Simplicity of $C_r^*(\mathbb{F}_2)$

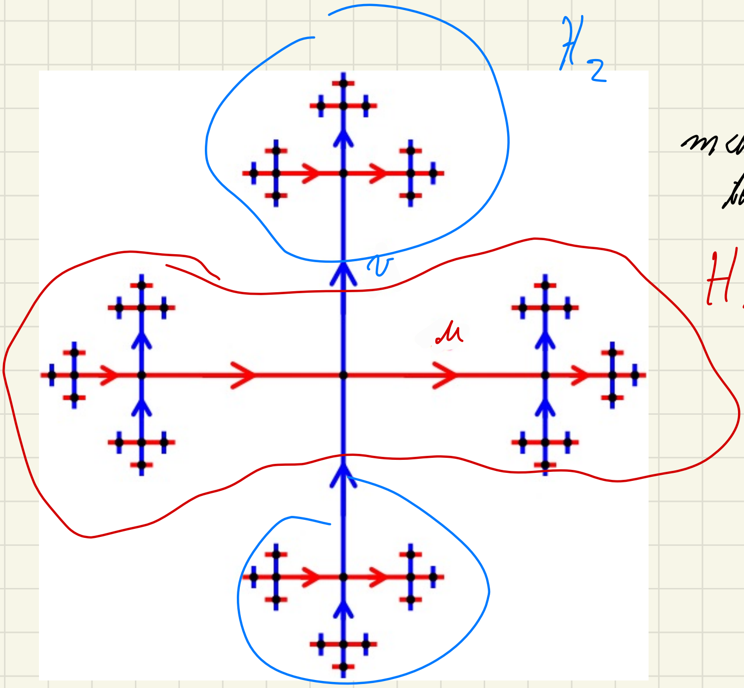
u, v

- We showed simplicity in Lecture 10 by approximating the conditional expectation

$\rightsquigarrow$ approximate the trace with $\frac{1}{n}\sum_{i=1}^{n} u^i x u^{-i}$

- This sum should converge to 0 for all elements other the identity

$\rightsquigarrow$ try this for a subset $H = \ell^2(\mathbb{F}_2)$



$H_1 \to H_2$ multiplication from the left by $v^{\pm 1}$

$H_2 \to H$
$u^{\pm 1}$

Lemma 1. *Let $H = H_1 \bigoplus H_2$ be a Hilbert space and H_1, H_2 orthogonal subspaces. Let $x \in B(H)$ with $x(H_1) \subseteq H_2$ and $U_j \in B(H)$, $1 \leq j \leq n$ be unitary operators such that $U_i^* U_j(H_2) \subseteq H_1$ for $i \neq j$. Then*

$$\| \sum_{j=1}^{n} U_j x U_j^* \| \leq 2\sqrt{n} \|x\|.$$

(1) If we could use Pythagoras in C^*-Algebras ($U_j x U_j^*$ were orthogonal)

$$\| \sum_{j=1}^{n} U_j x U_j^* \|^2 \;\text{"="}\; \sum_{j=1}^{n} \| U_j x U_j^* \|^2$$

$$= n \|x\|^2$$

(2) Almost Pythagoras

if $x^* y = y^* x = 0$

$$\| x + y \|^2 \leq \|x\|^2 + \|y\|^2$$

(3) Use $y := \sum_{j=1}^{n-1} U_n^* U_j x U_j^* U_n$ $\rightsquigarrow$ induction + projections

For $\nu = 0, 1$ consider the linear map $T_n^\nu : C_r^*(\mathbb{F}_2) \to C_r^*(\mathbb{F}_2)$ given by

$$T_n^\nu(y) := \frac{1}{n}\sum_{i=1}^{n} u_\nu^i y u_\nu^{-i}.$$

Then T_n^ν is a contraction for all $n \in \mathbb{N}$, hence bounded.

Lemma 2. *For $\nu = 0, 1$ let $\mathbb{F}_2^\nu \subseteq \mathbb{F}_2$ be the set of all reduced words in $u_\nu^{\pm 1}$. Then*

$$\lim_{n\to\infty} T_n^\nu(y) = \begin{cases} y & \text{if } y \in \mathbb{F}_2^\nu \\ 0 & \text{else} \end{cases}$$

for any $y \in \mathbb{F}_2$.

① If $y \in \mathbb{F}_2^\nu \Rightarrow y = u_\nu^k$, $k \in \mathbb{Z} \Rightarrow T_n^\nu(y) = y$

② If $y \in \mathbb{F}_2 \setminus \mathbb{F}_2^\nu \Rightarrow t \in \mathbb{F}_2$ starting and ending with $v_\nu^{\pm 1}$

$\Rightarrow y = u_\nu^k \, t \, u_\nu^l$, $k, l \in \mathbb{Z}$

set $u_i := u_\nu^i \in B(\ell^2(\mathbb{F}_2))$

③ Apply Lemma 1

$\| T_n^\nu(y) \| = \| u_\nu^k T_n^\nu(t) u_\nu^l \|$

$= \| T_n^\nu(t) \|$

$\underset{\text{lemma 1}}{\leq} \frac{2}{\sqrt{n}} \|t\| = \frac{2}{\sqrt{n}} \|y\|$

$\longrightarrow 0$

$(n \to \infty)$