An introduction to the classification of $\mathbb{C}^*$-dynamics

ISem24 $\mathbb{C}^*$-algebras and dynamics – Final workshop

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Objects of interest: C^*-dynamical systems (A, α, G), where

A is a C^*-algebra

G is a locally compact group

$\alpha : G \curvearrowright A$ is a continuous action.

Overarching goal: Classify certain C^*-dynamics up to cocycle conjugacy, using *nice* invariants.

Definition

Let $\alpha : G \curvearrowright A$ be an action.

- An α-cocycle is a strictly continuous map $\mathbb{u} : G \to \mathcal{U}(\mathcal{M}(A))$ with $\mathbb{u}_{gh} = \mathbb{u}_g \alpha_g(\mathbb{u}_h)$ for all $g, h \in G$. In this case, $\alpha^{\mathbb{u}}_\bullet := \mathrm{Ad}(\mathbb{u}_\bullet) \circ \alpha_\bullet$ is another action. (Note: $\mathbf{1} \in A \implies \mathcal{M}(A) = A$.)
- α is said to be (cocycle) conjugate to $\beta : G \curvearrowright B$, if there is an isomorphism $\varphi : A \to B$ (and an α-cocycle $\mathbb{u}$) such that
$$\alpha^{\mathbb{u}}_g = \varphi^{-1} \circ \beta_g \circ \varphi, \quad g \in G.$$

In this talk C^*-algebras shall be unital and groups discrete. (convenience!)

Definition (Reminder)

Given $\alpha : G \curvearrowright A$, its (universal) crossed product is the universal C^*-algebra $A \rtimes_\alpha G$ generated by a copy of A and the range of a unitary representation $u^\alpha : G \to \mathcal{U}(A \rtimes_\alpha G)$ subject to the relation

$$u_g^\alpha a u_g^{\alpha *} = \alpha_g(a), \quad a \in A,\ g \in G.$$

Note: If $\mathbb{u} : G \to \mathcal{U}(A)$ is a cocycle for $\alpha : G \curvearrowright A$, then one can observe for all $a \in A$ and $g, h \in G$ that

$$(\mathbb{u}_g u_g^\alpha) a = \mathbb{u}_g \alpha_g(a) u_g^\alpha = \alpha_g^{\mathbb{u}}(a)(\mathbb{u}_g u_g^\alpha)$$

and

$$(\mathbb{u}_g u_g^\alpha)(\mathbb{u}_h u_h^\alpha) = \mathbb{u}_g \alpha_g(\mathbb{u}_h) u_{gh}^\alpha = \mathbb{u}_{gh} u_{gh}^\alpha.$$

This yields a canonical isomorphism $\Phi : A \rtimes_{\alpha^{\mathbb{u}}} G \to A \rtimes_\alpha G$ with $\Phi|_A = \mathrm{id}_A$ and $\Phi(u_g^{\alpha^{\mathbb{u}}}) = \mathbb{u}_g u_g^\alpha$.

Corollary

Cocycle conjugate actions have isomorphic crossed products.

The "overarching goal" to classify up to cocycle conjugacy is meant as an extension of the Elliott program, i.e., the more established attempts to classify C^*-algebras up to isomorphism should correspond to $G = \{1\}$.

In particular A is often simple nuclear Jiang–Su-stable...

In order to introduce you to this subject, I would like to preview the important slogan (or meta-idea) that I choose to focus on.

> When classifying a class of C^*-dynamics, first understand how to classify the underlying C^*-algebras. Then find a way to reduce *dynamical classification* to *non-dynamical classification* by means of an *averaging process* that exploits *amenability*.

In a bit, we will discuss the classification of finite group actions with the Rokhlin property, where this theme can be nicely demonstrated with not too involved arguments.

Before looking at C^*-dynamics, first we need to go through some basics.

