The Reduced Group C*-Algebra of a Free Group and C*-Simplicity

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The Plan

- The following is a brief summary of what we will each be covering.
  1. **Joseph:** Properties of group C*-algebras and the canonical faithful, tracial state on the reduced group C*-algebra of a discrete group.
  2. **Milan:** An introduction to free groups and the notion of amenability of groups.
  3. **Moritz:** Two lemmas which will be useful in proving the main result.
  4. **Marcel:** One further lemma, culminating in the proof of the main result.
  5. **Ayoub:** The Furstenberg boundary characterisation of C*-simplicity, providing an alternate perspective on the previously explored results.

The *-Algebra $\mathbb{C}[G]$

- Fix an arbitrary discrete group G.
- We define $\mathbb{C}[G]$ as follows:

$$\mathbb{C}[G] := \{f : G \to \mathbb{C} \mid f \text{ is a finitely supported function}\}.$$

  $\mathbb{C}[G]$ is a $\mathbb{C}$-vector space under pointwise addition and scalar multiplication.
- $\mathbb{C}[G]$ has a linear basis $\{\delta_t \mid t \in G\}$, where

$$\delta_t(s) = \begin{cases} 1, & \text{if } s = t \\ 0, & \text{otherwise} \end{cases}$$

  for all $t, s \in G$. So an arbitrary element of $\mathbb{C}[G]$ has the form $\sum_{t \in G} \alpha_t \delta_t$ where each $\alpha_t \in \mathbb{C}$ and only finitely many are non-zero.

The *-Algebra $\mathbb{C}[G]$ cont.

- We define a multiplication and involution (respectively) on $\mathbb{C}[G]$ as follows:
  - $\left(\sum_{t\in G}\alpha_t\delta_t\right)\left(\sum_{s\in G}\beta_s\delta_s\right) := \sum_{t,s\in G}\alpha_t\beta_s\delta_{ts}$,
  - $\left(\sum_{t\in G}\alpha_t\delta_t\right)^* := \sum_{t\in G}\overline{\alpha_t}\delta_{t^{-1}}$.
- $\mathbb{C}[G]$, equipped with these operations, is a *-algebra.

The Reduced Group C*-Algebra $C_r^*(G)$

- Define a unitary representation of G via

$$\lambda : G \to U\left(\ell^2(G)\right), \lambda_t(\delta_s) = \delta_{ts} \text{ for all } t, s \in G.$$

  λ is known as the **left regular representation** of G.

- λ canonically extends to a *-homomorphism $\tilde{\lambda}$ defined as follows:

$$\tilde{\lambda} : \mathbb{C}[G] \to B\left(\ell^2(G)\right), \tilde{\lambda}\left(\sum_{t \in G} \alpha_t \delta_t\right) = \sum_{t \in G} \alpha_t \lambda_t.$$

- $\tilde{\lambda}$ is injective, so $\|x\|_r := \|\tilde{\lambda}(x)\|$ for all $x \in \mathbb{C}[G]$ defines a C*-norm on $\mathbb{C}[G]$.
- The **reduced group C*-algebra** of G is defined to be:

$$C_r^*(G) := \overline{\tilde{\lambda}(\mathbb{C}[G])}.$$

The Full Group C*-Algebra $C_f^*(G)$

- Now let $\pi : G \to U(H)$ be any unitary representation of G. As before, π canonically extends to a *-homomorphism $\tilde{\pi} : \mathbb{C}[G] \to B(H)$ defined via $\tilde{\pi}\left(\sum_{t\in G} \alpha_t \delta_t\right) = \sum_{t\in G} \alpha_t \pi(t)$. Observe:

$$\|\tilde{\pi}\left(\sum_{t\in G} \alpha_t \delta_t\right)\| = \|\sum_{t\in G} \alpha_t \pi(t)\| \leq \sum_{t\in G} |\alpha_t|.$$

- Hence we have a C*-norm on $\mathbb{C}[G]$ defined by:

$$\|x\|_f := \sup_{\tilde{\pi}} \|\tilde{\pi}(x)\|, \text{ for all } x \in \mathbb{C}[G],$$

  where the supremum is taken over all unital *-representations $\tilde{\pi}$ of $\mathbb{C}[G]$.

- The **full group C*-algebra** of G is the completion of $\mathbb{C}[G]$ with respect to $\|\cdot\|_f$, and it is denoted by $C_f^*(G)$.

The Universal Property of $C_f^*(G)$

- $C_f^*(G)$ has the following universal property: for any unitary representation $\pi : G \to U(H)$ of G, there exists a unique *-homomorphism $\tilde{\pi} : C_f^*(G) \to B(H)$ satisfying $\tilde{\pi}(\delta_t) = \pi(t)$.
- Applying this universal property to the left regular representation of G yields a surjective *-homomorphism $\Phi : C_f^*(G) \to C_r^*(G)$. Is Φ injective in general?
- Finally, G is said to be **C*-simple** if $C_r^*(G)$ is a simple C*-algebra.

The Canonical Faithful, Tracial State

Theorem

Let G be a discrete group. The map $\tau : C_r^*(G) \to \mathbb{C}$ defined by $\tau(x) = \langle \delta_e, x\delta_e \rangle$ for each $x \in C_r^*(G)$ is a faithful, tracial state. It is the unique such state satisfying $\tau(\lambda_e) = 1$ and $\tau(\lambda_t) = 0$ for each $t \in G \setminus \{e\}$.

Proof:

- Linearity and positivity are immediate. Uniqueness is similarly immediate, as every element of $C_r^*(G)$ is a norm limit of finite sums $\sum_{t\in G} \alpha_t \lambda_t$.
- τ acts on elements of $\tilde{\lambda}(\mathbb{C}[G])$ as follows:

$$\tau\left(\sum_{t\in G} \alpha_t \lambda_t\right) = \sum_{t\in G} \alpha_t \langle \delta_e, \lambda_t \delta_e \rangle = \sum_{t\in G} \alpha_t \langle \delta_e, \delta_t \rangle = \alpha_e.$$

Thus $\tau(\lambda_e) = 1, \tau(\lambda_t) = 0$ for all $t \in G \setminus \{e\}$, and $\|\tau\| = \tau(\lambda_e) = 1$.

The Canonical Faithful, Tracial State cont.

Proof (cont.):

- For all $t, s \in G$ we have:

$$\tau\left(\lambda_t \lambda_s\right) = \tau\left(\lambda_{ts}\right) = \langle \delta_e, \delta_{ts} \rangle = \langle \delta_e, \delta_{st} \rangle = \tau\left(\lambda_{st}\right) = \tau\left(\lambda_s \lambda_t\right).$$

  Extending by linearity and continuity yields $\tau(xy) = \tau(yx)$ for all $x, y \in C_r^*(G)$.

- Faithfulness has been proven on the whiteboard, so we are done. $\square$