Project 9 The Calkin Algebra

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Outline

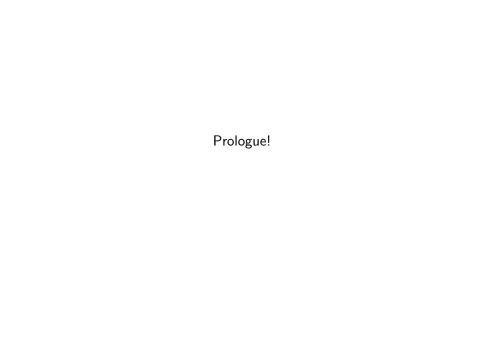
Prologue: Weyl-von Neumann Theorem as an invariant

Act 1: The Calkin Algebra and Fredholm Operators

Act 2: Essential normality: A first look

Main Act: The BDF-Theorem and Extensions

Special Case – essential Spectrum T



In all following: ${\cal H}$ separable, infinite dimensional Hilbert Space.

Theorem (Spectral Theorem for compact s.a. Operators)

Let $K \in \mathcal{K}(\mathcal{H})$ be self-adjoint. Then $0 \in \sigma(K)$ and $\sigma(K) \setminus \{0\}$ consists only of isolated eigenvalues of finite multiplicity.

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Question: Is this true for compact perturbations of general self-adjoint operators (instead of 0)?

Prologue: Weyl's Theorem



Theorem (Weyl, 1909)

Let $T \in \mathcal{B}(\mathcal{H})$ self-adjoint and $K \in \mathcal{K}(\mathcal{H})$ self-adjoint. Then

$$\tilde{\sigma}(T) = \tilde{\sigma}(T + K).$$

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Theorem (Weyl, 1909)

Let $T_1, T_2 \in \mathcal{B}(\mathcal{H})$ self-adjoint. Then

$$T_1 = T_2 + K$$
 for a $K \in \mathcal{K}(\mathcal{H}) \implies \tilde{\sigma}(T_1) = \tilde{\sigma}(T_2)$.



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▶ Does not hold without allowing change of basis, since $\tilde{\sigma}(U^*T_1U) = \tilde{\sigma}(T_1)$ for every unitary U.

Definition (Unitary Equivalence up to Compacts)

For $T_1,T_2\in\mathcal{B}(\mathcal{H})$, we write $T_1\sim_{\mathcal{K}}T_2$ if there is unitary $U\in\mathcal{B}(\mathcal{H})$ and $K\in\mathcal{K}(\mathcal{H})$ s.t.

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We then call T_1 and T_2 unitarily equivalent up to a compact or essentially equivalent.

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▶ Such a theorem describes a "complete invariant" for self-adjoint operators up to $\sim_{\mathcal{K}}$.

Prologue: Berg's Generalisation

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Theorem (Weyl, von Neumann, Berg, 1971)

Let $T_1, T_2 \in \mathcal{B}(\mathcal{H})$ normal. Then

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Definition

For
$$T \in \mathcal{B}(\mathcal{H})$$

$$\sigma_{\mathrm{e}}(T) \coloneqq \sigma(\pi(T))$$

is called the essential spectrum of T.

Weyl-von Neumann-Berg - revisit

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The Definition of Fredholm Operators

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 $T \in \mathcal{B}(\mathcal{H})$ is Fredholm if $\pi(T)$ is invertible in $\mathcal{A}(\mathcal{H})$. The set of all Fredholm operators on \mathcal{H} is denoted by $\mathcal{F}(\mathcal{H})$.

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For $T \in \mathcal{B}(\mathcal{H})$, $\pi(T)$ is invertible in $\mathcal{A}(\mathcal{H}) \iff \dim \mathsf{N}(T)$, $\dim \mathsf{N}(T^*) < \infty$.

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Therefore ind(S) = -1.

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Especially, for $T\in \mathcal{F}(\mathcal{H})$ and $K\in \mathcal{K}(\mathcal{H})$ we have $\mathrm{ind}(T)=\mathrm{ind}(T+K).$

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Theorem (Weyl, von Neumann, Berg, 1971)

Let $T_1, T_2 \in \mathcal{B}(\mathcal{H})$ normal and $K, K' \in \mathcal{K}(\mathcal{H})$. Then

$$T_1 + K \sim_{\mathcal{K}} T_2 + K' \iff \sigma_{\mathsf{e}}(T_1) = \sigma_{\mathsf{e}}(T_2).$$

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$$\inf(S) = -1$$

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So class of essentially normal operators is more than just compact perturbations of normal operators!

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... but even this weaker property fails to hold.

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Outline

Prologue: Weyl—von Neumann Theorem as an invariant

Act 1: The Calkin Algebra and Fredholm Operators

Act 2: Essential normality: A first look

Main Act: The BDF-Theorem and Extensions

Special Case – essential Spectrum T

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Let T essentially normal and $\Delta = \sigma_{\mathsf{e}}(T) \subset \mathbb{R}$.

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- Weyl-von Neumann: All ess. normal operators with ess. spectrum Δ are essentially equivalent to T.

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Write $\operatorname{Ext}(T) := (E, \Phi)$.

