Project 8 - Tensor products of C*-algebras

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Overview

Introduction

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- Multilinear algebra–which initially appeared more complicated than linear algebra–is subsumed by linear algebra.
- Vectors in $H \otimes K$ are sums of $x \otimes y$ for $x \in H$, $y \in K$.

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1. $\chi_{(x_1+x_2,y)} - \chi_{(x_1,y)} - \chi_{(x_2,y)}$
2. $\chi_{(x,y_1+y_2)} - \chi_{(x,y_1)} - \chi_{(x,y_2)}$,
3. $\lambda\chi_{(x,y)} - \chi_{(\lambda x,y)}$ and
4. $\lambda\chi_{(x,y)} - \chi_{(x,\lambda y)}$.

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3. $\lambda\chi_{(x,y)} - \chi_{(\lambda x,y)}$ and
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The image of the characteristic function over the point $(x, y) \in H \times K$, (an element) $\chi(x, y) \in C_c(H \times K)$, under the canonical quotient map $C_c(H \times K) \longrightarrow H \otimes K$ is called an *elementary tensor* and is denoted $x \otimes y$.

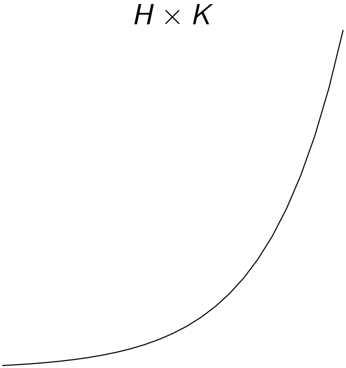
Universality

The other crucial fact about tensor products is their universal property; they are designed to turn bilinear maps $H \times K \longrightarrow L$ into linear maps $H \otimes K \longrightarrow L$. Moreover, $H \otimes K$ is the unique vector space (up to isomorphism), with this property. Before making this precise first note that the natural mapping

$$H \times K \longrightarrow H \otimes K, \ (x, y) \longmapsto x \otimes y$$

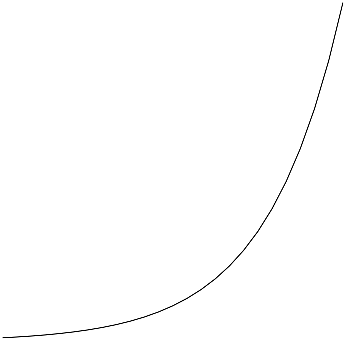
is not linear $-$ it is bilinear.

Geometrically

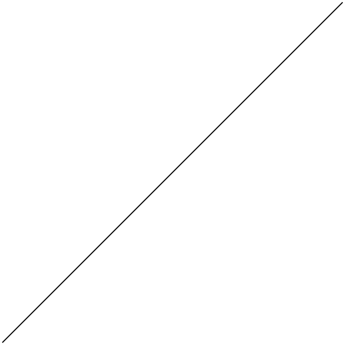


Geometrically

$H \times K$



$H \otimes K$



Universality

Proposition

For any vector space L and any bilinear map $\sigma : H \times K \longrightarrow L$, there exists a unique linear map $\sigma' : H \otimes K \longrightarrow L$ such that

$$\begin{tikzcd} H \times K \arrow[rr, "\sigma"] \arrow[dr, "i"'] & & L \\ & H \otimes K \arrow[ur, dotted, "\sigma'"'] & \end{tikzcd}$$

commutes (i.e. $\sigma'(x \otimes y) = \sigma((x, y))$ for all $x \in H$, $y \in K$).

Calculating with Tensors

Proposition

The following identities hold for all vectors and scalars:

1. $(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y$ *and* $x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2$.
2. $\lambda(x \otimes y) = (\lambda x) \otimes y = x \otimes (\lambda y)$

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2. $\lambda(x \otimes y) = (\lambda x) \otimes y = x \otimes (\lambda y)$

This is rigged so that
a linear map $H \otimes K \longrightarrow L$ is the same as a bilinear map $H \times K \longrightarrow L$

Calculating with Tensors

Example

- $\mathbb{R}^2 = \langle e_1, e_2 \rangle$.
- $\mathbb{R}^2 \otimes \mathbb{R}^2$ is spanned by $e_1 \otimes e_1,\ e_1 \otimes e_2,\ e_2 \otimes e_1,\ e_2 \otimes e_2$

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- Observe how these add:

$$(e_1 \otimes e_1) + (e_1 \otimes e_2) = e_1 \otimes (e_1 + e_2)$$

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- Observe how these add:

$$(e_1 \otimes e_1) + (e_1 \otimes e_2) = e_1 \otimes (e_1 + e_2)$$

Warning:

You cannot combine $(e_1 \otimes e_1) + (e_2 \otimes e_2)$.