Definition

Two $*$-homomorphisms $\varphi, \psi : A \to B$ between separable C^*-algebras are called approximately unitarily equivalent, if there is a sequence of unitaries $v_n \in \mathcal{U}(B)$ such that $\psi(a) = \lim_{n\to\infty} v_n \varphi(a) v_n^*$ for all $a \in A$.

In every single attempt to abstractly classify certain C^*-algebras by some nice invariant "Inv", it is vital to understand two things:

$\rightsquigarrow$ Does every arrow $\mathrm{Inv}(A) \to \mathrm{Inv}(B)$ come from a map $A \to B$?
$\rightsquigarrow$ Does $\mathrm{Inv}(\varphi) = \mathrm{Inv}(\psi)$ imply $\varphi \approx_{\mathrm{u}} \psi$?

Example: The ordered K_0-groups do the job if A and B are AF algebras

The positive answers to such questions are referred to as *existence and uniqueness theorems*. As I will briefly outline, the prevalence of this phenomenon always implies that the relevant class of C^*-algebras is in fact classified by "Inv".

Theorem (Elliott intertwining)

Let A and B be two separable C^*-algebras. Suppose there are $*$-homomorphisms $\varphi : A \to B$ and $\psi : B \to A$ with $\psi \circ \varphi \approx_{\mathrm{u}} \mathrm{id}_A$ and $\varphi \circ \psi \approx_{\mathrm{u}} \mathrm{id}_B$. Then φ and ψ are approximately unitarily equivalent to mutually inverse isomorphisms.

Idea: Inductively pick unitaries $u_n \in A$, $v_n \in B$ so that with $\varphi = \mathrm{Ad}(v_n) \circ \varphi$ and $\psi_n = \mathrm{Ad}(u_n) \circ \psi$, the diagram

$$\begin{tikzcd}
\dots \arrow[rr, "\mathrm{id}_A"] & & A \arrow[rr, "\mathrm{id}_A"] \arrow[dr, "\varphi_n"] & & A \arrow[r, "\mathrm{id}_A"] \arrow[dr] & \dots \\
\dots \arrow[r, "\mathrm{id}_B"] & B \arrow[ur, "\psi_n"] \arrow[rr, "\mathrm{id}_B"] & & B \arrow[ur, "\psi_{n+1}"] \arrow[rr, "\mathrm{id}_B"] & & \dots
\end{tikzcd}$$

approximately commutes as one goes further to the right, with 1-summable speed of convergence. Then $\Phi = \lim_{n\to\infty} \varphi_n$ is an isomorphism with inverse $\Psi = \lim_{n\to\infty} \psi_n$. □

Fortunately for us, there is an easy dynamical analog when G is finite.

Definition

Let two actions $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be given. Two equivariant $*$-homomorphisms $\varphi, \psi : (A, \alpha) \to (B, \beta)$ are approximately G-unitarily equivalent, $\varphi \approx_{\mathrm{u},G} \psi$, if there is a sequence of unitaries $v_n \in \mathcal{U}(B^\beta)$ such that $\psi(a) = \lim_{n\to\infty} v_n \varphi(a) v_n^*$ for all $a \in A$.

By copying the non-dynamical proof almost verbatim, one gets:

Theorem (dynamical Elliott intertwining for finite groups)

Let G be a finite group, and let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be two actions on separable C^*-algebras. Suppose there are equivariant $*$-homomorphisms $\varphi : (A, \alpha) \to (B, \beta)$ and $\psi : (B, \beta) \to (A, \alpha)$ with $\psi \circ \varphi \approx_{\mathrm{u},G} \mathrm{id}_A$ and $\varphi \circ \psi \approx_{\mathrm{u},G} \mathrm{id}_B$. Then φ and ψ are approximately G-unitarily equivalent to mutually inverse conjugacies.

For the applications yet to come, I use as a black box the existence/uniqueness theorems underpinning the modern Elliott program. For the purpose of this talk it is *not* necessary to have any prior knowledge about this subject. Let $\mathfrak{E}$ denote the class of separable unital simple nuclear Jiang–Su-stable C^*-algebras satisfying the UCT.

