C^*-simplicity and the Furstenberg boundary

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Goal: dynamical characterization of C^*-simplicity

Recall: for G locally compact group

G C^*-simple $\underset{\text{def}}{\Longleftrightarrow}$ $C^*_r(G)$ simple C^*-algebra.

Preliminary 1: the Furstenberg boundary

Def: G discrete group, X compact G-space (with action $\sigma : G \to \mathrm{Homeo}(X)$)

- the action is minimal $\overset{\text{def}}{\iff}$ $\forall x \in X$ $\overline{G.x} = X$
- " " is proximal $\overset{\text{def}}{\iff}$ $\forall x,y \in X$, $\exists (t_i)_{i\in I}$ net in G s.t. $\lim t_i \cdot x = \lim t_i \cdot y$
- " " is strongly proximal $\overset{\text{def}}{\iff}$ $\left[(\sigma\#) : G \to \mathrm{Homeo}(P(X))\right.$ is a proximal action $\left. G \curvearrowright P(X)\right]$

where: $\cdot\, P(X) := \{\text{probability measures on } X\}$ with weak$*$ topology

$\cdot\, [(\sigma\#)(g)](\mu) := \sigma(g)\#\mu$ (pushforward measure $(\sigma(g)\#\mu)(E) = \mu(\sigma(g)^{-1}E)$)

Preliminary 1: the Furstenberg boundary

Def: X is G-boundary $\underset{\text{def}}{\Longleftrightarrow}$ X is $\begin{cases} \text{minimal} \\ \text{strongly proximal} \end{cases}$ G-space

we <u>admit</u> the following:

<u>**Theorem and definition**</u> for any discrete group G,

- $\exists$ (G-boundary) $\partial_F G$ s.t.: $\forall X$ G-boundary $\exists \varphi : \partial_F G \to X$ G-equivariant surjective continuous
- such G-boundary is unique up to G-equivariant homeomorphism
- $\partial_F G$: the Furstenberg boundary of G

Main theorem:

Reminder: $G \curvearrowright X$ free $\overset{\text{def}}{\Longleftrightarrow}$ $\forall g \in G \setminus \{e\}$: $\{x \in X \text{ s.t. } g \cdot x = x\} = \emptyset$

$G \curvearrowright X$ top. free $\overset{\text{def}}{\Longleftrightarrow}$ $\forall g \in G \setminus \{e\}$: $\mathrm{Int}(\{x \in X \text{ s.t. } g \cdot x = x\}) = \emptyset$

Theorem (Kalantar, Kennedy, 2014)

G discrete countable group. TFAE:

(1) G C^*-simple

(2) $G \curvearrowright \partial_F G$ is free

(3) $\exists X$ G-boundary s.t. $G \curvearrowright X$ is topologically free.

this theorem answers our question.

we outline the proof of (2) $\Rightarrow$ (1),

following Breuillard-Kalantar-Kennedy-Ozawa (2014)

H. Furstenberg Boundary theory and stochastic processes ... (1973)

Preliminary 2: Some properties of $\partial_F G$:

Proposition F.1: $\partial_F G$ is extremally disconnected.

Proposition F.2: (rigidity):

- $Id : C(\partial_F G) \to C(\partial_F G)$ is the only G-equivariant unital positive linear map $C(\partial_F G) \to C(\partial_F G)$
- If A G-C^* algebra, $\varphi : C(\partial_F G) \to A$ G-equivariant unital positive linear

  then φ $\boxed{\text{isometric}}$

Proposition F.3: D unital G-C^*-algebra,

$A \subset B$ closed unital selfadjoint G-invariant subspaces of D.

$\varphi : A \to C(\partial_F G)$ G equivariant unital positive

$\Rightarrow \exists \psi : B \to C(\partial_F G)$ G-equivariant unital positive st. $\psi|_A = \varphi$

Preliminary 3: Completely positive maps:

Def: $\varphi : A \to B$ linear, A, B C^*algebras

φ completely positive $\overset{\text{def}}{\iff}$ $\forall n \geq 0$ $\varphi : M_n(A) \to M_n(B)$ positive

$\iff \forall n$ $\forall (a_i) \in A^n \forall (b_i) \in B^n$: $\sum_{i,j} b_i^* \varphi(a_i^* a_j) b_j \geq 0$

