Irreducible representations and pure states

Project 5

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Representation

Let A be a C*-algebra. A Representation of A is a pair (π, H), where H is Hilbert space and

$$\pi : A \to B(H) ,$$

is $*$-homomorphism.

We also say that π is a representation.

- Let A be a *-algebra and let (π, H) be a representation of A. A subspace $N \subset H$ is said to be invariant if $\pi(a)N \subset N$ for all $a \in A$.
- A representation (π, H) of A is called non-degenerate if $\pi(A)H$ is dense in H, otherwise the representation is called degenerate.

Irreducible Representation

Definition

A representation of a*-algebra is irreducible if the only closed invariant subspaces are $\{0\}$ and H, otherwise it is reducible.

- A representation (π, H) is a cyclic representation if there is a cyclic vector in H for $\pi(A)$.
- A cyclic representation is non-degenerate.

Definition

Positive linear functional:- Let A be C*-algebra, A linear functional φ on A is positive, written $\varphi \geq 0$, if $\varphi(x) \geq 0$ whenever $x \geq 0$.

Definition (State)

A state on A is a positive linear functional of norm 1. We denote $S(A)$ the set of all states on A, called the state space of A.

Example

If A is a concrete C^*-algebra of operators acting non-degenerately on H and $\xi \in H$ and $\varphi_\xi(x) = \langle x\xi, \xi \rangle$ for $x \in A$, then φ_ξ is a positive linear functional on A of norm $||\xi||^2$, so φ_ξ is a state if $||\xi|| = 1$. Such a state is called a vector state.

GNS Representation

- *GNS-construction*: For $\varphi \in S(A)$ there are a Hilbert space H_φ, a *representation* $\pi_\varphi : A \to B(H_\varphi)$ and a *cyclic vector* $x_\varphi \in H_\varphi$ such that $\varphi(a) = \langle \pi_\varphi(a)x_\varphi, x_\varphi \rangle$ for all $a \in A$
  - Construction of H_φ:
    $\langle a, b \rangle := \varphi(b^* a) \to$ mod out elements of zero norm $\to$ complete
  - π_φ is left multiplication viewed in H_φ

Definition (Pure states $PS(A)$)

A state φ on a C^*-algebra A, is pure if it has the property that whenever ρ is a positive linear functional on A such that $\rho \leq \varphi$, necessarily there is a number $t \in [0, 1]$ such that $\rho = t\varphi$.

Theorem

Let $\varphi \in S(A)$:
$\varphi \in PS(A) \iff (H_\varphi, \pi_\varphi)$ *is irreducible.*

Idea of proof from the commutative case

- For A commutative (and unital), $A = C(X)$, with X compact (G-N)

- Riesz-Markov: States correspond to Radon measures of mass one:

$$\varphi(f) = \int_X f \, d\mu$$

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- GNS-construction w.r.t. Dirac measure?
  $\rightarrow \mathbb{C}$ with complex multiplication: clearly irreducible.

Idea of proof from the commutative case

Proof of the other direction (by contrapositive: assume that φ is not pure, show that (H_φ, π_φ) is reducible):

- GNS-constr. of $\varphi \in S(A)$, with corresponding measure μ, is $L^2(X, \mu)$

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- GNS-constr. of $\varphi \in S(A)$, with corresponding measure μ, is $L^2(X, \mu)$
- When μ is not a Dirac measure:
  $\implies$ the support of μ has a proper subset $Y \subset X$
  $\implies$ $L^2(Y, \mu)$ is a reducing subspace. $\square$

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$$L^2(Y, \mu) = \chi_Y H_\varphi$$

Existence of projections $\to$ Invariant subspaces

Characterisation of irreducibility with commutant

Definition (Commutant)

$\pi_\varphi(A)' := \{v \in B(H_\varphi) \text{ s.t. } v \text{ commutes with all } \pi_\varphi(a) \in \pi_\varphi(A)\}$

Lemma

$\pi_\varphi(A)$ *is irreducible* $\iff \pi_\varphi(A)' = \mathbb{C}1$

Proof.

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Proof.

- $\pi_\varphi(A)$ is irreducible $\iff \pi_\varphi(A)'$ has no nontrivial projections
- $\iff \pi_\varphi(A)' = \mathbb{C}1$ because $\pi_\varphi(A)'$ is a vNA, and thus a linear span of its projections.

