Smale spaces and C*-algebras

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Topological dynamical systems

A topological dynamical system is a pair $((X, d), \varphi)$ consisting of a compact metric space X and a homeomorphism $\varphi : X \to X$.

Given a point $x \in X$, its orbit is the sequence $\mathrm{orb}(x) := (\varphi^n(x))_{n\in\mathbb{Z}}$.

A point $x \in X$ is called

- a fixed point if $\varphi^n(x) = x$ for every $n \in \mathbb{Z}$;
- periodic if $\varphi^n(x) = x$ for some $n \in \mathbb{Z}$;
- aperiodic if $\varphi^n(x) \neq x$ for every $n \in \mathbb{Z}$.

Stability

A set $S \subset X$ of points is stable if, for any two points in S, the distance between their forward orbits approaches zero. That is, for every $x, y \in S$,

$$d(\varphi^n(x), \varphi^n(y)) \to 0,$$

as $n \to \infty$.

A set $U \subset X$ of points is unstable if, for any two points in U, the distance between their backward orbits approaches zero. That is, for every $x, y \in U$,

$$d(\varphi^{-n}(x), \varphi^{-n}(y)) \to 0,$$

as $n \to \infty$.

Smale spaces - a non-definition

A Smale space is a dynamical system $((X, d), \varphi)$ which is locally hyperbolic in the sense that at every point $x \in X$, the space can be decomposed into a stable set and unstable set.

Example: The Full 2-shift

Let $X := \{0,1\}^{\mathbb{Z}}$ be the space of sequences in 0's and 1's.

For $x = (x_n)_{n\in\mathbb{Z}}, y = (y_n)_{n\in\mathbb{Z}}$ define

$d(x, y) = \inf\{2^{-n} \mid n \geq 0, x_i = y_i \text{ for all } |i| < n\}$.

Let $\sigma : X \to X$ be the left shift, that is, $\sigma((x_n))_{n\in\mathbb{Z}} = (x_{n+1})_{n\in\mathbb{Z}}$.

Here, X is a Cantor set, but the shift is far from being minimal. For example, the sequences of all 1's and of all 0's are fixed points. It is also easy to see that there are many periodic points.

2-Shift: local stable and unstable sets

Let $\epsilon = 2^{-(k+1)} \leq 1/2$ and fix $x \in X$.

Define $S(x, \epsilon) := \{y \in X \mid d(x, y) \leq \epsilon \text{ and } y_i = x_i \text{ for all } i \geq -k\}$,

and $U(x, \epsilon) := \{y \in X \mid d(x, y) \leq \epsilon \text{ and } y_i = x_i \text{ all } i \leq k\}$.

$\mathbb{Z}$	… -11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	
x	… 0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	…
y	… y_{-11}	y_{-10}	y_{-9}	y_{-8}	y_{-7}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	…
z	… 0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	z_7	z_8	z_9	z_{10}	z_{11}	z_{12}	z_{13}	…

2-Shift: local stable and unstable sets

For $y \in S(x, \epsilon)$ we have that $d(\sigma(y), \sigma(x)) \leq \frac{1}{2} d(y, x)$ and $\lim_{n \to \infty} d(\sigma^n(y), \sigma^n(x)) = 0$.

On the other hand, for $y \in U(x, \epsilon)$, we have that $d(\sigma^{-1}(y), \sigma^{-1}(x)) \leq \frac{1}{2} d(y, x)$ and $\lim_{n \to \infty} d(\sigma^{-n}(y), \sigma^{-n}(x)) = 0$

Furthermore, $S(x, \epsilon) \cap U(x, \epsilon) = \{x\}$ and $S(x, \epsilon) \times U(x, \epsilon)$ is homeomorphic to the ϵ ball centered at x.

Thus, there exists an ϵ_X such that, for every $x \in X$, whenever $\epsilon \leq \epsilon_X$, we can identify two coordinates whose origin is x, such that the system is contracting along one coordinate and expanding along the other.

2-Shift: Global (un)stable equivalence

If $y \in S(x, \epsilon)$ then $d(\sigma^n(x), \sigma^n(y)) \to 0$ as $n \to \infty$.

However

$$d(\sigma^n(x), \sigma^n(y)) \to 0, \quad n \to \infty$$

need not imply $y \in S(x, \epsilon)$.

E.g.: if $\epsilon = 2^{-(k+1)}$ and $y_i = x_i$ for every $i > 2k$, but $x_i \neq y_i$ for some $|i| \leq k$. Then $d(\sigma^n(x), \sigma^n(y)) \to 0$ but $y \notin S(x, \epsilon)$.

Similarly, $d(\sigma^{-n}(x), \sigma^{-n}(y)) \to 0$, $n \to \infty$ need not imply $y \in U(x, \epsilon)$.

