^1}, P_{v^2} have to be mutually orthogonal!

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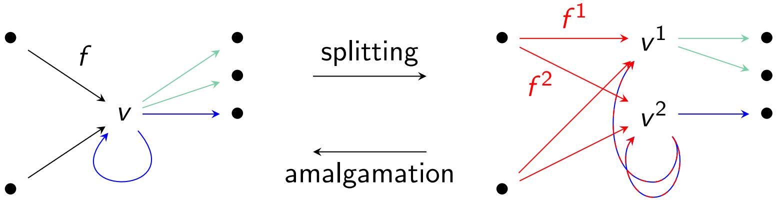

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Shift Conjugacies Induce C*-Isomorphisms

Proposition

Let G, H be finite graphs without sinks or sources and X_G, X_H their both-sided edge shifts. Then X_G and X_H are conjugate if and only if H can be obtained by successive application of in- and out-splitting/amalgamation.

Theorem ([CK80, Proposition 2.17])

Let $A, B \in \{0,1\}^{n\times n}$ have no zero columns or rows and assume that E_A, E_B both satisfy condition (I). If the one-sided shifts $\hat{X}_A, \hat{X}_B$ are conjugate, then there is an isomorphism $\psi : \mathcal{O}_A \to \mathcal{O}_B$ mapping $\mathcal{D}_A$ to $\mathcal{D}_B$ such that $\Phi_A|_{\mathcal{D}_A} \circ \psi = \Phi_B|_{\mathcal{D}_B}$.

Condition (I): Every vertex in E_A is reachable by a path from another vertex such that the latter has two disjoint cycles through it. $\implies$ *every cycle has an entry*

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Question (cont'd): ...or at least stably isomorphic, i.e. $C^*(E) \otimes \mathcal{K} \cong C^*(F) \otimes \mathcal{K}$ where $\mathcal{K}$ are the compact operators on a separable Hilbert space?

Theorem

Let $A, B \in \{0,1\}^{n\times n}$ have no zero columns or rows and assume that

- *E_A, E_B are single cycles or*
- *E_A, E_B are both satisfy condition (I) or*
- *A, B are both acyclic.*

If the one-sided shifts $\hat{X}_A, \hat{X}_B$ are flow-equivalent, *then*

$$(\mathcal{K} \otimes \mathcal{O}_A, \mathcal{C} \otimes \mathcal{D}_A) \cong (\mathcal{K} \otimes \mathcal{O}_B, \mathcal{C} \otimes \mathcal{D}_B)$$

where $\mathcal{C}$ is a maximal commutative subalgebra of $\mathcal{K}$.

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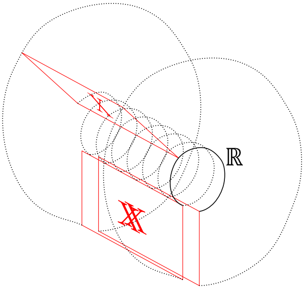
Flow-Equivalence



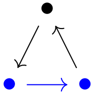
suspension of a both-sided shift (X, σ):

$$SX := X \times \mathbb{R} \big/ _{\langle (\sigma(x), t) \sim (x, t+1) \rangle} = X \times [0,1] \big/ _{\langle (\sigma(x), 0) \sim (x, 1) \rangle}$$

For one-sided shift: $\mathbb{R}_+$ instead of $\mathbb{R}$
$\rightarrow$ has natural $\mathbb{R}$ resp. $\mathbb{R}_+$ action

flow-equivalence := homeomorphism $\psi : SX \rightarrow SY$ mapping orbits to orbits

Flow equivalence is generalization of conjugacy,
on graph level: in/out-splitting/amalgamation + expansion

$M_2(\mathbb{C}) \otimes C(\mathbb{T})$  $\xrightarrow{\text{exp.}}$  $M_3(\mathbb{C}) \otimes C(\mathbb{T})$

$\rightsquigarrow$ are stably isomorphic
in/out-splitting/amalg. cannot be applied

Invariants of Graph Algebras

Question: How to show that two graph algebras are **not** *isomorphic, given their graphs?*

Can construct invariants (under isomorphisms) for graph algebras, and compute them using the adjacency matrix A_E

Example: **Extension semigroup** $\mathrm{Ext}(\mathcal{A})$

Main idea: $\mathrm{Ext}(\mathcal{A})$ = equivalence classes of injective *-homomorphisms into Calkin-algebra $\sigma : \mathcal{A} \to \mathcal{B}(H)/\mathcal{K}(H)$

Group 4
Graph Algebras
Shift Spaces
Connections

Recall **Condition (I):** Every vertex in E_A is reachable by a path from another vertex such that the latter has two disjoint cycles through it.