$\square$

Proof: ($\varphi \in PS(A) \iff (H_\varphi, \pi_\varphi)$ is irreducible)

$\implies$: (assume φ is pure, show that $\pi_\varphi(A)' = \mathbb{C}1$).

- Let $v \in \pi_\varphi(A)', 0 \leq v \leq 1$. Define a positive linear functional $\rho(a) := \langle v\pi_\varphi(a)x_\varphi, x_\varphi \rangle$
- We have $\rho \leq \varphi$, thus $\rho = t\varphi \implies \cdots \implies v = t1$
- $\implies \pi_\varphi(A)' = \mathbb{C}1$ as element of $B(H)$ can be decomposed into a linear combination of positive elements

Proof: ($\varphi \in PS(A) \iff (H_\varphi, \pi_\varphi)$ is irreducible)

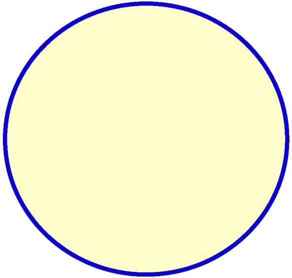
$\Longleftarrow$: (assume $\pi_\varphi(A)' = \mathbb{C}1$, show that φ is pure)

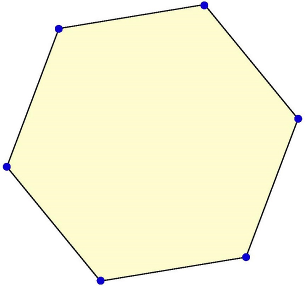
- Let $\rho \leq \varphi$ be a positive linear functional
- $\implies$ There is $v \in \pi_\varphi(A)', 0 \leq v \leq 1$, s.t. $\rho(a) = \langle v\pi_\varphi(a)x_\varphi, x_\varphi\rangle$ (cf. Radon-Nikodym theorem for absolutely continuous measures)
- By the assumption $v = \lambda 1, \lambda \in \mathbb{C}$
- Moreover $\lambda \in [0, 1]$ because $0 \leq v \leq 1 \implies \rho = \lambda\varphi \quad \square$

Overview

- Preliminaries: The Krein-Milman theorem
- Motivation: States and Pure States on $C(X)$
- $PS(A)$ are extreme points of $S(A)$
- Example: States and Pure States in Quantum Mechanics.

Preliminaries: The Krein-Milman theorem

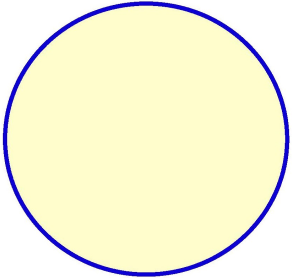


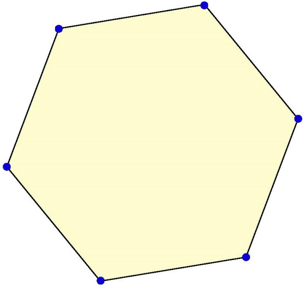


Definition

Let $\mathcal{X}$ be a locally convex space and $S \subset \mathcal{X}$. We define a *closed convex hull* of S, denoted by $\overline{\mathrm{co}}(S)$, to be the smallest closed convex set containing S.

Preliminaries: The Krein-Milman theorem





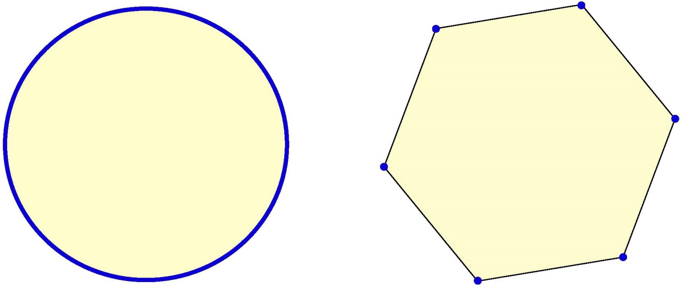
Definition

Let K be a convex subset of a vector space $\mathcal{X}$. We say that $x \in K$ is an *extreme point* of K if

$$x = ty + (1-t)z\,,\ t \in (0,1)\,,\ y,z \in K \implies x = y = z\,.$$

We denote the set of all extreme points of K by $\mathrm{ext}(K)$.