Global (un)stable equivalence

If $\lim_{n\to\infty} d(\sigma^n(x), \sigma^n(y)) = 0$ we say x, y are globally stably equivalent, written $x \sim_s y$.

The global stable set of x is $S(x) := \{y \in X \mid x \sim_s y\}$.

Global unstable equivalence for two points x, y is defined similarly and denoted $x \sim_u y$.

The global unstable set of x is $U(x) = \{y \in X \mid y \sim_u x\}$

Homoclinic equivalence

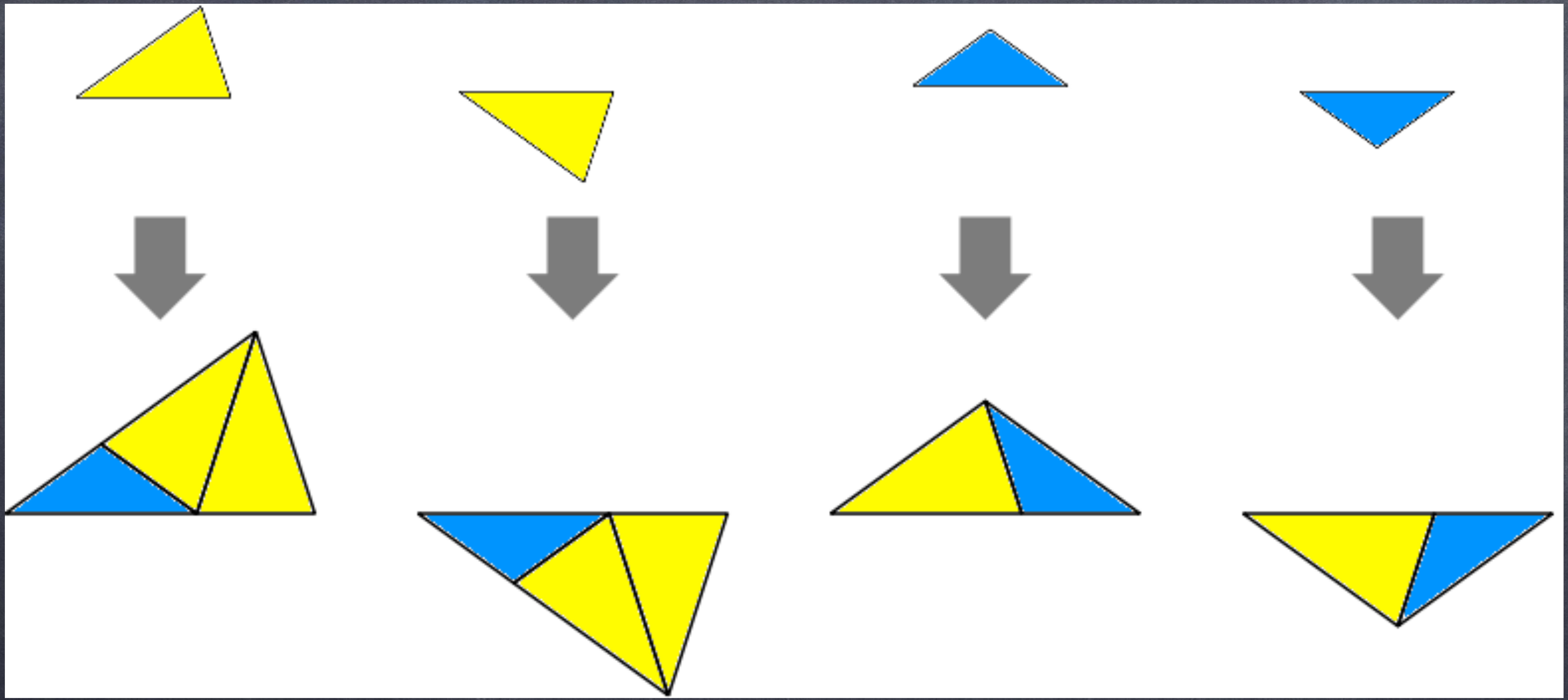
As their names suggest, $\sim_s$ and $\sim_u$ are both equivalence relations on X.

We also have the homoclinic equivalence relation: $x \sim_h y$ if $d(\sigma^n(x), \sigma^n(y)) \to 0$ as $n \to \pm\infty$. Let $H(x) := \{y \in X \mid y \sim_h y\}$.

