Project 9
The Calkin Algebra

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Outline

Prologue!

Prologue: Spectral Theorem

In all following: $\mathcal{H}$ separable, infinite dimensional Hilbert Space.

Theorem (Spectral Theorem for compact s.a. Operators)

Let $K \in \mathcal{K}(\mathcal{H})$ be self-adjoint.
Then $0 \in \sigma(K)$ and $\sigma(K) \setminus \{0\}$ consists only of isolated eigenvalues of finite multiplicity.

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Question: Is this true for compact perturbations of general self-adjoint operators (instead of 0)?

Prologue: Weyl's Theorem

Theorem (Weyl, 1909)

Let $T \in \mathcal{B}(\mathcal{H})$ *self-adjoint and* $K \in \mathcal{K}(\mathcal{H})$ *self-adjoint. Then*

$$\tilde{\sigma}(T) = \tilde{\sigma}(T + K).$$

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Let $T_1, T_2 \in \mathcal{B}(\mathcal{H})$ self-adjoint. Then

$$T_1 = T_2 + K \text{ for a } K \in \mathcal{K}(\mathcal{H}) \implies \tilde{\sigma}(T_1) = \tilde{\sigma}(T_2).$$

Prologue: Von Neumann's complete Invariant

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- Does not hold without allowing change of basis, since $\tilde{\sigma}(U^*T_1U) = \tilde{\sigma}(T_1)$ for every unitary U.

Prologue: Von Neumann's complete Invariant

Definition (Unitary Equivalence up to Compacts)

For $T_1, T_2 \in \mathcal{B}(\mathcal{H})$, we write $T_1 \sim_{\mathcal{K}} T_2$ if there is unitary $U \in \mathcal{B}(\mathcal{H})$ and $K \in \mathcal{K}(\mathcal{H})$ s.t.

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We then call T_1 and T_2 unitarily equivalent up to a compact or essentially equivalent.

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- Such a theorem describes a "complete invariant" for self-adjoint operators up to $\sim_{\mathcal{K}}$.

Prologue: Berg's Generalisation

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Theorem (Weyl, von Neumann, Berg, 1971)

Let $T_1, T_2 \in \mathcal{B}(\mathcal{H})$ normal. Then

$$T_1 \sim_{\mathcal{K}} T_2 \iff \tilde{\sigma}(T_1) = \tilde{\sigma}(T_2).$$

- Such a theorem describes a "complete invariant" for normal operators up to $\sim_{\mathcal{K}}$.

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- the **spectrum** up to isolated eigenvalues of finite multiplicity.

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- $\mathcal{A}(\mathcal{H})$ is a unital C^*-algebra.
- $\mathcal{K}(\mathcal{H})$ is the only nontrivial closed ideal in $\mathcal{B}(\mathcal{H})$.

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Definition

For $T \in \mathcal{B}(\mathcal{H})$

$$\sigma_{\mathsf{e}}(T) \coloneqq \sigma(\pi(T))$$

is called the essential spectrum of T.

Weyl-von Neumann-Berg - revisit

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The Definition of Fredholm Operators

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$T \in \mathcal{B}(\mathcal{H})$ is Fredholm if $\pi(T)$ is invertible in $\mathcal{A}(\mathcal{H})$. The set of all Fredholm operators on $\mathcal{H}$ is denoted by $\mathcal{F}(\mathcal{H})$.

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For $T \in \mathcal{B}(\mathcal{H})$,
$\pi(T)$ is invertible in $\mathcal{A}(\mathcal{H}) \iff \dim \mathsf{N}(T),\ \dim \mathsf{N}(T^*) < \infty.$

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$S^*\colon (a_0, a_1, a_2, \cdots) \longmapsto (a_1, a_2, a_3, \cdots)$
Therefore $\mathsf{ind}(S) = -1$.

Properties of the Fredholm index (1)

Theorem

If $S, T \in \mathcal{F}(\mathcal{H})$, then we have

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Let $S \in \mathcal{B}(\ell^2)$ be the right shift operator, then we have

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Corollary

Let $(F_t)_{t\in[0,1]}$ a continuous path in the set of Fredholm operators, then we have $\text{ind}(F_0) = \text{ind}(F_1)$.
Especially, for $T \in \mathcal{F}(\mathcal{H})$ and $K \in \mathcal{K}(\mathcal{H})$ we have $\text{ind}(T) = \text{ind}(T + K)$.

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- $[0, 1] \longrightarrow \mathbb{Z},\ t \longmapsto \mathrm{ind}(F_t)$ is continuous.
- $(T + tK)_{t\in[0,1]}$ is a continuous path in $\mathcal{F}(\mathcal{H})$.

Outline

Weyl-von-Neumann-Berg Theorem as an Invariant

Theorem (Weyl, von Neumann, Berg, 1971)

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$$T_1 \sim_{\mathcal{K}} T_2 \iff \sigma_{\mathsf{e}}(T_1) = \sigma_{\mathsf{e}}(T_2).$$

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So class of essentially normal operators is more than just compact perturbations of normal operators!

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  ... but even this weaker property fails to hold.

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Suppose $\pi(T)\pi(S) = \pi(B)\pi(T)$ for $\pi(T) \in \mathcal{A}(\mathcal{H})$ invertible.

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- Weyl-von Neumann: All ess. normal operators with ess. spectrum Δ are essentially equivalent to T.

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Write $\mathrm{Ext}(T) \coloneqq (E, \Phi)$.

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Equivalence of extensions

Let $(E_1, \Phi_1), (E_2, \Phi_2)$ be extensions of $\mathcal{K}(\mathcal{H})$ by $C(X)$.

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- Let T_1, T_2 be essentially normal with $\sigma_{\mathsf{e}}(T_1) = \sigma_{\mathsf{e}}(T_2)$. Then $\mathrm{Ext}(T_1) \equiv \mathrm{Ext}(T_2) \iff T_1 \sim_{\mathcal{K}} T_2$.

Example – essential spectrum in a simple arc

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- Let N be normal with $\sigma_{\mathsf{e}}(N) = \Lambda$, $(E_2, \Phi_2) := \mathsf{Ext}(N)$.

Example – essential spectrum in a simple arc

We have seen $(E_1, \eta^* \circ \Phi_1) \equiv (E_2, \eta^* \circ \Phi_2)$.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K}(\mathcal{H}) \arrow[r] & E_i \arrow[r, "\Phi_i"] \arrow[dr, "\eta^* \circ \Phi_i"'] & C(\Lambda) \arrow[r] \arrow[d, "\eta^*"] & 0 \\
& & & C(\Delta) \arrow[r] & 0
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- If $T_2 \in \mathcal{B}(\mathcal{H})$ ess. normal with $\sigma_{\mathsf{e}}(T_2) = \Lambda$ then $T_1 \sim_{\mathcal{K}} T_2$.

Outline

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Theorem (Wold's decomposition)

Any isometry $V \in \mathcal{B}(\mathcal{H})$ *can be written as*

$$V = (S \otimes 1) \oplus U$$

where $S \otimes 1$ *is an amplification of the unilateral shift and* U *unitary.*

A surprising Lemma

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$\implies S \oplus U \sim_{\mathcal{K}} S$

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- $\pi(T^*T - I) = \pi(T)^*\pi(T) - I = 0$

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