Preliminaries: The Krein-Milman theorem



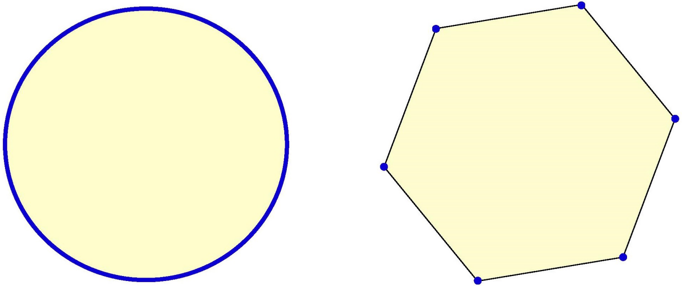
Definition

Let K be a convex subset of a vector space $\mathcal{X}$ and let $F \subset K$. If $x \in F$, $y, z \in K$ and

$$x = ty + (1-t)z \text{, for some } t \in (0,1) \implies y, z \in F,$$

then F is called a *face* of F.

Preliminaries: The Krein-Milman theorem



Theorem (Krein-Milman)

Let $\mathcal{X}$ be a locally convex space and $K \subset \mathcal{X}$ non-empty, compact, convex subset. Then $\operatorname{ext}(K) \neq \varnothing$ and

$$K = \overline{\operatorname{co}}(\operatorname{ext}(K)) .$$

States and Pure States on $C(X)$

- Let X be a compact Hausdorff space.
- $\mathcal{M}(X)$ = space of all Radon measures on X.
- Riesz representation theorem: $C(X)^* \simeq \mathcal{M}(X)$

We have *wk** topology on $\mathcal{M}(X)$ induced from $C(X)^*$ and a measure μ acts on $C(X)$ by

$$\mu(f) = \int_X f(x) d\mu(x)$$

.

States and Pure States on $C(X)$

- Recall that by Gelfan-Naimark theorem unital abelian C^* algebra A is isometrically $*$-isomorphic to $C(\mathrm{Spec}(A))$
  $X = \mathrm{Spec}(A) =$ space of all characters on A

- State ϕ on $\mathcal{A}$ corresponds to Radon probability measure μ_ϕ on X

$$\phi(f) = \int_X f(x) d\mu_\phi(x).$$

States and Pure States on $C(X)$

Theorem

1. *The set $\mathcal{P}(X)$ of Radon probability measures on X is a wk^*-compact convex subset of $\mathcal{M}(X)$.*
2. *The extreme points of $\mathcal{P}(X)$ are exactly the Dirac measures δ_x which assign mass 1 at the point $x \in X$ and zero everywhere else.*
3. *The map $x \mapsto \delta_x \in \mathrm{ext}(\mathcal{P}(X))$ is a homeomorphism from X to the space $(\mathrm{ext}(\mathcal{P}(X)), wk^*)$*

States and Pure States on $C(X)$

Theorem

Let τ be a state on an abelian C^* algebra A. Then τ is pure if and only if it is a character on A, i.e. $\mathrm{Spec}(A) = PS(A)$.

- Since Dirac measures correspond to characters, they correspond to pure states.
- We expect an analogous result of pure states being extreme points of the state space in the non-commutative case! And that is our goal.

Classical probability	Quantum Probability
Probability measure μ on a cpt. Hausdorff space	ρ a on unital C^* alg. A
$X \ni x \mapsto \delta_x \in \mathcal{P}(X)$ homeomorphism	$PS(A) = \mathrm{Spec}(A)$ for abelian
$\mathcal{P}(X)$ is wk^*-cpt. and convex	$S(A)$ is wk^*-cpt. and convex
δ_x are extreme pts. of $\mathcal{P}(X)$	$PS(A)$ are extreme pts. of $S(A)$

$PS(A)$ are extreme points of $S(A)$

Theorem

If A is a C^* algebra, then the set S of norm decreasing positive linear functionals on A forms a convex wk^*-compact set and $\mathrm{ext}(S) = \{0\} \cup PS(A)$.

Proof

S is wk^*-compact and convex.

- Let $\phi_i \longrightarrow \phi \implies \phi_i(a^*a) \longrightarrow \phi(a^*a) \implies \phi$ is positive.
- $|\phi_i(a)| \leq \|a\| \implies |\phi(a)| \leq \|a\| \implies \|\phi\| \leq 1 \implies S$ is wk^*-closed.
- Banach-Alaoglu thm: S is wk^*-compact.
- Convexity is clear.

$PS(A)$ are extreme points of $S(A)$

$0 \in \mathrm{ext}(S)$.

