Noncommutative geometry, operator systems and state spaces.

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(joint with Alain Connes)

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A spectral approach to geometry

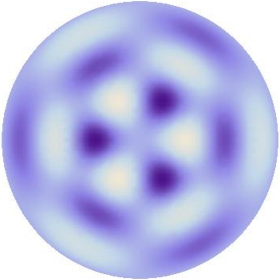
"Can one hear the shape of a drum?" (Kac, 1966)

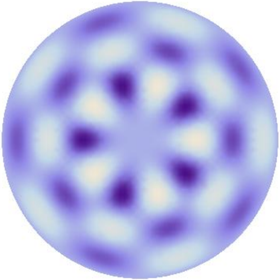
Or, more precisely, given a Riemannian manifold M, does the spectrum of wave numbers k in the Helmholtz equation

$$\boxed{\Delta_M u = k^2 u}$$

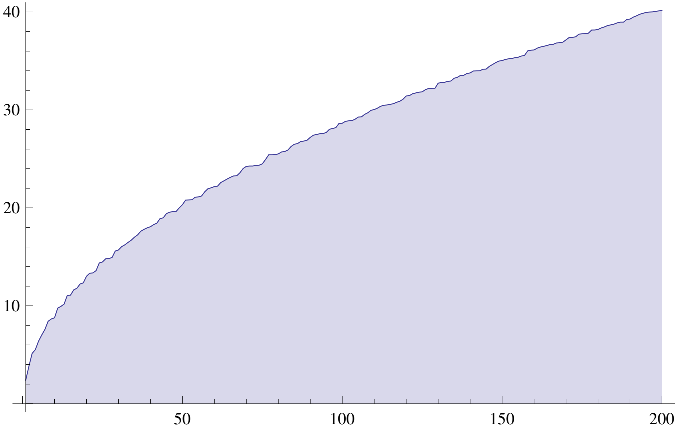
determine the geometry of M?

The disc

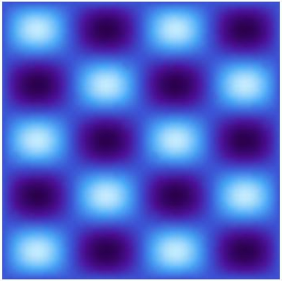


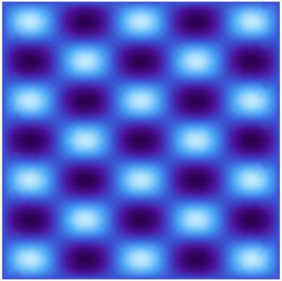


Wave numbers on the disc

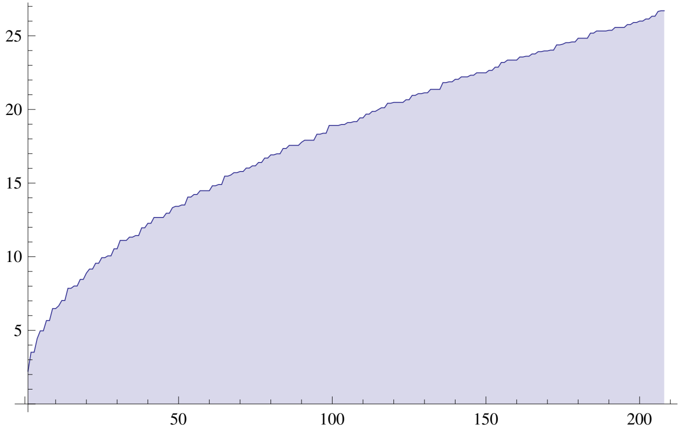


The square



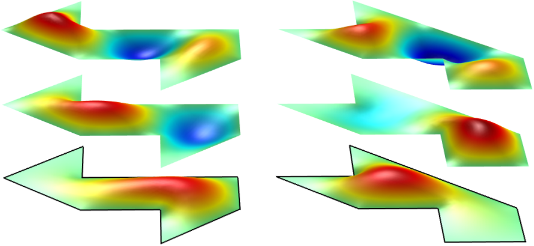


Wave numbers on the square



Isospectral domains

But, there are isospectral domains in $\mathbb{R}^2$:



(Gordon, Webb, Wolpert, 1992)