Calculating with Tensors

Remark

Note that the vector space structures on $H \otimes K$ and $H \times K$ are completely different. For example, in $H \times K$ we have $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ while there is no way to "simplify" $x_1 \otimes y_1 + x_2 \otimes y_2$ (in general).

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Remark

In many proofs involving tensor products it will suffice to consider only elementary tensors. But this is because they form a spanning set for $H \otimes K$ and one must not forget that $H \otimes K$ contains a lot more than just the elementary tensors.

Tensor Product vs Cartesian Product

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Different than the cartesian product of vector spaces,

$$dim(H \times K) = (dimH) + (dimK)$$

Tensor Product Maps

Proposition

If $u : H \longrightarrow H'$ *and* $v : K \longrightarrow K'$ *are linear maps between vector spaces, then by elementary linear algebra there exists a unique linear map:*

$$u \otimes v : H \otimes K \longrightarrow H' \otimes K'$$

such that $(u \otimes v)(x \otimes y) = u(x) \otimes v(y)$ *for all* $x \in H$ *and all* $y \in K$.

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Remark

The map $(u, v) \mapsto u \otimes v$ is bilinear.

Corollary (Tensor product functionals)

If τ, ρ are linear functionals on the vector spaces H, K respectively, then there is a unique linear functional $\tau \otimes \rho$ on $H \otimes K$ such that

$$(\tau \otimes \rho)(x \otimes y) = \tau(x)\rho(y) \qquad (x \in H,\ y \in K)$$

since the function $H \times K \longrightarrow \mathbb{C}, \quad (x, y) \mapsto \tau(x)\rho(y)$, is bilinear.

Linear Independence

Question: How can one show that a set of elementary tensors is linearly independent ?

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Proposition

If $\{x_1, \cdots, x_n\} \subset H$ *are linearly independent,* $\{y_1, \cdots, y_n\} \subset K$ *are arbitrary and*

$$0 = \sum_{i=1}^{n} x_i \otimes y_i \in H \otimes K$$

then $y_1 = y_2 = \cdots = 0$.

Sketch of proof:

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- $\{\tau_1, \cdots, \tau_n\} \subset H^*$ a dual set of functionals (i.e. $\tau_j(x_i) = \delta_{i,j}$) and $\rho \in K^*$.

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- $\{\tau_1, \cdots, \tau_n\} \subset H^*$ a dual set of functionals (i.e. $\tau_j(x_i) = \delta_{i,j}$) and $\rho \in K^*$.
- For $1 \leq j \leq n$

$$
\begin{aligned}
0 &= \tau_j \otimes \rho\Big(\sum_{i=1}^{n} x_i \otimes y_i\Big) \\
0 &= \rho(y_j).
\end{aligned}
$$

$\square$

There are many norms on $H \otimes K$

Remark

- If H and K are normed, then there are in general many possible norms on $H \otimes K$ which are related in a suitable manner to the norms on H and K, and indeed it is this very lack of uniqueness that creates the difficulties of the theory, as we shall see in the case that H and K are C^*-algebras.
- When the spaces are Hilbert spaces, however, matters are simple.

Tensor Product of Hilbert Spaces

Theorem

Let H and K be Hilbert spaces. Then there is a unique inner product $< \cdot, \cdot >$ on $H \otimes K$ such that

$$< x \otimes y, x' \otimes y' > = < x, x' >< y, y' > \qquad (x, x' \in H, \quad y, y' \in K).$$

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- $x \in H$, let τ_x be the conjugate-linear functional defined by setting $\tau_x(y) =< x, y >$.
- X be the vector space of all conjugate-linear functionals on $H \otimes K$. The map

$$H \times K \longrightarrow X, \quad (x, y) \mapsto \tau_x \otimes \tau_y,$$

is bilinear.

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- there is a unique linear map $M : H \otimes K \longrightarrow X$ such that $M(x \otimes y) = \tau_x \otimes \tau_y$ for all x and y.