- Assume $0 = t\tau + (1-t)\rho$, where $0 < t < 1$ and $\tau, \rho \in S$.
- $\forall a \in A\,, 0 \leq (1-t)\rho(a^*a) = -t\tau(a^*a) \leq 0 \implies \tau(A^+) = \rho(A^+) = 0 \implies \tau = \rho = 0\,.$

$PS(A) \subset \mathrm{ext}(S)$.

- Let $\rho \in PS(\mathcal{A})$ and $\rho = t\tau + (1-t)\tau'$ for $0 < t < 1$ and $\tau, \tau\prime \in S$.
- $0 \leq \rho - t\tau$ i.e $t\tau \leq \rho \implies t\tau = t'\rho$ for some $t' \in [0,1]\,.$
- $1 = \|\rho\| = t\|\tau\| + (1-t)\|\tau'\| \implies \|\tau\| = \|\tau'\| = 1\,. \implies t = \|t\tau\| = \|t'\rho\| = t'$ so $\tau = \rho$ and similarly $\tau' = \rho$. Thus $\rho \in \mathrm{ext}(S)\,.$

$PS(A)$ are extreme points of $S(A)$

$\mathrm{ext}(S) \setminus \{0\} \subset PS(A)$.

- Let $\rho \in \mathrm{ext}(S)$ be non zero. Then

$$\rho = \|\rho\|(\rho/\|\rho\|) + (1 - \|\rho\|)0$$

  and $0, \rho/\|\rho\| \in S$.

- Therefore, $\|\rho\| = 1$ as $\rho \in \mathrm{ext}(S)$.
- Let τ be a nonzero, positive, linear functional s.t. $\tau \lneq \rho$. Then $\|\tau\| = t \in (0, 1)$.
- Since $1 - t = \|\rho - \tau\|$ we write

$$\rho = t(\tau/\|\tau\|) + (1 - t)(\rho - \tau)/\|\rho - \tau\|$$

$$\implies \rho = \tau/\|\tau\| \implies \rho \in PS(A).$$

$PS(A)$ are extreme points of $S(A)$

Corollary

$S = \overline{\mathrm{co}}(\{0\} \cup PS(A))$.

Corollary

Let A be a non–zero C^*-algebra and $a \in A^+$. Then there is $\rho \in PS(A)$ such that $\|a\| = \rho(a)$.

$PS(A)$ are extreme points of $S(A)$

Corollary

Let A be a C^*-algebra. Then $\mathrm{ext}(S(A)) = PS(A)$.

Proof.

- $S(A)$ is a face of S: Let $\phi \in S(A)$, $\tau\rho \in S$ and $0 < t < 1$.

$$\phi = t\tau + (1-t)\rho \implies 1 = t\|\tau\| + (1-t)\|\rho\| \implies \|\tau\| = \|\rho\| = 1$$

- Therefore, $\mathrm{ext}(S(A)) \subset \mathrm{ext}(S) \implies \mathrm{ext}(S(A)) \subset PS(A)$
- $PS(A) \subset \mathrm{ext}(S(A))$ similarly as before. $\square$

$PS(A)$ are extreme points of $S(A)$

Theorem

If A is a unital C^*-algebra, then $S(A) = \overline{\text{co}}(PS(A))$.

Proof.

- $S(A)$ is clearly convex
- $(\tau_i)_{i \in I}$ net in $S(A)$ and $\tau_i \to \tau$ in (A^*, wk^*).
  $\implies \tau(1) = \lim_i \tau_i(1) = \lim_i 1 = 1 \implies \tau \in S(A) \implies S(A)$ is wk^*-closed
- Banach-Alaouglu thm: $S(A)$ is wk^*-compact
- Krein-Milman thm: $S(A) = \overline{\text{co}}(\text{ext}(S(A)))$
- $\text{ext}(S(A)) = PS(A)$. □

Example: Mixed and Pure states in Quantum Mechanics

- *Physical system* = unital separable C^*-algebra A.
- *Observables* = self-adjoint elements of A
- *States of the system* = states on A.

Example: Mixed and Pure states in Quantum Mechanics

Classical mechanics:

- A is abelian. Gelfand representation $\implies A \simeq C(X)$ where X is compact and Hausdorff.
- Observables $=$ real continuous functions.
- States $=$ probability measures on X.
- Pure states $=$ Dirac measures $=$ points in X .