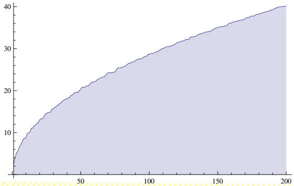
so the answer to Kac's question is no

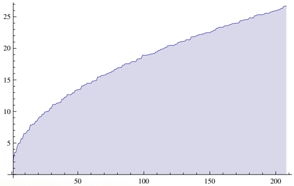
Weyl's estimate

Nevertheless, certain information can be extracted from spectrum, such as dimension d of M:

$$N(\Lambda) = \#\text{wave numbers } \leq \Lambda$$
$$\sim \frac{\Omega_d \mathrm{Vol}(M)}{d(2\pi)^d}\Lambda^d$$

For the disc and square this is confirmed by the parabolic shapes ($\sqrt{\Lambda}$):

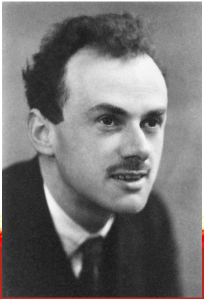




Analysis: Dirac operator

Recall that k^2 is an eigenvalue of the Laplacian in the Helmholtz equation.

- The Dirac operator is a ‘square-root’ of the Laplacian, so that its spectrum give the wave numbers k.
- First found by Paul Dirac in flat space, but exists on any Riemannian spin manifold M.
- Let us give some examples.



The circle

- The Laplacian on the circle $\mathbb{S}^1$ is given by

$$\Delta_{\mathbb{S}^1} = -\frac{d^2}{dt^2}; \qquad (t \in [0, 2\pi))$$

- The Dirac operator on the circle is

$$D_{\mathbb{S}^1} = -i\frac{d}{dt}$$

with square $\Delta_{\mathbb{S}^1}$.

The 2-dimensional torus

- Consider the two-dimensional torus $\mathbb{T}^2$ parametrized by two angles $t_1, t_2 \in [0, 2\pi)$.
- The Laplacian reads

$$\Delta_{\mathbb{T}^2} = -\frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial t_2^2}.$$

- At first sight it seems difficult to construct a differential operator that squares to $\Delta_{\mathbb{T}^2}$:

$$\left(a\frac{\partial}{\partial t_1} + b\frac{\partial}{\partial t_2}\right)^2 = a^2\frac{\partial^2}{\partial t_1^2} + 2ab\frac{\partial^2}{\partial t_1 \partial t_2} + b^2\frac{\partial^2}{\partial t_2^2}$$

- This puzzle was solved by Dirac who considered the possibility that a and b be complex *matrices*:

$$a = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \qquad b = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

  then $a^2 = b^2 = -1$ and $ab + ba = 0$
- The Dirac operator on the torus is

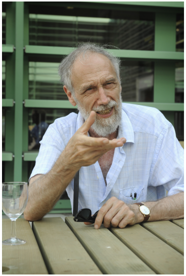
$$D_{\mathbb{T}^2} = \begin{pmatrix} 0 & \frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2} \\ -\frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2} & 0 \end{pmatrix},$$

  which satisfies $(D_{\mathbb{T}^2})^2 = -\frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial t_2^2}$.

The 4-dimensional torus

- Consider the 4-torus $\mathbb{T}^4$ parametrized by t_1, t_2, t_3, t_4 and the Laplacian is
$$\Delta_{\mathbb{T}^4} = -\frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial t_2^2} - \frac{\partial^2}{\partial t_3^2} - \frac{\partial^2}{\partial t_4^2}.$$
- The search for a differential operator that squares to $\Delta_{\mathbb{T}^4}$ again involves matrices, but we also need quaternions:
$$i^2 = j^2 = k^2 = ijk = -1.$$
- The Dirac operator on $\mathbb{T}^4$ is
$$D_{\mathbb{T}^4} = \begin{pmatrix} 0 & \frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2} + j\frac{\partial}{\partial t_3} + k\frac{\partial}{\partial t_4} \\ -\frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2} + j\frac{\partial}{\partial t_3} + k\frac{\partial}{\partial t_4} & 0 \end{pmatrix}$$
- The relations $ij = -ji$, $ik = -ki$, *et cetera* imply that its square coincides with $\Delta_{\mathbb{T}^4}$.

Noncommutative geometry



If combined with the C^*-algebra $C(M)$, then the answer to Kac' question is affirmative.