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is bilinear.
- there is a unique linear map $M : H \otimes K \longrightarrow X$ such that $M(x \otimes y) = \tau_x \otimes \tau_y$ for all x and y.
- The map $< \cdot, \cdot >: (H \otimes K)^2 \longrightarrow \mathbb{C}, \quad (z, z') \mapsto M(z)(z')$, is a sesquilinear form on $H \otimes K$ such that
$$< x \otimes y, x' \otimes y' > = < x, x' >< y, y' > \qquad (x, x' \in H,\ y, y' \in K).$$

$\square$

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- If $z \in H \otimes K$, then $z = \sum_{j=1}^{n} x_j \otimes y_j$ for some $x_1, \cdots, x_n \in H$ and $y_1, \cdots, y_n \in K$.

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- Let $e_1, \cdots, e_m$ be an orthonormal basis for linear span of $y_1, \cdots, y_n$. Then $z = \sum_{j=1}^{n} x_j' \otimes e_j$ for some $x_1', \cdots, x_m' \in H$

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- Let $e_1, \cdots, e_m$ be an orthonormal basis for linear span of $y_1, \cdots, y_n$. Then $z = \sum_{j=1}^{n} x'_j \otimes e_j$ for some $x'_1, \cdots, x'_m \in H$
- $$\begin{aligned} < z, z > &= \sum_{i,j=1}^{m} < x'_i \otimes e_i, x'_j \otimes e_j > \\ &= \sum_{i,j=1}^{m} < x'_i, x'_j >< e_i, e_j > \\ &= \sum_{j=1}^{m} \|x'_j\|^2. \end{aligned}$$

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- Let $e_1, \cdots, e_m$ be an orthonormal basis for linear span of $y_1, \cdots, y_n$. Then $z = \sum_{j=1}^{n} x_j' \otimes e_j$ for some $x_1', \cdots, x_m' \in H$
- $$\begin{aligned} < z, z > &= \sum_{i,j=1}^{m} < x_i' \otimes e_i, x_j' \otimes e_j > \\ &= \sum_{i,j=1}^{m} < x_i', x_j' >< e_i, e_j > \\ &= \sum_{j=1}^{m} \|x_j'\|^2. \end{aligned}$$
- $< \cdot, \cdot >$ is an inner product.

$\square$

Remark

If H and K are as in the previous Theorem, we shall always regard $H \otimes K$ as a pre-Hilbert space with the above inner product. The Hilbert space completion of $H \otimes K$ is denoted by $H \hat{\otimes} K$, and called the *Hilbert space tensor product* of H and K. Note that

$$\|x \otimes y\| = \|x\| \|y\|.$$

Tensor Product of Operators

Theorem

Let H, K be Hilbert spaces, $u \in B(H)$ and $v \in B(K)$. Then there is a unique operator $u \hat{\otimes} v \in B(H \hat{\otimes} K)$ with

$$(u \hat{\otimes} v)(x \otimes y) = u(x) \otimes v(y) \text{ and } \|u \hat{\otimes} v\| = \|u\| \|v\|.$$

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Proof.

- $z \in H \otimes K$, $z = \sum_{j=1}^{n} x_j \otimes y_j$ with pairwise orthogonal $y_1, \ldots, y_n$
- $\|(u \otimes v)(z)\|^2 = \| \sum_{j=1}^{n} u(x_j) \otimes v(y_j)\|^2 = \sum_{j=1}^{n} \|u(x_j) \otimes v(y_j)\|^2$
  $= \sum_{j=1}^{n} \|u(x_j)\|^2 \|v(y_j)\|^2 = \sum_{j=1}^{n} \|x_j\|^2 \|y_j\|^2 = \|z\|^2$
- $\|u \otimes v\| = 1$
- $u \hat{\otimes} v$ extension of $u \otimes v$

Tensor Product of Operators

to show:

$\|u\hat{\otimes}v\| = \|u\|\|v\|$

Proof.

"$\leq$":

- $B(H) \to B(H\hat{\otimes}K), u \mapsto u\hat{\otimes}id_K$ and $B(K) \to B(H\hat{\otimes}K), v \mapsto id_H\hat{\otimes}v$ injective *-homomorphisms $\Longrightarrow$ isometric
- $\|u\hat{\otimes}v\| = \|(u\hat{\otimes}id_K)(id_H\hat{\otimes}v)\| \leq \|u\hat{\otimes}id_K\|\|id_H\hat{\otimes}v\| = \|u\|\|v\|$

"$\geq$":

- $u, v \neq 0$ and $0 < \varepsilon < \min(\|u\|, \|v\|)$
- unit vectors x, y with $\|u(x)\| > \|u\| - \varepsilon > 0$ and $\|v(y)\| > \|v\| - \varepsilon > 0$
- $\|(u\hat{\otimes}v)(x\otimes y)\| = \|u(x)\otimes v(y)\| = \|u(x)\|\|v(y)\| > (\|u\|-\varepsilon)(\|v\|-\varepsilon)$

