$$T_n^\nu(y) = \frac{1}{n}\sum_{i=1}^{n} u_\nu^i y u_\nu^{-i}$$

$$T_n^0(y) = \frac{1}{n}\sum_{i=1}^{n} u_0^i y u_0^{-i} \qquad T_n^1(y) = \frac{1}{n}\sum_{i=1}^{n} u_1^i y u_1^{-i}$$

Lemma. *For $\nu = 0, 1$ let $\mathbb{F}_2^\nu \subseteq \mathbb{F}_2$ be the set of all reduced words in $u_\nu^{\pm 1}$. With the notation as above we have*

$$\lim_{n\to\infty} T_n^\nu(y) = \begin{cases} y & \text{if } y \in \mathbb{F}_2^\nu \\ 0 & \text{else} \end{cases}$$

Recall that canonical trace is defined as follows: $\tau : C_r^*(G) \to \mathbb{C}$

$$\tau(x) = \langle \delta_e, x\delta_e \rangle .$$

If $x = \sum_{t\in G} \alpha_t \delta_t \in \mathbb{C}[G] \subseteq C_r^*(G)$ then we get

$$\tau(x) = \sum_{t\in G} \alpha_t \tau(\delta_t) = \alpha_e,$$

Given $n \in \mathbb{N}$ we define a linear map $T_n : C_r^*(\mathbb{F}_2) \to C_r^*(\mathbb{F}_2)$ by

$$T_n(y) := T_n^1 T_n^0(y) = \frac{1}{n^2}\sum_{i,j=1}^{n} u_1^i u_0^j y u_0^{-j} u_1^{-i}.$$

contraction

$y \in I \Rightarrow T_n(y) \in I$

δ_e

Lemma. *Let $x \in C_r^*(\mathbb{F}_2)$ be arbitrary. With the notation as above we have*

$$T_n(x) \to \tau(x)1 =: T(x)$$

in norm as $n \to \infty$. That is, $\lim_{n\to\infty} T_n = T$ in the point norm topology.

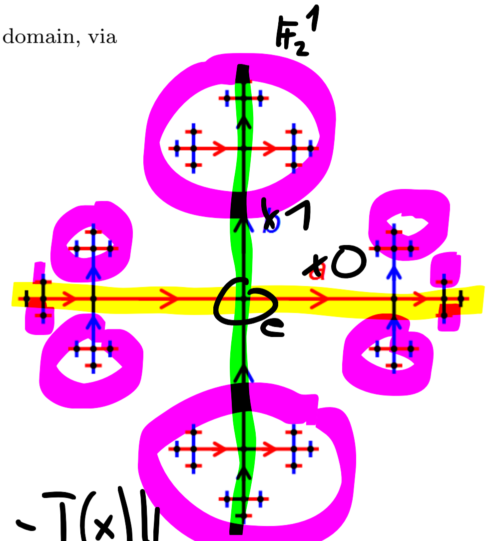

FIGURE 1. Public Domain, Jim.belk at English Wikipedia, Public domain, via Wikimedia Commons

$x \in \mathbb{C}[\mathbb{F}_2] \qquad x = x_0 + x_1$

$$x_0 = \sum_{t\in\mathbb{F}_2^1} \alpha_t \delta_t = \sum_{t\in\mathbb{F}_2^1\setminus\{e\}} \alpha_t\delta_t + \underbrace{\alpha_e \delta_e}_{\tau(x)\cdot 1} \qquad \mathbb{F}_2^0$$

$$x_1 = \sum_{t\in\mathbb{F}_2\setminus\mathbb{F}_2^1} \alpha_t \delta_t$$

$$\|T_n(x) - T(x)\| = \|T_n^1(T_n^0(x) - x_0 + x_0) - T(x)\|$$

$$\leq \|T_n^1(\underbrace{T_n^0(x) - x_0})\| + \|T_n^1(x_0) - T(x)\|$$

T_n^1 contracting

$T_n^0(x) \to x_0 \qquad T_n^1(x_0) \to \alpha_e\delta_e = \tau(x)1 = T(x)$

$T_n^0(x) - x_0 \to 0$

$x \in C_r^*[\mathbb{F}_2] \qquad x' \in \mathbb{C}[\mathbb{F}_2]$ close to x

$$\|T_n(x) - T(x)\| = \|T_n(x) - T_n(x')\| + \|T_n(x') - T(x')\| + \|T(x') - T(x)\|$$

Lemma (Lemma 2.6 from ISEM notes). *Let A be a unital Banach algebra. If $x \in A$ with $\|1-x\| < 1$, then x is invertible and $x^{-1} = \sum_{n=0}^{\infty}(1-x)^n$.*

Proposition (From Joseph's part). *For every group G the canonical trace $\tau : C^*_{\mathrm{r}}(G) \to \mathbb{C}$ is faithful.*

Theorem. *The reduced group C^* -algebra $C^*_{\mathrm{r}}(\mathbb{F}_2)$ of the free group on two generators is simple.*

$x \in C^*_r(\mathbb{F}_2) \quad x \neq 0 \qquad I_x = C^*_r(\mathbb{F}_2)$

$\tau(x^*x) \neq 0$

$T_n(x^*x) \longrightarrow \tau(x^*x)\,1 \in I_x$

Corollary. *The canonical map $C^*_{\mathrm{f}}(\mathbb{F}_2) \to C^*_{\mathrm{r}}(\mathbb{F}_2)$ is not an isomorphism.*

not simple   simple

$\varepsilon : C^*_f(\mathbb{F}_2) \longrightarrow \mathbb{C}$

$\delta_{w_1} \longmapsto 1$

$\varepsilon(w_1 - w_2) = 0$