Connes' reconstruction theorem [2008]:

$$(C(M), D_M) \longleftrightarrow (M, g)$$

Spectral data

- This mathematical reformulation of geometry in terms of spectral data requires the knowledge of all eigenvalues of the Dirac operator.
- From a physical standpoint this is not very realistic: detectors have limited energy ranges and resolution.

> *We develop the mathematical formalism for (noncommutative) geometry with only part of the spectrum.*

This is in line with earlier work of [D'Andrea–Lizzi–Martinetti 2014], [Glaser–Stern 2019], [Berendschot 2019] and based on [arXiv:2004.14115]

The "usual" story

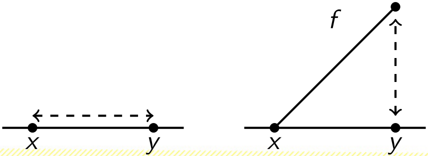
Given cpt Riemannian spin manifold (M, g) with spinor bundle S on M.

- the C^*-algebra $C(M)$
- the self-adjoint Dirac operator D_M
- both acting on Hilbert space $L^2(M, S)$

$\rightsquigarrow$ spectral triple: $(C(M), L^2(M, S), D_M)$

Reconstruction of distance function [Connes 1994]:

$$d(x, y) = \sup_{f \in C(M)} \{|f(x) - f(y)| : \|[D_M, f]\| \leq 1\}$$



Spectral triples

More generally, we consider a triple $(A, \mathcal{H}, D)$

- a C^*-algebra A
- a self-adjoint operator D with compact resolvent and bounded commutators $[D, a]$ for $a \in \mathcal{A} \subset A$
- both acting (boundedly, resp. unboundedly) on Hilbert space $\mathcal{H}$

Generalized distance function:

- States are positive linear functionals $\phi : A \to \mathbb{C}$ of norm 1
- Pure states are extreme points of state space
- Distance function on state space of A:

$$d(\phi, \psi) = \sup_{a \in A} \{|\phi(a) - \psi(a)| : \|[D, a]\| \leq 1\}$$

Spectral truncations

Given $(A, \mathcal{H}, D)$ we project onto part of the spectrum of D:

- $\mathcal{H} \mapsto P\mathcal{H}$, projection onto closed Hilbert subspace
- $D \mapsto PDP$, still a self-adjoint operator
- $A \mapsto PAP$, this is not an algebra any more (unless $P \in A$)

Instead, PAP is an operator system: $(PaP)^* = Pa^*P$.

So, we turn to study $(PAP, P\mathcal{H}, PDP)$.

We expect:

- state and pure states on PAP
- a distance formula on states of PAP.
- a rich symmetry: isometries of $(A, \mathcal{H}, D)$ remain isometries of $(PAP, P\mathcal{H}, PDP)$

Operator systems

Definition (Choi-Effros 1977)
An operator system is a $*$-closed vector space E of bounded operators.

For convenience we take E to be finite-dimensional and to contain the identity operator.

- E is ordered: cone $E_+ \subseteq E$ of positive operators, in the sense that $T \in E_+$ iff
$$\langle \psi, T\psi \rangle \geq 0; \qquad (\psi \in \mathcal{H}).$$
- in fact, E is completely ordered: cones $M_n(E)_+ \subseteq M_n(E)$ of positive operators on $\mathcal{H}^n$ for any n.

Maps between operator systems E, F are complete positive maps in the sense that their extensions $M_n(E) \to M_n(F)$ are positive for all n.

Isomorphisms are complete order isomorphisms

States spaces of operator systems

- The existence of a cone $E_+ \subseteq E$ of positive elements allows to speak of states on E as positive linear functionals of norm 1.
- Also, the dual E^d of an operator system is an operator system, with

$$E^d_+ = \{\phi \in E^d : \phi(T) \geq 0, \forall T \in E_+\}$$

  and similarly for the complete order.
- We have $(E^d)^d_+ \cong E_+$ as cones in $(E^d)^d \cong E$.
- It follows that we have the following useful correspondence:

  pure states on $E \longleftrightarrow$ extreme rays in $(E^d)_+$

  and the other way around.

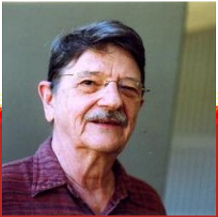
C^*-envelope of a unital operator system

Arveson introduced the notion of C^*-envelope for operator systems in 1969, Hamana established existence and uniqueness in 1979.