Example: Mixed and Pure states in Quantum Mechanics

<u>Quantum mechanics</u>:

Theorem (Gelfand-Naimark)

Let A be a unital separable C^*-algebra. Then there is a separable Hilbert space H such that

- *A is isometrically *-isomorphic to a C^*-subalgebra of $B(H)$, via $\pi : A \longrightarrow B(H)$.*
- *$\psi \in S(A)$ iff there exists a positive trace-class operator ρ_ψ such that $\mathrm{Tr}\rho_\psi = 1$ and $\psi(a) = \mathrm{Tr}(\rho_\psi \pi(a))$.*

Example: Mixed and Pure states in Quantum Mechanics

- ρ_ψ compact and self-adjoint $\implies$ $\rho_\psi = \sum_{n=1}^{\infty} \alpha_n P_n$, where P_n is a projection on $\mathbb{C}e_n$ and $\{e_n\}$ is a base for $\mathcal{H}$
- ρ_ψ positive $\implies$ $\alpha_n \geq 0$
- $\mathrm{Tr}(\rho_\psi) = 1 \implies \sum_{n=1}^{\infty} \alpha_n = 1 \implies$
- ρ_ψ is in a closed convex hull of one-dimensional projections.

Kaplansky's density theorem

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Notation

We denote for any set $S \subseteq A$ a C^*-algebra:

- $S_{sa} = \{a \in S \mid a^* = a\}$
- $S_+ = \{a \in S_{sa} \mid a \geq 0\}$
- $\mathrm{ball}(S) = \{a \in S \mid ||a|| \leq 1\}$

Main Theorem

Statement

Let H be a Hilbert Space, A a sub C^*-algebra of $B(H)$ and $B = \overline{A}^{\mathrm{SOT}}$ be the SOT-closure of A in $B(H)$. Then

1. A_{sa} is SOT-dense in B_{sa}.

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2. $\mathrm{ball}(A_{\mathrm{sa}})$ is SOT-dense in $\mathrm{ball}(B_{\mathrm{sa}})$.

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3. $\mathrm{ball}(A_+)$ is SOT-dense in $\mathrm{ball}(B_+)$.

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4. $\mathrm{ball}(A)$ is SOT-dense in $\mathrm{ball}(B)$.

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3. $\mathrm{ball}(A_+)$ is SOT-dense in $\mathrm{ball}(B_+)$.
4. $\mathrm{ball}(A)$ is SOT-dense in $\mathrm{ball}(B)$.
5. If A is unital, $U(A)$ is SOT-dense in $U(B)$.

Sketch of Proof

Strong Continuity

$f : \mathbb{R} \to \mathbb{C}$ is strongly continuous if for every Hilbert space H and net $(T_i) \in B(H)$ of self adjoint operators such that $T_i \to T$ in SOT, we have $f(T_i) \to f(T)$ in SOT.

Lemma 1

If $f : \mathbb{R} \to \mathbb{C}$ is bounded continuous, then f is strongly continuous.

Sketch of Proof

Statement 1

A_{sa} is SOT-dense in B_{sa}.

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Proof

Let $b \in B_{\mathrm{sa}}$. There is a net (a_i) in A such that $a_i \to b$ in SOT.

- Thus $a_i \to b$ in WOT.

Sketch of Proof

Statement 1

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Proof

Let $b \in B_{\mathrm{sa}}$. There is a net (a_i) in A such that $a_i \to b$ in SOT.

- Thus $a_i \to b$ in WOT.
- The map $Re : x \mapsto Re(x)$ is WOT continuous, hence $Re(a_i) \to Re(b)$ in WOT.

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- Thus $a_i \to b$ in WOT.
- The map $Re : x \mapsto Re(x)$ is WOT continuous, hence $Re(a_i) \to Re(b)$ in WOT.
- By convexity of A_{sa}, $b \in \overline{A_{\mathrm{sa}}}^{\mathrm{WOT}} = \overline{A_{\mathrm{sa}}}^{\mathrm{SOT}}$.

Sketch of Proof

Statement 2

$\mathrm{ball}(A_{\mathrm{sa}})$ is SOT-dense in $\mathrm{ball}(B_{\mathrm{sa}})$.

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Sketch of Proof

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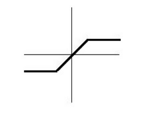
Proof

Let $b \in \mathrm{ball}(B_{\mathrm{sa}})$, by (1), we have $(a_i) \in A_{\mathrm{sa}}$ such that $a_i \to b$ in SOT.