Involution and Multiplication on Tensor Products

Remark

For $u, u' \in B(H)$ and $v, v' \in B(K)$ we have

$$(u \hat{\otimes} v)(u' \hat{\otimes} v') = uu' \hat{\otimes} vv'$$

and

$$(u \hat{\otimes} v)^* = u^* \hat{\otimes} v^*.$$

Involution and Multiplication on Tensor Products

Theorem

If A and B are algebras, there is a unique multiplication on $A \otimes B$ such that

$$(a \otimes b)(a' \otimes b') = aa' \otimes bb'$$

for all $a, a' \in A$ and $b, b' \in B$. We call $A \otimes B$ endowed with this multiplication the algebra tensor product of the algebras A and B.

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Proof.

- $L_a(x) := ax$ for $a, x \in A$, analogously L_b for $b \in B$
- bilinear map $A \times B \to X, \quad (a, b) \mapsto L_a \otimes L_b,$
- unique linear map $M : A \otimes B \to X, \quad a \otimes b \mapsto L_a \otimes L_b$
- $(A \otimes B)^2 \to A \otimes B, \quad (c, d) \mapsto cd := M(c)(d)$
  unique multiplication on $A \otimes B$

Involution and Multiplication on Tensor Products

Theorem

*If A and B are *-algebras, then there is a unique involution on $A \otimes B$ such that $(a \otimes b)^* = a^* \otimes b^*$ for all $a \in A$ and $b \in B$. We call $A \otimes B$ with this involution the *-algebra tensor product of A and B.*

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Proof.

- to show: $\sum_{j=1}^n a_j \otimes b_j = 0 \Longrightarrow \sum_{j=1}^n a_j^* \otimes b_j^* = 0$
- $b_j = \sum_{i=1}^m \lambda_{ij} c_i$ for linearly independent $c_1, \ldots, c_m$
- $\sum_{i,j} \lambda_{ij} a_j \otimes c_i = 0 \Longrightarrow \sum_{j=1}^n \lambda_{ij} a_j = 0 \ (i = 1, \ldots, m)$
- $\sum_{j=1}^n a_j^* \otimes b_j^* = \sum_{i,j} a_j^* \otimes \overline{\lambda_{ij}} c_i^* = \sum_{i=1}^m (\sum_{j=1}^n \overline{\lambda_{ij}} a_j^*) \otimes c_i^*$

$$= \sum_{i=1}^m 0 \otimes c_i^* = 0$$

Involution and Multiplication on Tensor Products

Remark

Let

- A, B, C *-algebras
- $\varphi : A \to C$, $\psi : B \to C$ *-homomorphisms.

Then

$$A \times B \to C, \quad (a, b) \mapsto \varphi(a)\psi(b)$$

is bilinear and so induces a unique linear map

$$\pi : A \otimes B \to C \text{ with } \pi(a \otimes b) = \varphi(a)\psi(b).$$

If the elements of $\varphi(A)$ and $\psi(B)$ commute this map is also a *-homomorphism.

C^*-Tensor Products

Construction of C^*-Tensor Products

Idea:

- Consider two C^*-algebras A and B.
- Find a C^*-norm γ on the $*$-algebra $A \otimes B$.
- Complete $A \otimes B$ with respect to γ to obtain a C^*-algebra $A \otimes_\gamma B$.

Note:

- There can be multiple γ with different completions $A \otimes_\gamma B$.
- Most important are the spatial C^*-norm $\|\cdot\|_*$ and the maximal C^*-norm $\|\cdot\|_{\max}$.

Comparison - Crossed Products

Recall:

- Let (A, G, α) be a C^*-dynamical system. Then the convolution algebra $C_c(G, A, \alpha)$ is a $*$-algebra.
- $C_c(G, A, \alpha)$ can be equipped with the reduced and full crossed product norms.
- Completions with respect to these norms result in the reduced and full crossed product $A \rtimes_{a,r} G$ and $A \rtimes_{a,f} G$.

$\implies$ Tensor products of C^*-algebras are constructed similarly.

Spatial C^*-Norm Construction

We begin with the spatial C^*-norm $\|\cdot\|_*$.

Idea:

- Find a faithful representation $\pi\colon A \otimes B \to B(H)$ for some Hilbert space H.
- Define $\|c\|_* := \|\pi(c)\|$ for $c \in A \otimes B$.
- Then $\|\cdot\|_*$ is a C^*-norm on $A \otimes B$, since the representation is faithful.