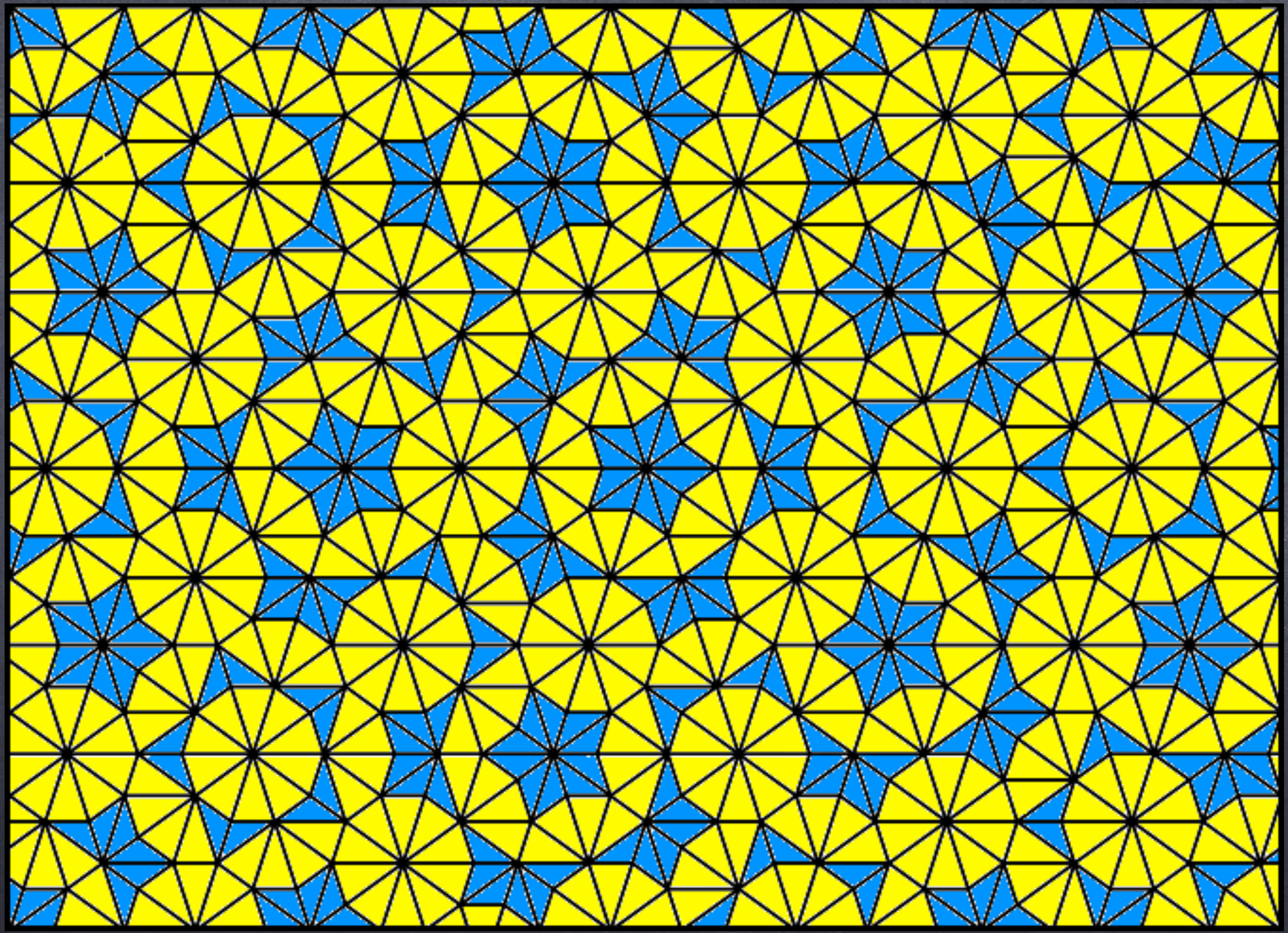
Note that $H(x) = S(x) \cap U(x)$.

Example: Aperiodic substitution tilings

Penrose tiles: We start with a number of “prototiles”, plus an inflation and substitution rule.



Repeatedly appplying the substitution rule eventually gives us a tiling of the plane. Let Ω be the set of all such filings.



Tilings: local stable and unstable sets

For two tilings $T, T' \in \Omega$, define

$d(T,T') = \inf(\{1/\sqrt{2}\} \cup \{\epsilon \mid \exists u_1, u_2 \in \mathbb{R}^2, \|u_i\| \leq \epsilon$ such that $T + u_i$ agree on $B(0,\epsilon^{-1})\}$.

The substitution rule induces a homeomorphism $\omega : \Omega \to \Omega$.

For a tiling $T \in \Omega$, the local stable set, $S(T,\epsilon)$, consists of those T' which agree on a large ball around the origin.

Local unstable sets, $U(T,\epsilon)$, consist of those T' which are small perturbations of T.