Def and proposition φ u.c.p. (unital completely positive)

$$D_\varphi := \left\{ a \in A \ ; \ \begin{array}{l} \varphi(a^*a) = \varphi(a)^*\varphi(a) \\ \varphi(a a^*) = \varphi(a)\varphi(a)^* \end{array} \right\} \text{ multiplicative domain}$$

then $\forall a \in D_\varphi \forall b \in A$: $\varphi(ba) = \varphi(b)\varphi(a)$

$\varphi(ab) = \varphi(a)\varphi(b)$

(how to prove? : use Stinespring theorem: if $\varphi : A \to B \hookrightarrow B(H)$ u.c.p

then $\exists V$ isometry, $\exists \pi$ $*$rep s.t. $\varphi(\cdot) = V^*\pi(\cdot)V$

then $a \in D_\varphi \Rightarrow \varphi(a)V^* = V^*\pi(a)$)

Theorem (Arveson): A unital C^*algebra, E operator subsystem
$\forall \varphi : E \to B(H)$ u.c.p, $\exists \tilde{\varphi} : A \to B(H)$ u.c.p extension

(how to prove? take E_λ finite dimensional projections $E_\lambda \xrightarrow[\lambda]{SOT} Id_H$
then $E_\lambda \varphi E_\lambda : E \to M_{n_\lambda}(\mathbb{C})$ gives $\psi_\lambda : M_{n_\lambda}(E) \to \mathbb{C}$ positive unital
extend ψ_λ to $\overline{\Psi}_\lambda : M_{n_\lambda}(A) \to \mathbb{C}$ unital contractive ($\rightsquigarrow$ positive)
we get $\overline{\varphi}_\lambda : A \to M_{n_\lambda}(\mathbb{C}) \hookrightarrow B(H)$ u.c.p, take a limit $\overline{\varphi}$)

Proposition X compact, $\varphi : A \to C(X)$ unital positive $\Rightarrow$ u.c.p

Outline of the proof (2) ⇒ (1)

suppose $G \curvearrowright \partial_F G$ free, let us show $\left(\begin{array}{l} \pi : C^*_r(G) \to B(H) \text{ unital } *\text{hom} \\ \Rightarrow \quad \pi \text{ injective} \end{array} \right)$

G acts on $C(\partial_F G)$ by $(t \cdot f)(\cdot) = f(t^{-1} \cdot)$, $t \in G$, $f \in C(\partial_F G)$

thus we can take the reduced crossed product $C(\partial_F G) \rtimes_r G$,

(fix $\rho : C(\partial_F G) \hookrightarrow B(H_0)$ $*$ rep.)

we have $C^*_r(G) \hookrightarrow C(\partial_F G) \rtimes_r G$ ($*$-hom)

$\lambda_t \to \bar{\lambda}_t$

(where $\bar{\lambda}_t \in B(\ell^2(G, H_0))$ is $(\bar{\lambda}_t \xi)(g) = \xi(t^{-1} g)$, for $\xi \in \ell^2(G, H_0)$)

by Arveson's theorem, extend π to

- $\phi : C(\partial_F G) \rtimes_r G \to B(H)$ u.c.p

$C_r^*(G) \subset D_\phi$

thus $\forall a \in C(\partial_F G) \rtimes_r G \ \forall y, x \in G$; $\phi(\bar{\lambda}_x a \bar{\lambda}_y) = \phi(\bar{\lambda}_x)\, \phi(a)\, \phi(\bar{\lambda}_y)$

$$(*) \quad \phi(\lambda_x a \lambda_y) = \pi(\lambda_x)\, \phi(a)\, \pi(\lambda_y)$$

we consider . $G \curvearrowright B(H)$ $\qquad x.T = \pi(\lambda_x) A \pi(\lambda_x^*)$

. $G \curvearrowright C(\partial_F G) \rtimes_r G$ $\qquad x.a = \bar{\lambda}_x a \bar{\lambda}_x^*$

(extends $G \curvearrowright C(\partial_F G)$)

then $\phi : C(\partial_F G) \rtimes_r G \to B(H)$ is G-equivariant by $(*)$

Fact : ϕ is faithful (i.e. $\phi(a^* a) = 0 \Rightarrow a = 0$)

(then π is injective)

proof of Fact:

$\phi : C(\partial_F G) \rtimes_r G \to B(H)$, $C(\partial_F G) \hookrightarrow C(\partial_F G) \rtimes_r G$

$f \to P(f)\lambda_e$

then $\phi|_{C(\partial_F G)}$ is G-equivariant unital positive

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- by F.2 $\phi' = \phi|_{C(\partial_F G)}^{\phi(C(\partial_F G))}$ isometric

- by F.3: $(\phi')^{-1}$ can be extended

$\tau : \mathrm{Im}\,\phi \to C(\partial_F G)$ G-equivariant unital positive.