- Let $f(t) = min\{max\{-1, t\}, 1\}$. By Lemma 1, f is strongly continuous and $f(a_i) \to f(b)$ in SOT.



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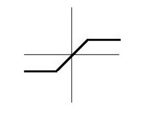

- $\sigma(b) \subseteq [-1, 1]$, so $f|_{\sigma(b)}(t) = t$ and $f(b) = b$.

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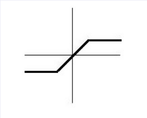

- $\sigma(b) \subseteq [-1, 1]$, so $f|_{\sigma(b)}(t) = t$ and $f(b) = b$.
- $f(a_i) \in \mathrm{ball}(A_{\mathrm{sa}})$ and hence $f(a_i) \to f(b) = b$ in SOT.

Sketch of Proof

Statement 3

$\mathrm{ball}(A_+)$ is SOT-dense in $\mathrm{ball}(B_{\mathrm{sa}})$.

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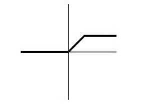

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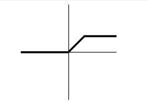

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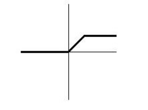

- $\sigma(b) \subseteq [0, 1]$, so $f|_{\sigma(b)}(t) = t$ and $f(b) = b$.
- $f(a_i)$ is self adjoint, of norm atmost 1, and positive, so $f(a_i) \in \mathrm{ball}(A_+)$.

Sketch of Proof

Statement 4

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- $\overline{b}$ is self adjoint and has $\|\overline{b}\| \leq 1$.

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- It is easy to check that $\overline{M_2(A)}^{\mathrm{SOT}} = M_2(B)$.
- By part (2), there is a net $(\overline{a_i}) \in M_2(A)_{\mathrm{sa}}$ that converges to $\overline{b}$ in SOT.

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Let $b \in \mathrm{ball}(B)$, then let $\overline{b} = \begin{bmatrix} 0 & b \\ b^* & 0 \end{bmatrix} \in M_2(B)$.

- $\overline{b}$ is self adjoint and has $\|\overline{b}\| \leq 1$.
- It is easy to check that $\overline{M_2(A)}^{\mathrm{SOT}} = M_2(B)$.
- By part (2), there is a net $(\overline{a_i}) \in M_2(A)_{\mathrm{sa}}$ that converges to $\overline{b}$ in SOT.
- $(\overline{a_i})_{1,2} \to b$ in SOT and $\|\overline{a_i}_{1,2}\| \leq \|\overline{a_i}\| \leq 1$.

Sketch of Proof

Statement 5

If A is unital, $U(A)$ is SOT-dense in $U(B)$.

Proof

Let $u \in B$ be unitary.

- From functional calculus, it follows that there exists a self adjoint $b \in B$ such that $u = e^{ib}$.

Sketch of Proof

Statement 5

If A is unital, $U(A)$ is SOT-dense in $U(B)$.

Proof

Let $u \in B$ be unitary.

- From functional calculus, it follows that there exists a self adjoint $b \in B$ such that $u = e^{ib}$.
- By (1), there is a net (a_λ) of self adjoint operators in A such that $a_\lambda \to b$ in SOT.

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Let $u \in B$ be unitary.

- From functional calculus, it follows that there exists a self adjoint $b \in B$ such that $u = e^{ib}$.
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If A is unital, $U(A)$ is SOT-dense in $U(B)$.

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Let $u \in B$ be unitary.

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- By (1), there is a net (a_λ) of self adjoint operators in A such that $a_\lambda \to b$ in SOT.
- The function $t \to e^{it}$ is strongly continuous.
- By Lemma 1, $e^{ia_\lambda} \to e^{ib} = u$ in SOT.

Sketch of Proof

In our context

- Let A be a C^*-algebra and $\pi : A \to B(H)$ be an irreducible representation of A. In particular it is non-degenerate, i.e. the set $\{\pi(a)h \mid a \in A, h \in H\}$ is dense in H.
- By the von Neumann bicommutant theorem, we know that $A'' = \overline{A}^{\mathrm{SOT}}$.
- By irreducibility, $\pi(A)' = \mathbb{C}I$ and $\pi(A)'' = B(H)$. So A is strongly dense in $B(H)$.
- Now we apply the theorem with $B = B(H)$.