Smale space definition

Definition 1.1. [**56**, Section 7.1] Let (X, d) be a compact metric space and let $\varphi : X \to X$ be a homeomorphism. The dynamical system (X, φ) is called a Smale space if there are two constants $\epsilon_X > 0$ and $0 < \lambda_X < 1$ and a map, called the bracket map,

$$[\cdot,\cdot] : X \times X \to X$$

which is defined for $x, y \in X$ such that $d(x, y) < \epsilon_X$. The bracket map is required to satisfy the following axioms:

B1. $[x, x] = x$,

B2. $[x, [y, z]] = [x, z]$,

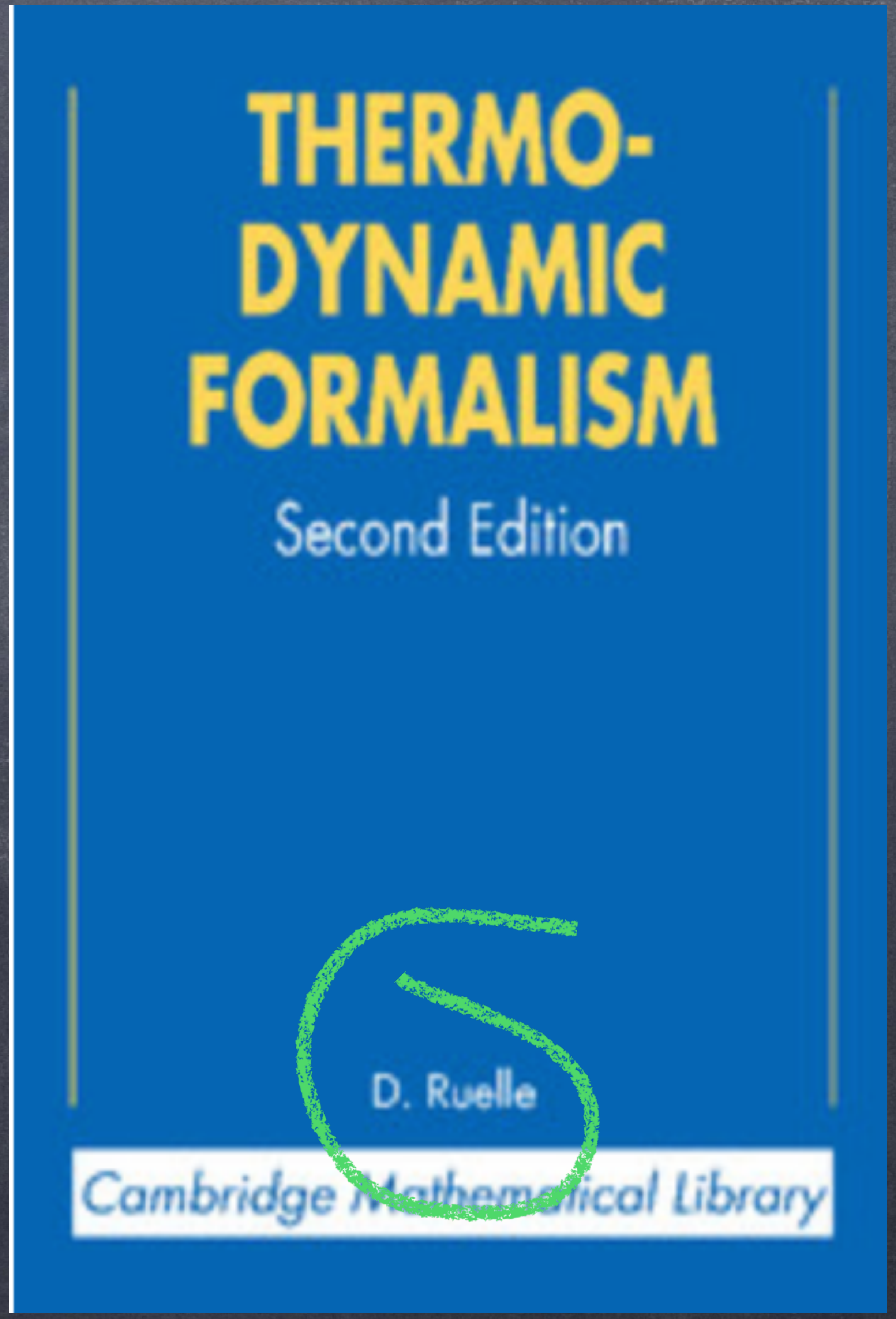
B3. $[[x, y], z] = [x, z]$,

B4. $\varphi[x, y] = [\varphi(x), \varphi(y)]$;

for $x, y, z \in X$ whenever both sides in the above equations are defined. The system also satisfies

C1. For $x, y \in X$ such that $[x, y] = y$, we have $d(\varphi(x), \varphi(y)) \leq \lambda_X d(x, y)$ and

C2. For $x, y \in X$ such that $[x, y] = x$, we have $d(\varphi^{-1}(x), \varphi^{-1}(y)) \leq \lambda_X d(x, y)$.



When X is zero-dimensional, then a Smale space is exactly a shift of finite type. Stay tuned for more on shifts of finite type in an upcoming lecture!

In fact, Smale spaces behave a lot like SFTS: they always admit Markov partitions.