Representations of $A \otimes B$

To obtain a faithful representation of $A \otimes B$ we need the following theorem.

Theorem

Suppose that (H, φ) and (K, ψ) are representations of the C^*-algebras A and B, respectively. Then there exists a unique $*$-homomorphism $\pi \colon A \otimes B \to B(H \hat{\otimes} K)$ such that

$$\pi(a \otimes b) = \varphi(a) \hat{\otimes} \psi(b) \qquad (a \in A, b \in B).$$

Moreover, if φ and ψ are injective, so is π.

The $*$-homomorphism π is also denoted by $\varphi \hat{\otimes} \psi$.

Representations of $A \otimes B$ - Proof

Proof:

Define the maps

$$\varphi' \colon A \to B(H \hat{\otimes} K), \qquad a \mapsto \varphi(a) \hat{\otimes} \mathrm{id}_K,$$
$$\psi' \colon B \to B(H \hat{\otimes} K), \qquad b \mapsto \mathrm{id}_H \hat{\otimes} \psi(b).$$

Then

- φ' and ψ' are $*$-homomorphisms.
- $\varphi'(a)$ and $\psi'(b)$ commute for all $a \in A, b \in B$.

By a previous result there exists a unique $*$-homomorphism π with

$$\pi(a \otimes b) = \varphi'(a)\psi'(b) = \varphi(a) \hat{\otimes} \psi(b) \qquad (a \in A, b \in B).$$

Representations of $A \otimes B$ - Proof

Assume φ and ψ are injective and let

$$c = \sum_{j=1}^{n} a_j \otimes b_j \in \ker \pi,$$

where $b_1, \ldots, b_n$ are linearly independent.

Then $\psi(b_1), \ldots, \psi(b_n)$ are linearly independent and

$$\pi(c) = \sum_{j=1}^{n} \varphi(a_j) \otimes \psi(b_j) = 0.$$

By a previous result we obtain $\varphi(a_1) = \cdots = \varphi(a_n) = 0$ such that $a_1 = \cdots = a_n = 0$ and $c = 0$. $\square$

Spatial Tensor Product

Definition (Spatial C^*-Norm)

Let A and B be C^*-algebras with universal representations (H, φ) and (K, ψ). Then

$$\|\cdot\|_* : A \otimes B \to \mathbb{R}^+, \quad c \mapsto \|(\varphi \hat{\otimes} \psi)(c)\|$$

is a C^*-norm on $A \otimes B$, called the *spatial C^*-norm*.

Definition (Spatial Tensor Product)

The completion of $A \otimes B$ with respect to $\|\cdot\|_*$ is called the *spatial tensor product* of A and B and is denoted by $A \otimes_* B$.

Spatial Tensor Product - Remarks

One can check directly that

- $\|a \otimes b\|_* = \|a\| \cdot \|b\|$ for all $a \in A$, $b \in B$.

With more work one can show:

- The spatial C^*-norm is independent of the faithful representation.
- The spatial C^*-norm is the minimal C^*-norm on $A \otimes B$:
  For every C^*-norm γ on $A \otimes B$ holds

$$\|c\|_* \leq \gamma(c) \quad (c \in A \otimes B).$$

Tensor Product Continuity

To construct another C^*-norm on $A \otimes B$ we need some preparations and consider general C^*-norms on $A \otimes B$.

Lemma

Let A, B be C^*-algebras and let γ be a C^*-norm on $A \otimes B$. Then for $a' \in A$ and $b' \in B$ the maps

$$\varphi\colon A \to A \otimes_\gamma B, \quad a \mapsto a \otimes b',$$
$$\psi\colon B \to A \otimes_\gamma B, \quad b \mapsto a' \otimes b$$

are continuous.

Tensor Product Continuity - Proof

Proof:

Consider $\varphi : A \to A \otimes_\gamma B$. Since A and $A \otimes_\gamma B$ are Banach spaces we can use the closed graph theorem to show that φ is continuous.

It remains to show: If a sequence (a_n) converges to 0 in A and $(a_n \otimes b')$ converges to c in $A \otimes_\gamma B$ then $c = 0$.

