The free group on 2 generators $\mathbb{F}_2$

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$\mathbb{F}_2$ is **set** of reduced words in a, a^{-1}, b, b^{-1}.

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$\mathbb{F}_S$ only depends on $|S|$, in particular unique free group on n generators $\mathbb{F}_n$ for any $n \in \mathbb{N}$.

Cayley Graph of $\mathbb{F}_2$

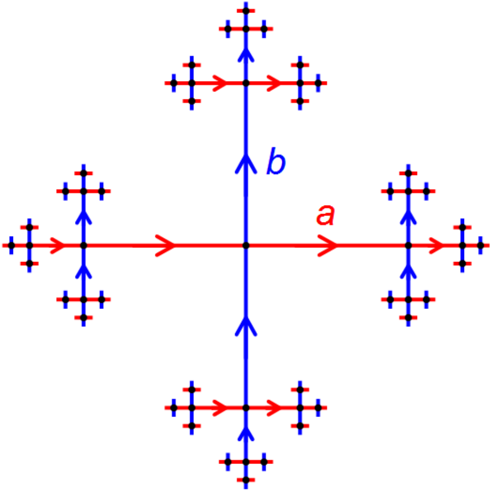

Figure: Source: Jim.belk,
`commons.wikimedia.org/wiki/File:F2_Cayley_Graph.png`

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\begin{tikzcd}
 & \mathbb{F}_S \arrow[d, dashed, "g"] \\
S \arrow[ur, "\iota"] \arrow[r, "f"'] & G
\end{tikzcd}
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- if $N \triangleleft G$ is a normal subgroup, then G is amenable if and only if both N and G/N are amenable;
- if $(G_n)_{n\in\mathbb{N}}$ is an increasing sequence of subgroups of G and $G = \cup_{n\in\mathbb{N}} G_n$, then G is amenable if and only if all G_n are amenable.

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Idea: split $\mathbb{F}_2$ in **disjoint** subsets G_a and G_b:

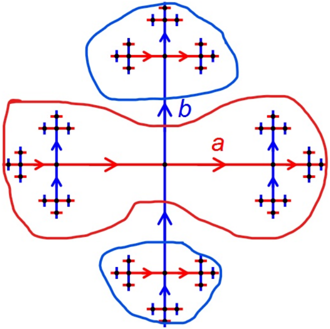

Figure: $\mathbb{F}_2 = G_a \cup G_b$

$\mathbb{F}_2$ is not amenable

Suppose $\mathbb{F}_2$ is amenable with mean μ. Since $\mathbb{F}_2 = G_a \cup G_b$ is a disjoint union, we have

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On the other hand, $aG_b \subset G_a$, hence

$$2\mu(G_a) \leq \mu(G_b) = \mu(aG_b) \leq \mu(G_a).$$

Then $\mu(G_a) = \mu(G_b) = 0$. Contradiction with (1).

Amenability and group C*-algebras

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As we will see: simplicity of $C_r^*(\mathbb{F}_2)$ implies that indeed $C_r^*(\mathbb{F}_2) \ncong C_f^*(\mathbb{F}_2)$.

It turns out that $C_r^*(G) \cong C_f^*(G)$ **if and only if** G **is amenable!**
In fact, for **any** dynamical system (A, G, α), we have an isomorphism $A \rtimes_{f,\alpha} G \cong A \rtimes_{r,\alpha} G$ when G is amenable.