A C^*-extension $\kappa : E \to A$ of an operator system E is given by a complete order isomorphism onto $\kappa(E) \subseteq A$ such that $C^*(\kappa(E)) = A$.

A C^*-envelope of an operator systems is a C^*-extension $\kappa : E \to A$ with the following universal property:

$$
\begin{tikzcd}
E \arrow[r, "\kappa"] \arrow[dr, "\lambda"'] & A \\
& B \arrow[u, dotted, two heads, "\exists!\rho"']
\end{tikzcd}
$$



Shilov boundaries

There is a useful description of C^*-envelopes using Shilov ideals.

Definition

Let $\kappa : E \to A$ be a C^*-extension of an operator system. A boundary ideal is given by a closed 2-sided ideal $I \subseteq A$ such that the quotient map $q : A \to A/I$ is completely isometric on $\kappa(E) \subseteq A$.

The Shilov ideal is the largest of such boundary ideals.

Proposition

Let $\kappa : E \to A$ be a C^*-extension. Then there exists a Shilov boundary ideal J and $C^*_{env}(E) \cong A/J$.

As an example consider the operator system of continuous harmonic functions $C_{\mathrm{harm}}(\overline{\mathbb{D}})$ on the closed disc. Then by the maximum modulus principle the Shilov boundary is S^1. Accordingly, its C^*-envelope is $C(S^1)$.

Propagation number of an operator system

One lets $E^{\circ n}$ be the norm closure of the linear span of products of $\leq n$ elements of E.

Definition

The propagation number $prop(E)$ of E is defined as the smallest integer n such that $\kappa(E)^{\circ n} \subseteq C^*_{env}(E)$ is a C^*-algebra.

Returning to harmonic functions in the disk we have $\mathrm{prop}(C_{\mathrm{harm}}(\mathbb{D})) = 1$.

Proposition

The propagation number is invariant under complete order isomorphisms, as well as under stable equivalence:

$$prop(E) = prop(E \otimes_{min} \mathcal{K})$$

More generally [Koot, 2021], we have

$$prop(E \otimes_{\mathrm{min}} F) = \max\{prop(E), prop(F)\}$$

Spectral truncation of the circle

Consider the circle $(C(S^1), L^2(S^1), D = -id/dt)$

- Eigenvectors of D are Fourier modes $e_k(t) = e^{ikt}$ for $k \in \mathbb{Z}$
- Orthogonal projection $P = P_n$ onto $\mathrm{span}_{\mathbb{C}}\{e_1, e_2, \ldots, e_n\}$
- The space $C(S^1)^{(n)} := PC(S^1)P$ is an operator system
- Any $T = PfP$ in $C(S^1)^{(n)}$ can be written as a Toeplitz matrix

$$PfP \sim \left(t_{k-l}\right)_{kl} = \begin{pmatrix} t_0 & t_{-1} & \cdots & t_{-n+2} & t_{-n+1} \\ t_1 & t_0 & t_{-1} & & t_{-n+2} \\ \vdots & t_1 & t_0 & \ddots & \vdots \\ t_{n-2} & & \ddots & \ddots & t_{-1} \\ t_{n-1} & t_{n-2} & \cdots & t_1 & t_0 \end{pmatrix}$$

We have: $C^*_{\mathrm{env}}(C(S^1)^{(n)}) \cong M_n(\mathbb{C})$ and $\mathrm{prop}(C(S^1)^{(n)}) = 2$ (for any n).

Dual operator system: Fejér–Riesz

We introduce the Fejér–Riesz operator system $C^*(\mathbb{Z})_{(n)}$:

- functions on S^1 with a finite number of non-zero Fourier coefficients:

$$a = (\ldots, 0, a_{-n+1}, a_{-n+2}, \ldots, a_{-1}, a_0, a_1, \ldots, a_{n-2}, a_{n-1}, 0, \ldots)$$

- an element a is positive iff $\sum_k a_k e^{ikx}$ is a positive function on S^1.

Proposition

1. *The Shilov boundary of the operator system $C^*(\mathbb{Z})_{(n)}$ is S^1.*
2. *The C^*-envelope of $C^*(\mathbb{Z})_{(n)}$ is given by $C^*(\mathbb{Z})$.*
3. *The propagation number is infinite.*

Lemma (Fejér, Riesz)

Let $I \subseteq [-m, m]$ be an interval of length $m+1$. Suppose that $p(z) = \sum_{k=-m}^{m} p_k z^k$ is a Laurent polynomial such that $p(\zeta) \geq 0$ for all $\zeta \in \mathbb{C}$ for which $|\zeta| = 1$. Then there exists a Laurent polynomial $q(z) = \sum_{k \in I} q_k z^k$ so that $p(\zeta) = |q(\zeta)|^2$ for all $\zeta \in S^1 \subset \mathbb{C}$.