First consider the case of an invertible dynamical system (M, ϕ). Let $\mathcal{P} = \{P_0, P_1, \ldots, P_{r-1}\}$ be a topological partition of M. For each $x \in X_{\mathcal{P},\phi}$ and $n \geqslant 0$ there is a corresponding nonempty open set

$$D_n(x) = \bigcap_{k=-n}^{n} \phi^{-k}(P_{x_k}) \subseteq M.$$

Definition 6.5.6. Let (M, ϕ) be an invertible dynamical system. A topological partition $\mathcal{P} = \{P_0, P_1, \ldots, P_{r-1}\}$ of M gives a *symbolic representation* of (M, ϕ) if for every $x \in X_{\mathcal{P},\phi}$ the intersection $\bigcap_{n=0}^{\infty} \overline{D}_n(x)$ consists of exactly one point. We call $\mathcal{P}$ a *Markov partition* for (M, ϕ) if $\mathcal{P}$ gives a symbolic representation of (M, ϕ) and furthermore $X_{\mathcal{P},\phi}$ is a shift of finite type.

SFTs and factor maps

If a topological dynamical system (X, φ) admits a Markov partition, there exists a shift of finite type (Σ, σ) and a factor map $\pi : (\Sigma, \sigma) \to (X, \varphi)$. In particular, this is the case for any Smale space.

From Smale spaces to étale groupoids

Let (X, φ) be a Smale space. We want to construct a C*-algebra from (X, φ).

We could go ahead and form the crossed product $C(X) \rtimes_\varphi \mathbb{Z}$, but Smale spaces will always have many fixed or periodic points (periodic points are dense!), so the crossed product will be too wild.

Furthermore, we are more interested in capturing the hyperbolic behavior of (X, φ).

We will instead construct groupoid C*-algebras. So what's a groupoid?

Groupoids

A groupoid G consists of

- A unary operation $G \to G, g \to g^{-1}$
- A distinguished subset $G^{(2)} \subset G \times G$ of composable pairs
- A partially defined multiplication map $G^{(2)} \to G, (g, h) \to gh$

such that

- $(g^{-1})^{-1} = g$ for every $g \in G$.
- if $(g, h), (h, k) \in G \implies (g, hk), (gh, k) \in G^{(2)}$ and $g(hk) = (gh)k$
- $(g, g^{-1}), (g^{-1}, g) \in G^{(2)}$ for every $g \in G$ and for every $(g, h) \in G^{(2)}$ we have $g^{-1}(gh) = h, \quad (gh)h^{-1} = g$

Étale groupoids

$G^{(0)} := \{gg^{-1} \mid g \in G\} \subset G$ is called the space of units of G.

We can then define range and source maps, $r, s : G \to G^{(0)}$ by $r(g) = gg^{-1}$ and $s(g) = g^{-1}g$.

Then $G^{(2)} = \{(g, h) \mid s(g) = r(h)\}$.

A topological groupoid is a groupoid G equipped with a locally compact topology making $G^{(0)}$ Hausdorff in the relative topology, and such that $r, s, {}^{-1} : G \to G$ are continuous, and $(g, h) \to gh$ is continuous with respect to $G^{(2)} \to G$.

If r, s are local homeomorphisms, then we say G is an étale groupoid.

Examples

- A group G is a groupoid with $G^{(2)} = G \times G$, $G^{(0)} = \{e\}$. It is an étale groupoid if G is a discrete group.

- Let X be a compact metric space and $\mathscr{R} \subset X \times X$ an equivalence relation. Let Define $\mathscr{R}^{(2)} = \{((x,y),(y,z)) \mid x,y,z \in X\}$, $(x,y)(y,z) = (x,z)$, and $(x,y)^{-1} = (y,x)$.

Then $r(x,y) = (x,x)$, $s(x,y) = (y,y)$ so we identify $\mathscr{R}^{(0)} \cong X$.

- Let $\alpha : G \times X \to X$ be an action. Define the transformation groupoid $G \times_\alpha X := \{(g\,.\,x, g, x) \in X \times G \times X\}$. Let $((g\,.\,x, g, x), (h\,.\,y, h, y)) \in (G \times_\alpha X)^{(2)} \iff x = h\,.\,y$, in which case $((g\,.\,(h\,.\,y), g, h\,.\,y)(h\,.\,y, h, y)) = (gh\,.\,y, gh, y)$. Let $(g\,.\,x, g, x)^{-1} = (x, g^{-1}, g\,.\,x)$.

Again we get that $(G \times_\alpha X)^{(0)} \cong X$.

Groupoids: The shorter version.

Definition: A groupoid is a small category where every morphism is invertible.