Further, we can assume a_n and b' are positive. Replace a_n by $a_n^* a_n$ and b' by $b'^* b'$ and observe

$$\begin{array}{rcl} (a_n) \to 0 & \Leftrightarrow & (a_n^* a_n) \to 0 \\ (a_n \otimes b') \to 0 & \Leftrightarrow & ((a_n \otimes b')^*(a_n \otimes b')) \to 0 \end{array}$$

where

$$(a_n \otimes b')^*(a_n \otimes b') = (a_n^* a_n) \otimes (b'^* b').$$

Tensor Product Continuity - Proof

Let τ be a positive linear functional on $A \otimes_\gamma B$ then

$$\rho : A \to \mathbb{C}, \quad a \mapsto \tau(a \otimes b')$$

is also a positive linear functional. It follows τ, ρ are continuous and

$$\tau(c) = \lim_{n\to\infty} \tau(a_n \otimes b') = \lim_{n\to\infty} \rho(a_n) = 0.$$

Since τ was arbitrary, we obtain $c = 0$.

Therefore, φ is continuous. A similar proof works for ψ. $\square$

Decomposition of Representation of Tensor Product

Theorem

Let A, B be non-zero C-algebras and suppose γ is the C*-norm on $A \otimes B$. Let (H, π) be a non-degenerate representation on $A \otimes_\gamma B$. Then there exists unique *-homomorphism $\varphi : A \to B(H)$ and $\psi : B \to B(H)$ such that*

$$\pi(a \otimes b) = \varphi(a)\psi(b) = \psi(b)\varphi(a).$$

Moreover, (H, φ) and (H, ψ) are non-degenerate.

Proof

<u>Proof</u>: Let $H_0 = \pi(A \otimes B)H$. For $z \in H_0$,it can be written as

$$z = \sum_{i=1}^{n} \pi(a_i \otimes b_i)x_i.$$

If $(v_\mu)_{\mu \in M}$ is an approximate unit in B, then

$$\pi(a \otimes v_\mu)z = \sum_{i=1}^{n} \pi(aa_i \otimes v_\mu b_i)x_i.$$

Using above lemma, the limit becomes

$$\lim_\mu \pi(a \otimes v_\mu)z = \sum_{i=1}^{n} \pi(aa_i \otimes b_i)x_i.$$

- Therefore we can construct a well-defined map $\varphi : H_0 \to H_0$ by $\varphi(a)z = \sum_{i=1}^{n} \pi(aa_i \otimes b_i)x_i$.
- φ is linear, by the previous lemma, $||\pi(a \otimes b)|| \leq M||b||$
- Since H_0 is dense, we can extend $\varphi(a)$ uniquely to a bounded linear map on H, denote it as $\varphi(a)$.

Proof (cont'd)

Let $(u_\lambda)_{\lambda\in\Lambda}$ be approximate unit of A,using the similar argument, we have $\lim_\lambda \pi(u_\lambda \otimes b)z = \sum_i^n \pi(a_i \otimes bb_i)x_i$. We can construct a well-defined linear map $\psi(b) : H_0 \to H_0$, which is bounded by previous lemma, and we can extend to H, denote it as $\psi(b)$.

- φ and ψ are *-homomorphism.
- Now suppose there exists $z \in H_0$ such that $\varphi(a)z = 0$, then $\pi(a \otimes b)z = 0$ for all $a \in A$, $b \in B$.
- by the non-degeneracy of (H, π), $z = 0$. So φ is non-degenerate.
- Similarly, ψ is non-degenerate.

For uniqueness, consider another pair of φ' and ψ' satisfy the equation. Using the notation above, $\varphi'(u_\lambda)$ converge to id_H strongly. Hence, $\pi(u_\lambda \otimes b)$ converge to $\psi(b)$ and $\psi'(b)$ strongly, so $\psi' = \psi$. Similarly, $\varphi' = \varphi$.

Maximal C*-Norm

Corollary

Let A and B be C-algebras and γ be C*-seminorm on $A \otimes B$, then*

$$\gamma(a \otimes b) \leq ||a||||b|| \text{ for } a \in A, b \in B$$

Proof: Let $\delta = max(\gamma, ||.||_*)$, if $\delta(c) = 0$ for $c \in A \otimes B$, since $||.||_*$ is a norm, $c = 0$. So δ is a C*-norm. Let (H, π) be a faithful and non-degenerate representation of $A \otimes_\delta B$, using theorem above, there exists π_A and π_B such that $\pi(a \otimes b) = \pi_A(a)\pi_B(b)$ for $a \in A, b \in B$. Then $\delta(a \otimes b) = ||\pi(a \otimes b)|| \leq ||\pi_A(a)||||\pi_B(b)|| \leq ||a||||b||$, so $\gamma(a \otimes b) \leq ||a||||b||$.