Proposition

1. *The extreme rays in $(C^*(\mathbb{Z})_{(n)})_+$ are given by the elements $a = (a_k)$ for which the Laurent series $\sum_k a_k z^k$ has all its zeroes on S^1.*
2. *The pure states of $C^*(\mathbb{Z})_{(n)}$ are given by $a \mapsto \sum_k a_k \lambda^k$ $(\lambda \in S^1)$.*

Pure states on the Toeplitz matrices

The duality between $C(S^1)^{(n)}$ and $C^*(\mathbb{Z})_{(n)}$ is given by

$$C(S^1)^{(n)} \times C^*(\mathbb{Z})_{(n)} \to \mathbb{C}$$
$$(T = (t_{k-l})_{k,l}, a = (a_k)) \mapsto \sum_k a_k t_{-k}$$

This duality was studied in [CS 2020] and more recently by Farenick.

Proposition

1. *The extreme rays in $C(S^1)^{(n)}_+$ are $\gamma(\lambda) = |f_\lambda\rangle\langle f_\lambda|$ for any $\lambda \in S^1$.*
2. *The pure states of $C(S^1)^{(n+1)}$ are given by functionals $T \mapsto \langle \xi, T\xi \rangle$ where the vector $\xi = (\xi_0, \ldots, \xi_n) \in \mathbb{C}^{n+1}$ is such that the polynomial $z \mapsto \sum_k \xi_k z^{n-k}$ has all its zeroes on S^1.*
3. *The pure state space $\mathcal{P}(C(S^1)^{(n+1)}) \cong \mathbb{T}^n / S_n$ is the quotient of the n-torus by the symmetric group on n objects.*

Let us illustrate this!

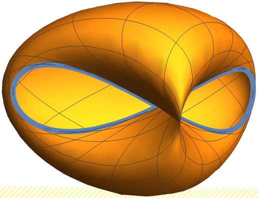
Spectral truncations of the circle ($n = 3$)

We consider $n = 3$ for which the Toeplitz matrices are of the form

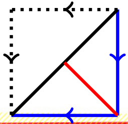
$$T = \begin{pmatrix} t_0 & t_{-1} & t_{-2} \\ t_1 & t_0 & t_{-1} \\ t_2 & t_1 & t_0 \end{pmatrix}$$

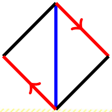
The pure state space is $\mathbb{T}^2/S_2$, given by vector states $|\xi\rangle\langle\xi|$ with

$$\xi \propto \begin{pmatrix} 1 \\ e^{ix} + e^{iy} \\ e^{i(x+y)} \end{pmatrix}$$



This is a Möbius strip!





An old factorization result of Carathéodory

Theorem
Let T be an $n \times n$ Toeplitz matrix. Then $T \geq 0$ if and only if T is of the following form:

$$\boxed{T = V \Delta V^*,}$$

where Δ is a positive diagonal matrix and V is a Vandermonde matrix,

$$\Delta = \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix}; \qquad V = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix},$$

for some $d_1, \ldots, d_n \geq 0$ and $\lambda_1, \ldots, \lambda_n \in S^1$.

Other curious results on Toeplitz matrices

Farenick continues to exploit the duality by showing:

- every positive linear map of the $n \times n$ complex matrices is completely positive when restricted to the operator subsystem of Toeplitz matrices
- every linear unital isometry of the $n \times n$ Toeplitz matrices into the algebra of all $n \times n$ complex matrices is a unitary similarity transformation.

Finite Fourier transform and duality

- Fourier transform on the cyclic group maps $l^\infty(\mathbb{Z}/m\mathbb{Z})$ to $\mathbb{C}[\mathbb{Z}/m\mathbb{Z}]$ and vice versa, exchanging pointwise and convolution product.
- This can be phrased in terms of a duality:

$$\mathbb{C}[\mathbb{Z}/m\mathbb{Z}] \times l^\infty(\mathbb{Z}/m\mathbb{Z}) \to \mathbb{C}$$
$$\langle c, g\rangle \mapsto \sum_{k,l} c_l g(k) e^{2\pi i kl/m}$$

  compatibly with positivity.
- Thus we may consider the above duality for Toeplitz matrices as some sort of generalization of Fourier theory to operator systems.
- However, note that for finite Fourier theory the symmetries are reduced from S^1 to $\mathbb{Z}/m\mathbb{Z}$.