Convolution algebra

- G an étale groupoid, $C_c(G)$ the vector space of compactly supported continuous functions on G

- $$f_1 f_2(g) := \sum_{h_1 h_2 = g} f_1(h_1) f(h_2), \quad f_1, f_2 \in C_c(G), g \in G\,.$$

- $$f^*(g) = \overline{f(g^{-1})}, \qquad f \in C_c(G), g \in G\,.$$

For every $x \in G^{(0)}$, define $\pi_x : C_c(G) \to B(\ell^2(s^{-1}(x))$ by

$$\pi_x(f)(\xi)(g) = \sum_{h_1 h_2 = g} f(h_1) f(h_2), \qquad f \in C_c(G)\xi \in \ell^2(s^{-1}(x)), g \in s^{-1}(x)\,.$$

Groupoid C*-algebras

Let

$$\|f\| = \sup_{x \in G^{(0)}} \|\pi_x(f)\|$$

The reduced groupoid C*-algebra of G is then defined to be

$$C_r(G) := \overline{C_c(G)}^{\|\cdot\|}.$$

Just as for group C*-algebras and crossed product C*-algebras, one is also able to define a full groupoid C*-algebra.

If G is not étale, but still locally compact with Hausdoff unit space, one can also construct a C*-algebra, provided G admits a Haar system.

Amenability

Furthermore , there is a notion of amenability for groupoids, extending the notion of amenability of groups (and actions).

Amenability $\implies$ $C_r^*(G) = C^*_{\text{full}}(G)$, and nuclearity.

However, unlike in the group case

$C_r^*(G) = C^*_{\text{full}}(G) \not\Longrightarrow$ Amenability! (Willett, 2015.)

Groupoids associated to a Smale space

There are three groupoids we can associate to any Smale space (X, φ):

- Stable equivalence relation $\mathcal{S} := \{(x, y) \in X \times X \mid x \sim_s y\}$
- Unstable equivalence relation $\mathcal{U} := \{(x, y) \in X \times X \mid x \sim_u y\}$
- Homoclinic equivalence relation $\mathcal{H} := \{(x, y) \in X \times X \mid x \sim_h y\}$

...but are they étale?

We can topologize $\mathcal{S}, \mathcal{U}, \mathcal{H}$, but beware: we do not give them the relative topology from the product topology!

For a given Smale space $\mathcal{S}, \mathcal{U}, \mathcal{H}$ are always amenable (Putnam–Spielberg 1999).

However, if $\dim(X) > 0$, then only $\mathcal{H}$ is étale.

$\mathcal{S}, \mathcal{U}$ are "too big"... but don't despair!