Maximal C*-Norm

We can define the maximal C*-norm. Let A, B be C*-algebras and Γ be the set of C*-norms on $A \otimes B$, we define the map $||.||_{max} : A \otimes B \to \mathbb{R}_+$ by $||c||_{max} = sup_{\gamma \in \Gamma} \gamma(c)$. This is called the maximal C*-norm.

Remark

The map is well-defined. The supremum exists because by the above corollary, $\gamma(\sum_{i=1}^{n} a_i \otimes b_i) \leq \sum_{i=1}^{n} ||a_i|| ||b_i||$ for $\gamma \in \Gamma$ and $a_i \in A, b_i \in B$.

Remark

Γ is non-empty, because $||.||_* \in \Gamma$

It follow from definition that maximal C*-norm is a C*-norm.

Maximal C*-Norm

We denote $A \otimes_{max} B$ be the completion of $A \otimes B$ under maximal C*-norm, called maximal tensor product. It has a useful property.

Theorem

Let A, B and C be C-algebras. Suppose $\varphi : A \to C$ and $\psi : B \to C$ are *-homomorphism such that $\varphi(A)$ commute with $\psi(B)$. Then there is a unique *-homomorphism $\pi : A \otimes_{max} B \to C$ such that*

$$\pi(a \otimes b) = \varphi(a)\psi(b) \ (a \in A, b \in B).$$

Maximal C*-Norm

Proof: By the remark in previous slide, there is a *-homomorphism $\pi : A \otimes B \to C$ satisfy the equation above. Since the function

$$\gamma : A \otimes B \to \mathbb{R}_+, \ c \to ||\pi(c)||$$

is a C*-seminorm, so $\gamma \leq ||.||_{max}$, so π is a norm-decreasing *-homomorphism on $A \otimes B$, so it extend to a norm-decreasing *-homomorphism on $A \otimes_{max} B$.
For uniqueness, two *-homomorphisms satisfy the equation must agree everywhere, so they must be the same *-homomorphism.

Nuclearity

Nuclear C*-Algebras

Definition

A C*-algebra A is called *nuclear* if, for all C*-algebras B, there is only one C*-norm on $A \otimes B$.

$\implies$ maximal C*-norm and spacial C*-norm on $A \otimes B$ coincide.

Remark

If a *-algebra C admits a complete C*-norm $\|\|$, then this is already the only C*-norm on C.

- γ different C*-norm on C
- $i : (C, \|\|) \hookrightarrow (\bar{C}, \gamma)$ injective *-hom

Example $M_n(\mathbb{C})$

Example

$M_n(\mathbb{C})$ is nuclear for each $n \in \mathbb{N}$.

Sketch of proof:

- arbitrary C*-algebra A
- Aim: $M_n(\mathbb{C}) \otimes A$ admits a complete C*-norm
- $\pi : M_n(\mathbb{C}) \otimes A \to M_n(A)$, $(\lambda_{ij})_{ij} \otimes a \mapsto (\lambda_{ij} a)_{ij}$ is a *-isomorphism
- $M_n(A)$ admits a complete C*-norm

Characterisation of finite-dimensional C*-Algebras

Theorem

Every non-zero finite-dimensional C-algebra is *-isomorphic to $M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$ for some integers $n_1, \ldots, n_k \in \mathbb{N}$.*

Sketch of proof: Let A be a non-zero finite-dimensional C*-algebra.

- if A is simple, then $A \cong M_n(\mathbb{C})$
- induction on the dimension of A
- suppose A is not simple
- $I \neq 0$ proper closed ideal of minimal dimension
- I simple $\implies I \cong M_{n_1}(\mathbb{C})$

Characterisation of finite-dimensional C*-Algebras

- I has a unit p
- $I = Ap$ and p commutes with all elements of A
- $A(1-p)$ is a C*-subalgebra of A

$$f : A \to Ap \oplus A(1-p), \qquad a \mapsto (ap, a(1-p))$$

- f is a *-isomorphism
- $Ap = I \cong M_{n_1}(\mathbb{C})$ and inductions hypothesis on $A(1-p)$
- $A \cong M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$

Characterisation of finite-dimensional C*-Algebras

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- $A \cong M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$

Nuclearity of finite-dimensional C*-Algebras

Theorem

Every finite-dimensional C-algebra is nuclear.*

Sketch of proof: Let A be a finite-dimensional C*-algebra.