Recall: we have a faithful expectation $E : C(\partial_F G) \rtimes_r G \to C(\partial_F G)$

$E(f\lambda_s) = 0$ for $s \neq e$, $E(f\lambda_e) = f$

we show that $\tau \circ \phi = E$

(this would imply ϕ faithful (and the theorem))

denote $\psi := \tau \circ \phi$, ψ is u.c.p.

- since $\tau|_{C(\partial_F G)} = (\phi')^{-1}$,

$$\psi(f) = f \quad \forall f \in C(\partial_F G)$$

in particular, $C(\partial_F G) \subset D_\psi$, thus

$$\forall f_1, f_2 \in C(\partial_F G), \quad \psi(f_1 \lambda_s f_2) = f_1 \psi(\lambda_s) f_2.$$

thus

- if $s \in G \setminus \{e\}$, $x \in \partial_F G$, $s \cdot x \neq x$

let $f = \begin{cases} 1 & \text{on } x \\ 0 & \text{on } s^{-1} \cdot x \end{cases}$

then $\psi(\lambda_s) f = \psi(\lambda_s f) = \psi((s \cdot f) \lambda_s) = (s \cdot f) \psi(\lambda_s)$

evaluating at x:

$$[\psi(\lambda_s)](x) = [\psi(\lambda_s)](x) f(x) = f(s^{-1} \cdot x) \psi(\lambda_s)(x) = 0$$

thus $\forall s \in G \setminus \{e\} \quad \psi(\lambda_s) = 0$

and $\forall f \in C(\partial_F G) \; \forall s \in G \setminus \{e\} : \psi(f\lambda_s) = f\psi(\lambda_s) = 0 = E(f\lambda_s)$

thus $\psi = E$

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Returning to $\mathbb{F}_2$

we construct a topologically free boundary ;

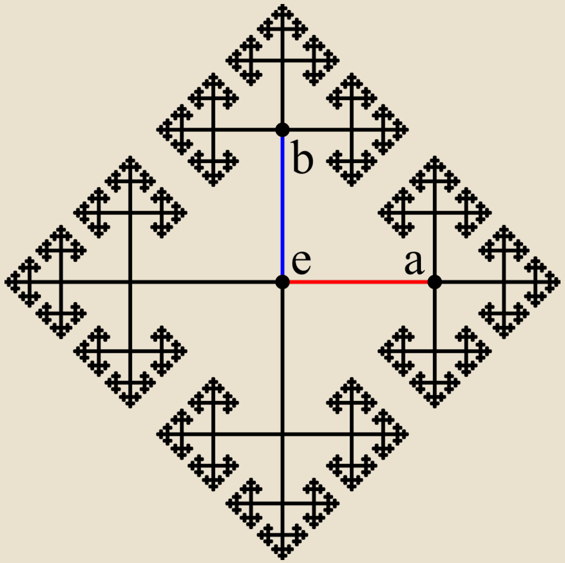
let $Y = \{a, a^{-1}, b, b^{-1}\}^{\mathbb{N}}$

$X = \{(w_n) \in Y \text{ s.t. } w_{n+1} \neq w_n^{-1} \ \forall n\}$

X : infinite paths of the Cayley graph

(boundary)

$\mathbb{F}_2 \curvearrowright X$ by "appending" and "reducing"



public domain

for γ generator ; $\gamma(w_1, w_2 \dots) = \begin{cases} (\gamma, w_1, w_2 \dots\dots) & \gamma^{-1} \neq w_1 \\ \text{or} & \\ (w_2 \dots\dots) & \gamma^{-1} = w_1 \end{cases}$

then $\mathbb{F}_2 \curvearrowright X$ is a topologically free boundary

References

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