Theorem (many hands)

Let $A, B \in \mathfrak{E}$ and let $\varphi, \psi : A \to B$ be two unital $*$-homomorphisms. Then $\varphi \approx_{\mathrm{u}} \psi$ if and only if $\underline{\mathrm{K}}\mathrm{T}_{\mathrm{u}}(\varphi) = \underline{\mathrm{K}}\mathrm{T}_{\mathrm{u}}(\psi)$.[†]

Theorem (many hands)

Let $A, B \in \mathfrak{E}$. For any morphism $\zeta : \underline{\mathrm{K}}\mathrm{T}_{\mathrm{u}}(A) \to \underline{\mathrm{K}}\mathrm{T}_{\mathrm{u}}(B)$, there exists a unital $*$-homomorphism $\varphi : A \to B$ with $\underline{\mathrm{K}}\mathrm{T}_{\mathrm{u}}(\varphi) = \zeta$.

In this precise form: Carrion–Gabe–Schafhauser–Tikuisis–White

Earlier similar results: (Elliott–)Gong–Lin–Niu, among others...

[†] This so-called "total invariant" is functorial and keeps track of information like K-theory, traces, and a natural interaction between them.

Now let us finally look at the classification of Rokhlin actions!

Definition (Izumi)

Let G be a finite group and A a separable unital C^*-algebra. An action $\alpha : G \curvearrowright A$ is said to have the Rokhlin property, if there exists a sequence of projections $e_n \in A$ such that

- $\|[a, e_n]\| \to 0$ for all $a \in A$
- $\sum_{g\in G} \alpha_g(e_n) \to \mathbf{1}_A$.

$(A, \alpha) \approx (A \otimes \mathcal{C}(G), \alpha \otimes \text{shift})$

Although there exist plenty of example of such actions, the Rokhlin property is quite restrictive. (In contrast to von Neumann algebras!) However, as shown in the work of Izumi, Rokhlin actions can be very effectively classified.

Example (Prototypical one)

Let G be a finite group with its left-regular representation $\lambda : G \to \mathcal{B}(\ell^2(G)) = M_{|G|}$. Then $\gamma = \mathrm{Ad}(\lambda)^{\otimes\infty} : G \curvearrowright M_{|G|^\infty}$ has the Rokhlin property.

Going forward, I wish to convince you that for Rokhlin actions, the previous existence/uniqueness theorems imply their own dynamical generalizations. This will ultimately give us dynamical classification.

We shall start with the following reduction principle regarding the uniqueness of $*$-homomorphisms.

Theorem

Let G be a finite group. Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be actions on separable unital C^*-algebras, and assume β has the Rokhlin property. For two unital equivariant $*$-homomorphisms $\varphi, \psi : (A, \alpha) \to (B, \beta)$, we have $\varphi \approx_{\mathrm{u},G} \psi$ if and only if $\varphi \approx_{\mathrm{u}} \psi$.

Theorem (continued)

Let G be a finite group. Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be actions on separable unital C^*-algebras, and assume β has the Rokhlin property. For two unital equivariant $*$-homomorphisms $\varphi, \psi : (A, \alpha) \to (B, \beta)$, we have $\varphi \approx_{\mathrm{u},G} \psi$ if and only if $\varphi \approx_{\mathrm{u}} \psi$.

Sketch of proof: Suppose $v_n \in \mathcal{U}(B)$ satisfies $\psi = \lim_{n\to\infty} \mathrm{Ad}(v_n) \circ \varphi$. Note that since φ and ψ were equivariant, one also has

$$\lim_{n\to\infty} \mathrm{Ad}(\beta_g(v_n)) \circ \varphi = \lim_{n\to\infty} \beta_g \circ \mathrm{Ad}(v_n) \circ \varphi \circ \alpha_g^{-1} = \beta_g \circ \psi \circ \alpha_g^{-1} = \psi.$$

Let $e_n \in B$ be a sequence of projections as required by the Rokhlin property. Without loss of generality we may assume $\|[e_n, v_n]\| \to 0$.