- $A \cong M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$
- Aim: $A \otimes B$ admits a complete C*-norm
- $\pi : A \otimes B \to (M_{n_1}(\mathbb{C}) \otimes B) \oplus \cdots \oplus (M_{n_k}(\mathbb{C}) \otimes B)$,
  $(a_1, \ldots, a_k) \otimes b \mapsto (a_1 \otimes b, \ldots, a_k \otimes b)$
- π is a *-isomorphism
- $M_n(\mathbb{C})$ nuclear $\implies$ $M_{n_i}(\mathbb{C}) \otimes B$ admits a unique C*-norm

AF-Algebra

Definition

An AF-algebra A is a C^*-algebra that contains an increasing sequence $(A_n)_{n=1}^{\infty}$ of finite-dimensional C^*-subalgebras such that $\bigcup\limits_{n=1}^{\infty} A_n$ is dense in A.

Theorem

An AF-algebra is nuclear.

Density and Nuclearity

Theorem

Let S be a non-empty set of C^*-subalgebras of a C^*-algebra A which is upwards-directed (i.e, if $B, C \in S$, then there exists $D \in S$ such that $B \subset D$ and $C \subset D$). In addition, We suppose that, $\bigcup\limits_{D \in S} D$ is dense in A and all the algebras in S are nuclear, then A is nuclear too.

Sketch of proof:

- Back to the definition, let B be an arbitrary C^*-algebra and let β, γ be C^*-norms on $A \otimes B$.
- $C = \bigcup_{D \in S} D \otimes B$ is a C^*-subalgebra of $A \otimes B$ and is dense in $A \otimes B$.
- On $D \otimes B$, $\beta = \gamma$. So on C too.
- $\pi : A \otimes_\beta B \to A \otimes_\gamma B$ is a *-isomorphism and on C it is the identity.
- With the density, we can conclude.

Example

Example

Let H a Hilbert space, The C^*-algebra of all compact operators of $B(H)$ (denoted $K(H)$) is nuclear.

Sketch of proof:

- Let E be an orthogonal basis for H and let I be the set of all finite non-empty subsets of E (upwards-directed).
- For $i \in I$, let p_i be the projection on the span of all elements in i.
- We denote $A_i = p_i K(H) p_i$ (finite-dimensional) and $S = \{A_i | i \in I\}$.
- With density of finite-rank operators in $K(H)$, $\bigcup_{A_i \in S} A_i$ is dense in $K(H)$.

AF-Algebra

Definition

An AF-algebra A is a C^*-algebra that contains an increasing sequence $(A_n)_{n=1}^{\infty}$ of finite-dimensional C^*-subalgebras such that $\bigcup_{n=1}^{\infty} A_n$ is dense in A.

Theorem

An AF-algebra is nuclear.

Idea:

- A finite-dimensional C^*-algebra is nuclear.
- The Previous theorem on the link between density and nuclearity.

Example

Example

Let H a Hilbert space, The C^*-algebra of all compact operators of $B(H)$ is an AF-algebra.

References

References

Xin Li, Christian Voigt and Moritz Weber, *C*-algebras and dynamics*, ISEM24, 2021.

A. Guichardet, *Tensor products of C^*-algebras. Part I, Finite tensor products*, 1969.

G. Loupias, *Les C^*-algèbres en physique théorique*, Les rencontres physiciens-mathématiciens de Strasbourg - RCP25, 1968, tome 4, "Conférences de R. Stora et textes sur les C*-Algèbres de S. Doplicher, A. Guichardet,D. Kastler et G. Loupias ", , exp. no6, p. 1-19

References

- **Gilles Pisier**, *Tensor products of C^*-algebras and operator spaces: The Connes-Kirchberg problem*, February 22, 2021.
- **Gerard J. Murphy**, *C^*-algebras and operator theory*. Academic Press, Inc., Boston, MA, 1990.
- **Nathanial P. Brown and Narutaka Ozawa**, *C^*-algebras and finite-dimensional approximations*, volume 88 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2008.
- **Richard V. Kadison and John R. Ringrose**, *Fundamentals of the theory of operator algebras, Vol. II*, volume 16 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 1997. Advanced theory, Corrected reprint of the 1986 original.

References

- **Michael Reed and Barry Simon**, *Methods of modern mathematical physics. I.* AcademicPress, Inc. [Harcourt Brace Jovanovich, Publishers], New York, second edition, 1980. Func-tional analysis.
- **M. Takesaki**, *Theory of operator algebras. I.* volume 124 of Encyclopaedia of Mathematical Sciences. Springer-Verlag, Berlin, 2002. Reprint of the first (1979) edition, Operator Algebras and Non-commutative Geometry, 5.

Thank you for your attention !