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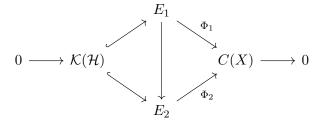
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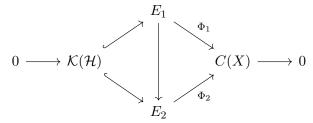


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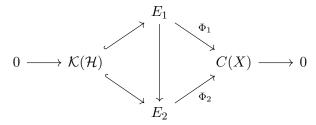
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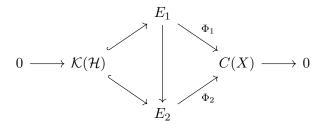
 $X \subset \mathbb{R}: (E_1, \Phi_1) \equiv (E_2, \Phi_2).$

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$$(E_1, \Phi_1) \equiv (E_2, \Phi_2)$$
 if

- ▶ $U \in \mathcal{B}(\mathcal{H})$ unitary such that $E_1 = U^*E_2U$



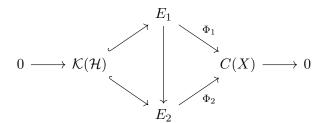
- $ightharpoonup X \subset \mathbb{R}: (E_1, \Phi_1) \equiv (E_2, \Phi_2).$
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- Let T_1, T_2 be essentially normal with $\sigma_{\mathsf{e}}(T_1) = \sigma_{\mathsf{e}}(T_2)$. Then $\mathsf{Ext}(T_1) \equiv \mathsf{Ext}(T_2) \iff T_1 \sim_{\mathcal{K}} T_2$.

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- ▶ Let N be normal with $\sigma_{\mathsf{e}}(N) = \Lambda$, $(E_2, \Phi_2) := \mathsf{Ext}(N)$.

We have seen $(E_1, \eta^* \circ \Phi_1) \equiv (E_2, \eta^* \circ \Phi_2)$.

$$0 \longrightarrow \mathcal{K}(\mathcal{H}) \longrightarrow E_i \xrightarrow{\Phi_i} C(\Lambda) \longrightarrow 0$$

$$\uparrow^{\eta^*} \qquad \qquad \downarrow^{\eta^*}$$

$$C(\Delta) \longrightarrow 0$$

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- Weyl-von Neumann-Berg: two normal operators with same ess. spectrum are ess. equivalent
- ▶ If $T_2 \in \mathcal{B}(\mathcal{H})$ ess. normal with $\sigma_{\mathbf{e}}(T_2) = \Lambda$ then $T_1 \sim_{\mathcal{K}} T_2$.

Outline

Prologue: Weyl—von Neumann Theorem as an invariant

Act 1: The Calkin Algebra and Fredholm Operators

Act 2: Essential normality: A first look

Main Act: The BDF-Theorem and Extensions

Special Case – essential Spectrum $\ensuremath{\mathbb{T}}$

Preparations

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Theorem (Polar decomposition)

Any bounded operator $T \in \mathcal{B}(\mathcal{H})$ can be written as

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for a partial isometry W.

Theorem (Wold's decomposition)

Any isometry $V \in \mathcal{B}(\mathcal{H})$ can be written as

$$V = (S \otimes 1) \oplus U$$

where $S\otimes 1$ is an amplification of the unilateral shift and U unitary.

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Let

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Application to the right shift

- ightharpoonup S essentially normal
- ightharpoonup U unitary

$$\implies S \oplus U \sim_{\mathcal{K}} S$$

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$$U \oplus S = U \oplus \big((D \oplus R) + K \big) = (U \oplus D \oplus R) + (0 \oplus K) \sim_{\mathcal{K}} S$$

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Let $T \in \mathcal{B}(\mathcal{H})$ essentially unitary.

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Outline of the proof:

- ► T compact perturbation of an isometry or coisometry
- ▶ use Wold's decomposition
- lacktriangle classify T up to $\sim_{\mathcal{K}}$ via the index

$$\blacktriangleright \pi(T^*T - I) = \pi(T)^*\pi(T) - I = 0$$

 $ightharpoonup T^*T - I$ compact

- $\begin{array}{ll} \blacktriangleright & T^*T-I \text{ compact} \\ \blacktriangleright & ((T^*T)^{\frac{1}{2}}-I)\cdot ((T^*T)^{\frac{1}{2}}+I)=T^*T-I \end{array}$

- ► $T^*T I$ compact ► $((T^*T)^{\frac{1}{2}} I) \cdot ((T^*T)^{\frac{1}{2}} + I) = T^*T I$

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- $\operatorname{dim} \mathsf{N}(W) \leq \operatorname{dim} \mathsf{N}(W^*) = \operatorname{dim} \mathsf{R}(W)^{\perp} < \infty.$
- $lackbox{ Modify }W$ to isometry V

Proof – Part 1: T compact Perturbation of (Co)isometry

- $ightharpoonup T^*T I$ compact
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- ▶ V W compact

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lacktriangleq If $\operatorname{ind}(T)=0$, the amplification is zero-dimensional

$$\implies T \sim_{\mathcal{K}} V = U$$

- lacktriangledown T compact perturbation of isometry V
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 $\blacktriangleright \ \mathsf{lf} \ \mathsf{ind}(T) > 0,$

$$\implies T \sim_{\mathcal{K}} V^* \sim_{\mathcal{K}} S^* \otimes 1$$

Theorem (Brown-Douglas-Fillmore, 1973)

Let $T_1, T_2 \in \mathcal{B}(\mathcal{H})$ essentially normal. Then $T_1 \sim_{\mathcal{K}} T_2$ iff

- $lack \Lambda \coloneqq \sigma_{\mathsf{e}}(T_1) = \sigma_{\mathsf{e}}(T_2)$ and
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- ▶ if $n := \operatorname{ind}(T_1) = \operatorname{ind}(T_2) = 0$, both are ess. equivalent to unitary operators by Weyl-von Neumann-Berg: $T_1 \sim_{\mathcal{K}} T_2$

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