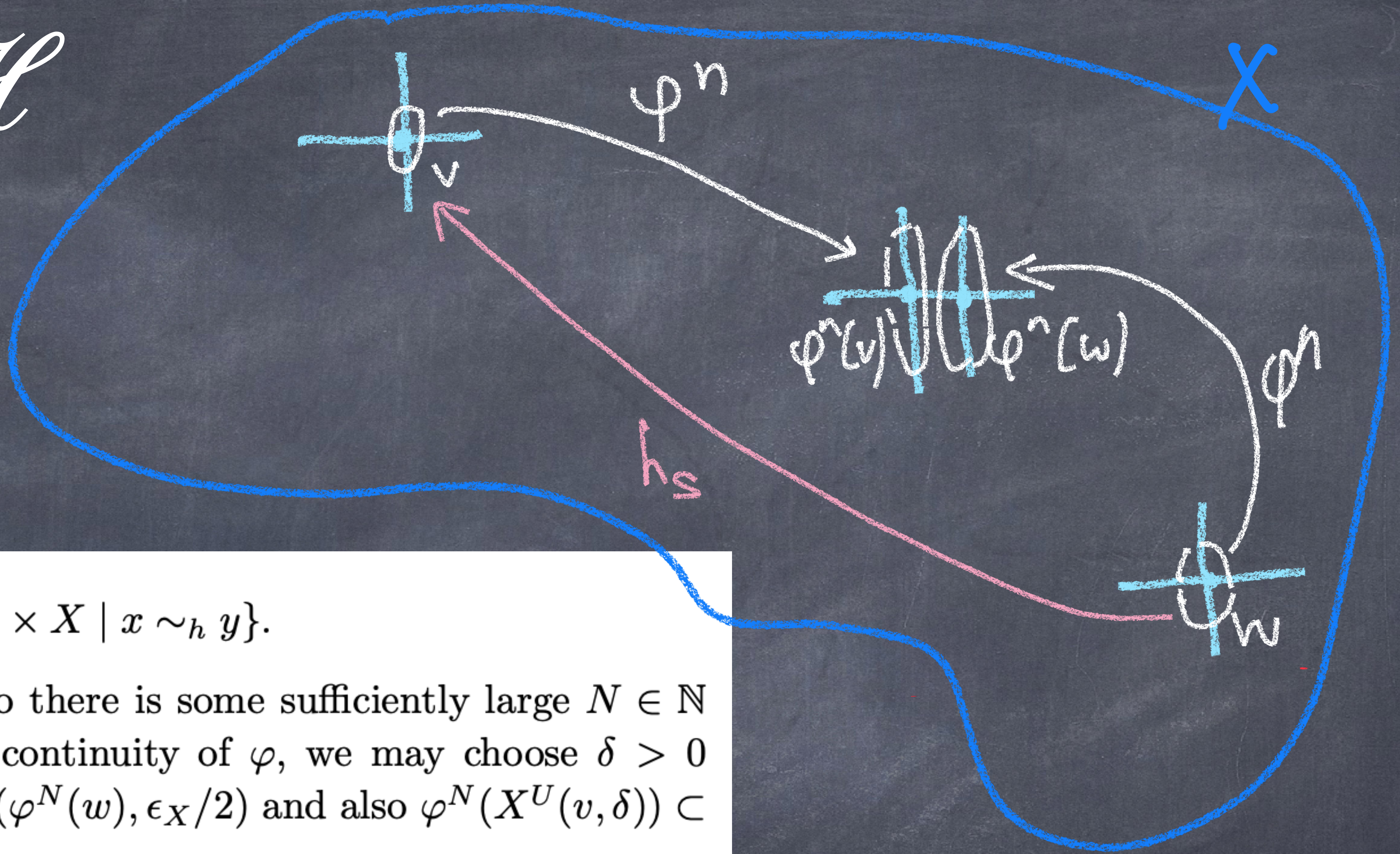
Cutting down by a transversal

We can "cut down" the equivalence classes in $\mathcal{S}$ and $\mathcal{U}$: Let P, Q be non-empty sets of periodic points satisfying $\varphi(P) \subset P, \varphi(Q) \subset Q$.

- $\mathcal{S}(P) := \{(x, y) \in \mathcal{S} \mid \exists p \in P : x, y \sim_u p\}$
- $\mathcal{U}(Q) := \{(x, y) \in \mathcal{U} \mid \exists q \in Q : x, y \sim_s q\}$

Now we can equip $\mathcal{S}(P), \mathcal{U}(Q)$ with an étale topology. The resulting C*-algebras will be stably isomorphic to $C^*(\mathcal{S}), C^*(\mathcal{U})$, respectively.

Topology on $\mathcal{H}$



$$\mathcal{G}_H := \{(x, y) \in X \times X \mid x \sim_h y\}.$$

Now, if $(v, w) \in X^S(P)$, then $v \sim_s w$ so there is some sufficiently large $N \in \mathbb{N}$ such that $d(\varphi^N(v), \varphi^N(w)) < \epsilon_X/2$. By continuity of φ, we may choose $\delta > 0$ small enough so that $\varphi^N(X^U(w, \delta)) \subset X^U(\varphi^N(w), \epsilon_X/2)$ and also $\varphi^N(X^U(v, \delta)) \subset X^U(\varphi^N(v), \epsilon_X/2)$. Then define

$$h^s := h^s(v, w, N, \delta) : X^U(w, \delta) \to X^U(v, \epsilon_X), \quad x \mapsto \varphi^{-N}([\varphi^N(x), \varphi^N(v)]).$$

By [**56**, Section 7.15] this is a local homeomorphism.

For any such v, w, δ, h, N, we then define an open set by

$$V(v, w, \delta, h^s, N) := \{(h^s(x), x) \mid x \in X^U(w, \delta)\} \subset \mathcal{G}_S(P).$$

Facts about Smale space C*-algebras

- $C^*(\mathscr{H})$ is unital and nuclear
- $C^*(\mathscr{S}), C^*(\mathscr{U})$ are
- $C^*(\mathscr{H}) \otimes \mathscr{K} \cong C^*(\mathscr{S}(P)) \otimes C^*(\mathscr{U}(Q))$

Definition: A topological dynamical system (X, φ) is mixing if , for every pair of open subsets $U, V \subset X$ there exists $n_0 \in \mathbb{N}$ such that $\varphi^n(U) \cap V \neq \emptyset$, for every $n \geq n_0$.

- If (X, φ) is a mixing Smale space, then $C^*(\mathscr{H}), C^*(\mathscr{S}), C^*(\mathscr{U})$ are simple.

Classification of Smale space C*-algebras

Theorem: Let A, B be simple, separable, unital, nuclear C*-algebras with finite nuclear dimension and which satisfy the UCT. Then, if $\varphi : \mathrm{Inv}(A) \to \mathrm{Inv}(B)$ is an isomorphism, there exists a *-isomorphism $\Phi : A \to B$, unique up to approximate unitary equivalence and satisfying $\mathrm{Inv}(\Phi) = \varphi$.

...Are Smale space C*-algebras covered by this theorem?

We already saw we'll have to stick to mixing Smale spaces. Also, the stable and unstable groupoid C*-algebras are not unital.