Then we find a sequence of unitaries $\mathcal{U}(B^\beta) \ni u_n \approx \sum_{g\in G} \beta_g(e_n v_n)$.

Then:
$$\mathrm{Ad}(u_n) \circ \varphi \approx \sum_{g\in G} \beta_g(e_n) \cdot \underbrace{\mathrm{Ad}(\beta_g(v_n)) \circ \varphi}_{\approx\psi} \approx \psi.$$

Thus the sequence u_n witnesses $\varphi \approx_{\mathrm{u},G} \psi$. □

Next we discuss the reduction principle regarding existence.

Theorem (Gardella–Santiago)

Let G be a finite group. Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be actions on separable unital C^*-algebras, and assume β has the Rokhlin property. Suppose $\varphi : A \to B$ is a unital $*$-homomorphism with $\varphi \circ \alpha_g \approx_{\mathrm{u}} \beta_g \circ \varphi$ for all $g \in G$. Then there exists a unital equivariant $*$-homomorphism $\psi : (A, \alpha) \to (B, \beta)$ with $\varphi \approx_{\mathrm{u}} \psi$.

Sketch of proof: For each $h \in G$ let $w_h \in \mathcal{U}(B)$ be some unitary such that

$$\beta_h \circ \varphi \circ \alpha_h^{-1} \approx \mathrm{Ad}(w_h) \circ \varphi, \quad h \in G.$$

Let $e \in B$ be a *good enough* projection as required by the Rokhlin property. Then we find a unitary $\mathcal{U}(B) \ni v \approx \sum_{h \in G} \beta_h(e) w_h$.

Set $\varphi_1 = \mathrm{Ad}(v) \circ \varphi$.

Sketch of proof: (continued)

We find a unitary $\mathcal{U}(B) \ni v \approx \sum_{h\in G} \beta_h(e)w_h$ and set $\varphi_1 = \mathrm{Ad}(v) \circ \varphi$.

We observe for all $g \in G$:

$$\begin{array}{rcl} \beta_g \circ \varphi_1 & \approx & \sum_{h\in G} \beta_{gh}(e) \cdot \beta_g \circ \underbrace{\mathrm{Ad}(w_h) \circ \varphi}_{\approx \beta_h \circ \varphi \circ \alpha_h^{-1}} \\ & \approx & \sum_{h\in G} \beta_{gh}(e) \cdot \beta_{gh} \circ \varphi \circ \alpha_h^{-1} \\ & = & \sum_{h\in G} \beta_h(e) \cdot \underbrace{\beta_h \circ \varphi \circ \alpha_h^{-1}}_{\approx \mathrm{Ad}(w_h)\circ\varphi} \circ \alpha_g \\ & \approx & \varphi_1 \circ \alpha_g. \end{array}$$

Repeat this inductively and get a sequence of maps $\varphi_1, \varphi_2, \varphi_3, \dots$ for which these approximations hold better and better. If one does this carefully, one can arrange the maps (φ_n) to be Cauchy in point-norm, which allows us to get the desired map as $\psi = \lim_{n\to\infty} \varphi_n$. □

As a consequence of all of this, we get the following classification result:

Theorem

Let G be a finite group. Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be two Rokhlin actions on classifiable C^*-algebras. Then α and β are conjugate if and only if

$$\underline{\mathrm{K}}\mathrm{T}_{\mathrm{u}}(\alpha) : G \curvearrowright \underline{\mathrm{K}}\mathrm{T}_{\mathrm{u}}(A) \quad \textit{and} \quad \underline{\mathrm{K}}\mathrm{T}_{\mathrm{u}}(\beta) : G \curvearrowright \underline{\mathrm{K}}\mathrm{T}_{\mathrm{u}}(B)$$

are conjugate.