Theorem (Bicommutant theorem)

Let $\pi : A \to B(H)$ be a non-degenerate representation. Then $\pi(A)$ is strongly dense in $\pi(A)''$.

Algebraic irreducibility

Let $\pi : A \to B(H)$ be a representation.

- *Topological* irreducibility: there are no *closed* invariant subspaces.

Question: Is algebraic irreducibility a stronger property than topological irreducibility?

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- $o \subsetneq K \subsetneq H$ closed and invariant yields a *reduction* $\pi|_K : A \to B(K)$.

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- $o \subsetneq K \subsetneq H$ closed and invariant yields a *reduction* $\pi|_K : A \to B(K)$.
- Call π *algebraically* irreducible if there are no invariant subspaces at all (except $\{o\}$ and H).

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Theorem (Kadison's transitivity theorem)

Let $\pi : A \to (H)$ be topologically irreducible, and $\xi_1, \ldots, \xi_d \in H$ be linearly independent. Then, for any $\eta_1, \ldots, \eta_d \in H$, there is an $a \in A$ such that $\pi(a)\xi_j = \eta_j$, $j = 1, \ldots, d$.

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- We even have "(almost) d-transitivity" (for *any* $d \in \mathbb{N}$).

Proving Kadison's transitivity theorem

Theorem (Kadison)

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Theorem

Let $\pi : A \to B(H)$ be topologically irreducible, $T \in B(H)$ and $X \subseteq H$ be finite-dimensional. There is an $a \in A$ such that $\pi(a)|_X = T|_X$ and $\|a\| \leq \|T\|$.

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- We only prove: for $\varepsilon > 0$, there is an $a = a_\varepsilon \in A$ with $\|a_\varepsilon\| \leq \|T\| + \varepsilon$ and $\pi(a_\varepsilon)|_X = T|_X$.

Proof

Theorem

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- By irreducibility, $\pi(A)$ is non-degenerate.

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- By irreducibility, $\pi(A)$ is non-degenerate.
- Bicommutant theorem: $\pi(A)$ is strongly dense in $B(H)$.
- **Idea:** approximate T with a sequence in $\pi(A)$.
- Let $\varepsilon > 0$. We find $a_1 \in A$ such that

$$\|(\pi(a_1) - T)|_X\| \leq \varepsilon/2\,;$$

by Kaplansky, we can require $\|a_1\| \leq \|T\|$.

Proof

- $\pi(A)$ is strongly dense in $B(H)$.
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- Consider $T_2 \in B(H)$ given by

$$T_2|_X = (T - \pi(a_1))|_X \ , \quad T_2|_{X^\perp} = 0 \ . \qquad \text{Then } \|T_2\| \leq \varepsilon/2$$

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- We find $a_2 \in A$ with $\|a_2\| \leq \|T_2\| \leq \varepsilon/2$ and

$$\|(\pi(a_1) + \pi(a_2) - T)|_X\| = \|(\pi(a_2) - T_2)|_X\| \leq \varepsilon/4 \ .$$

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- Inductively, we find $a_n \in A$ with $\|a_n\| \leq \varepsilon/2^{n-1}$ and
$$\left\| \left(\sum_{k=1}^{n} \pi(a_k) - T \right) |_X \right\| \leq \varepsilon/2^n \ , \quad n \geq 2 \, .$$

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- Then $\sum a_n$ is absolutely convergent in A, and $a_\varepsilon = a = \sum a_n$ satisfies

∞

Remarks

Theorem

Let $\pi : A \to B(H)$ be topologically irreducible, $T \in B(H)$ and $X \subseteq H$ be finite-dimensional. There is an $a \in A$ such that $\pi(a)|_X = T|_X$ and $\|a\| \leq \|T\|$. Moreover, if T is self-adjoint [resp. positive], we may choose $a \in A$ to be self-adjoint [resp. positive].

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- We can get $\|a_\varepsilon\| \leq \|T|_X\| + \varepsilon$ $(\leq \|T\| + \varepsilon)$ for free.
- We need the stronger denseness property from Kaplansky's theorem to control the norm of the approximating operators a_n, and get convergence of the series.

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$$\pi_\varphi(a)(x + N_\varphi) = ax + N_\varphi\,, \quad a, x \in A\,.$$

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- $A/N_\varphi \subseteq H_\varphi$ $\pi(A)$-invariant $\implies$ $A/N_\varphi = H_\varphi$. □

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