Theorem ([CK80, Theorem 5.1])

Suppose $A \in \{0,1\}^{n \times n}$ has no zero columns or rows and satisfies condition (I). Then the semigroups $\operatorname{Ext} \mathcal{O}_A$ and $\mathbb{Z}^n/(1-A)\mathbb{Z}^n$ are isomorphic.

Furthermore, by Elementarteilersatz we have

$$\mathbb{Z}^n/(1-A)\mathbb{Z}^n \cong \mathbb{Z}^n/D\mathbb{Z}^n \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \mathbb{Z}/d_2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_n\mathbb{Z}$$

for suitable $d_i \in \mathbb{N}_0$.

Bowen and Franks showed that $\mathbb{Z}^n/(1-A)\mathbb{Z}^n$ is invariant under flow equivalence of topological Markov chains.

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One Final Example

Consider the graphs (Condition (I) ✓)

$$E_1 = \begin{tikzcd} & 1 \arrow[loop right] \\ 2 \arrow[r, bend left] & 3 \arrow[u] \arrow[l, bend left] \arrow[loop right] \end{tikzcd} \qquad E_2 = \begin{tikzcd} 1 \arrow[r] \arrow[dr] & 4 \arrow[d] \arrow[loop right] \\ 2 \arrow[u] & 3 \arrow[l] \end{tikzcd}$$

Their adjacency matrices are

$$A_1 - 1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad A_2 - 1 = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\implies \operatorname{Ext} \mathcal{O}_{A_1} \cong \mathbb{Z}^3/(1 - A_1)\mathbb{Z}^3 \cong \mathbb{Z}$$
$$\operatorname{Ext} \mathcal{O}_{A_2} \cong \mathbb{Z}^4/(1 - A_2)\mathbb{Z}^4 \cong 0$$

$\implies C^*(E_1) \cong \mathcal{O}_{A_1}$ and $C^*(E_2) \cong \mathcal{O}_{A_2}$ are not isomorphic!

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Consider the graphs (Condition (I) ✓)

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$$\implies C^*(E_1) \cong \mathcal{O}_{A_1} \text{ and } C^*(E_2) \cong \mathcal{O}_{A_2} \text{ are not isomorphic!}$$

Outlook

- Ideal structure of graph algebras can be read off their graphs! E.g.: A irreducible $\implies$ $\mathcal{O}_A$ simple
- K-Theory of graph algebras (row-finite, no sources): $K_0(C^*(E)) \cong \operatorname{coker}(1-A_E^\top)$, $\quad K_1(C^*(E)) \cong \ker(1-A_E^\top) \subset \bigoplus_{E_0} \mathbb{Z}$
- Graph algebras with infinite receivers (non-row-finite graphs)
- Can reconstruct $\mathcal{K} \otimes \mathcal{O}_A$ from $\mathcal{D}_A$ as huge double crossed product with group of homeomorphisms of "unstable manifold" W

$$\mathcal{K} \otimes \mathcal{O}_A = r^{\mathbb{Z}} \ltimes \left(\frac{(Homeo_{u.f.d.}(W) \ltimes C_0(W))}{\langle \hat{u}P_B - \hat{v}P_B : u|_B = v|_B,\ B \text{ cpt. open}\rangle} \right)$$

where $r \in Homeo_{u.f.d.+index\ shift}(W)$ arbitrary.

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Sources

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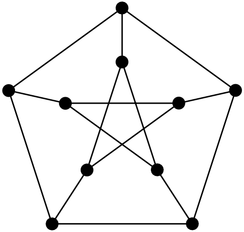
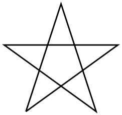
THANK YOU FOR YOUR ATTENTION!

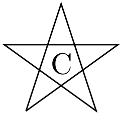
How to make a graph C^*-algebra:



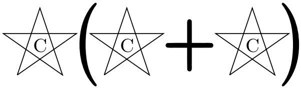
Step 1: Draw the Petersen graph



Step 2: Erase the outside



Step 3: Add a "C"



Step 4: Do algebra!