Finite nuclear dimension of irreducible Smale space C*-algebras

A topological dynamical system is irreducible if, for every order pair of open sets $U, V \subset X$, there exists $n \in \mathbb{N}$ such that $\varphi^n(U) \cap V \neq \emptyset$.

Every irreducible Smale space can be written as the disjoint union of finitely many cloven subspaces $X_1, \ldots, X_n$ which are cyclically permitted by φ and such that $(X_i, \varphi|_{X_i})$ are mixing Smale spaces.

Corollary 3.8. *Let (X, φ) be an irreducible Smale space and P a finite set of φ-invariant periodic points. Then the stable, unstable and homoclinic* C*-*algebras each have finite nuclear dimension.*

Classification of Smale space C*-algebras

Theorem 4.7. *The homoclinic algebras associated to mixing Smale spaces are contained in a class of* C**-algebras that is classified by the Elliott invariant. In particular,*

(1) *the homoclinic algebra associated to a mixing Smale space is approximately subhomogeneous,*
(2) *if A and B are the homoclinic algebras associated to the mixing Smale spaces (X, φ) and (Y, ψ), then an isomorphism*

$$\phi : (K_0(A), K_0(A)_+, [1_A], K_1(A)) \to (K_0(B), K_0(B)_+, [1_B], K_1(B))$$

lifts to a *-isomorphism

$$\Phi : A \to B$$

inducing ϕ.

Hyperbolic toral automorphisms

Let $A \in M_d(\mathbb{Z})$ with $|\det(A)| = 1$ and such that none of the eigenvalues of A have modulus 1.

Then A is invertible in $M_d(\mathbb{Z})$ and after identifying $\mathbb{T}^d \cong \mathbb{R}^d/\mathbb{Z}^d$, we have a well-defined homeomorphism $\varphi : \mathbb{T}^d \to \mathbb{T}^d$, defined by $\varphi(x) = Ax \mod \mathbb{Z}^d$

Let A = $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

Eigenvalues are $\lambda_1 = (3+\sqrt{5})/2, \quad \lambda_2 = (3-\sqrt{5})/2$. There exists $\alpha, \beta \in \mathbb{R}$ such that

$$v_1 = \begin{pmatrix} 1 \\ \alpha \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ \beta \end{pmatrix}$$

are the corresponding eigenvectors.

Let $\theta = e^{2\pi i(-\beta^{-1})}$ and let P be a finite set of periodic points with $\varphi(P) \subset P$. Let $p \in C^*(\mathcal{S}(P))$ be a nontrivial projection.

Then $pAp \cong C(\mathbb{T}) \rtimes_\theta \mathbb{Z} \cong A_\theta$ is the irrational rotation algebra!

Group actions on dynamical systems

Let (X, φ) be a topological dynamical system and let
$G \to \mathrm{Homeo}(X)$ be a continuous group homomorphism such that

$$g \,.\, \varphi(x) = \varphi(g \,.\, x),$$

for every $x \in X, g \in G$.

An action is free if, for every $x \in X$, $g \,.\, x = x$ if and only if $g = e$.

Group actions on Smale spaces

Let (X, σ) be the 2-shift. The automorphism group is very large! For example, it contains every finite group and the free group on two generators.

However, free actions on a Smale space (X, φ) are rare: If the group G has an element of infinite order, then G cannot act freely on (X, φ).

Any group action on a Smale space induces a group action on its C*-algebras. (One can always take the set of φ-invariant periodic points to be G-invariant.)

Examples of induced actions on C*-algebras

- Let (X, φ) be a mixing Smale space with $\dim(X) = 0$. If G is a finite group acting freely on (X, φ), then $G \curvearrowright C^*(\mathscr{H})$ has the Rokhlin property.

- Let (X, φ) be a Smale space. The homeomorphism φ induces a $\mathbb{Z}$-action on (X, φ), hence a $\mathbb{Z}$-action on $C^*(\mathscr{H})$, $C^*(\mathscr{S})$, $C^*(\mathscr{U})$, and we can consider their crossed products. If (X, φ) is mixing $C^*(\mathscr{S}) \rtimes \mathbb{Z}$, $C^*(\mathscr{U}) \rtimes \mathbb{Z}$ are simple and purely infinite.

Group actions on Smale space C*-algebras

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...and finally... Spectral triples

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Journal of Noncommutative Geometry

Spectral triples for hyperbolic dynamical systems

Michael F. Whittaker*

Abstract. Spectral triples are defined for C^*-algebras associated with hyperbolic dynamical systems known as Smale spaces. The spectral dimension of one of these spectral triples is shown to recover the topological entropy of the Smale space.

Some further reading...

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4. Ian F. Putnam, *C^*-algebras from Smale spaces*, Canad. J. Math. **48** (1996), no. 1, 175–195. MR 1382481
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...and many more!

Thanks for listening!

...and make sure to stay tuned for more on shifts of finite type.