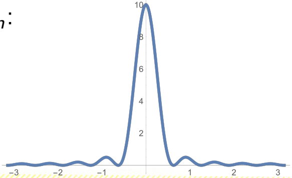
Convergence to the circle

In a recent paper I analyze the Gromov–Hausdorff convergence of the state spaces $\mathcal{S}(C(S^1)^{(n)})$ with the distance function d_n to the circle.

- The map $R_n : C(S^1) \to C(S^1)^{(n)}$ given by compression with P_n allows to pull-back states from $C(S^1)^{(n)}$ to the circle
- There is an approximate inverse $S_n : C(S^1)^{(n)} \to C(S^1)$:

$$R_n(S_n(T)) = T_n \odot T; \qquad S_n(R_n(f)) = F_n * f$$

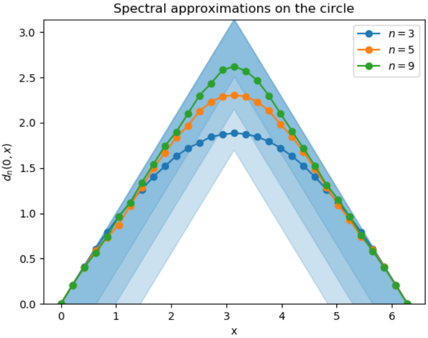
in terms of a Schur–Hadamrd product with a matrix T_n and the convolution with the Fejér kernel F_n:



- The fact that S_n is an approximate inverse of R_n allows one to prove

$$d_{S^1}(\phi, \psi) - 2\gamma_n \leq d_n(\phi \circ S_n, \psi \circ S_n) \leq d_{S^1}(\phi, \psi)$$

  where $\gamma_n \to 0$ as $n \to \infty$.
- Some (basic) Python simulations for point evaluation on S^1:



Gromov–Hausdorff convergence

Recall Gromov–Hausdorff distance between two metric spaces:

$$d_{\mathrm{GH}}(X,Y) = \inf\{d_H(f(X), g(Y)) \mid f : X \to Z, g : Y \to Z \text{ isometric}\}$$

and

$$d_H(X,Y) = \inf\{\epsilon \geq 0; X \subseteq Y_\epsilon, Y \subseteq X_\epsilon\}$$

- Using the maps R_n, S_n we can equip $\mathcal{S}(C(S^1)) \amalg \mathcal{S}(C(S^1)^{(n)})$ with a distance function that bridges (in the sense of Rieffel) the given distance functions on $\mathcal{S}(C(S^1))$ and $\mathcal{S}(C(S^1)^{(n)})$ within ϵ for large n.

Proposition (S21, Hekkelman 2021)

The sequence of state spaces $\{(\mathcal{S}(C(S^1)^{(n)}), d_n)\}$ converges to $(\mathcal{S}(C(S^1)), d_{S^1})$ in Gromov–Hausdorff distance.

Outlook: spaces at finite resolution (in progress)

Let (X, d) be a path metric space and consider the tolerance relation:

$$\mathcal{R}_\epsilon := \{(x, y) \in X \times X : d(x, y) < \epsilon\}$$

If X comes equipped with a measure μ of full support, then we define $E(\mathcal{R}_\epsilon)$ to be the operator system spanned by integral operators $\pi(F)$ on $L^2(X, \mu)$ with $F \in L^2(\mathcal{R}_\epsilon)$.

Proposition

Let (X, d) be a complete, locally compact path metric measure space and μ a measure on X with full support. Then $C^*_{env}(E(\mathcal{R}_\epsilon)) = \mathcal{K}(L^2(X))$ and

$$\mathrm{prop}(E(\mathcal{R}_\epsilon)) = \lceil diam(X)/\epsilon \rceil$$

The pure states of $E(\mathcal{R}_\epsilon)$ are given by vector states $|\psi\rangle\langle\psi|$ where the essential support of $\psi \in L^2(X)$ is ϵ-connected.

Thanks!