Proof: Assume that $\zeta : \underline{\mathrm{K}}\mathrm{T}_{\mathrm{u}}(A) \to \underline{\mathrm{K}}\mathrm{T}_{\mathrm{u}}(B)$ is an equivariant isomorphism. By the black box, we find $*$-homomorphisms $\varphi_0 : A \to B$ and $\psi_0 : B \to A$ lifting ζ and ζ^{-1}, respectively. Since ζ is equivariant, it follows from the black box that these maps are equivariant modulo $\approx_{\mathrm{u}}$. By the reduction trick, we may find equivariant $*$-homomorphisms $\varphi : (A, \alpha) \to (B, \beta)$ and $\psi : (B, \beta) \to (A, \alpha)$ lifting ζ and ζ^{-1}. Using again the black box and the other reduction trick, we see $\psi \circ \varphi \approx_{\mathrm{u},G} \mathrm{id}_A$ and $\varphi \circ \psi \approx_{\mathrm{u},G} \mathrm{id}_B$. Elliott intertwining takes care of the rest. □

A special example of a classifiable C^*-algebra is the Cuntz algebra $\mathcal{O}_2$, which has the same invariant as the zero algebra. It therefore holds that $A \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ whenever A is classifiable. (Kirchberg–Phillips)

Example

For any finite group G, there is a unique Rokhlin action $G \curvearrowright \mathcal{O}_2$. For example, the two actions

$$\alpha : \mathbb{Z}_2 \curvearrowright \mathcal{O}_2 = \mathrm{C}^*(s_1, s_2), \quad \alpha(s_j) = (-1)^j s_j$$

and

$$\beta : \mathbb{Z}_2 \curvearrowright \mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2, \quad \beta(x_1 \otimes x_2) = x_2 \otimes x_1$$

are conjugate.

Example

If we recall the prototypical Rokhlin action $\gamma : G \curvearrowright M_{|G|^\infty}$, then the two actions above are further conjugate to

$$\gamma \otimes \mathrm{id}_{\mathcal{O}_2} : G \curvearrowright M_{|G|^\infty} \otimes \mathcal{O}_2 \cong \mathcal{O}_2.$$

To round off this mini-introduction I would like to say a few words about the issues surrounding the classification of more general C^*-dynamics.

Warning 1: The theory of Rokhlin actions might lead you to believe that ultimately, nice actions on classifiable C^*-algebras are determined by how they act on the Elliott invariant. Although the analogous statement is true for actions on von Neumann algebras, this expectation fails spectacularly in the general C^*-context.

Example (Izumi)

Let $\mathcal{O}_\infty^{\mathrm{st}} \subset \mathcal{O}_\infty$ be the corner spanned by an inclusion $\mathcal{O}_2 \subset \mathcal{O}_\infty$. For some $q \in \mathcal{O}_\infty^{\mathrm{st}}$ that is the range projection of an isometry in $\mathcal{O}_\infty$, we consider the order 2 automorphism

$$\gamma = \bigotimes_{\mathbb{N}} \mathrm{Ad}(2q - \mathbf{1}) : \mathbb{Z}_2 \curvearrowright \bigotimes_{\mathbb{N}} \mathcal{O}_\infty^{\mathrm{st}} \cong \mathcal{O}_2.$$

Then $\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_2 \cong \mathcal{O}_\infty^{\mathrm{st}} \otimes M_{2^\infty}$ and $\mathcal{O}_2^\gamma \cong \mathcal{O}_\infty \otimes M_{2^\infty}$.

Example

Given a unital Kirchberg algebra A, we can induce an action

$$\alpha = \mathrm{id}_A \otimes \gamma : \mathbb{Z}_2 \curvearrowright A \otimes \mathcal{O}_2 \cong \mathcal{O}_2.$$

Since the crossed products of such actions can be computed to have many possible K-groups, this yields uncountably many outer actions $\mathbb{Z}_2 \curvearrowright \mathcal{O}_2$ that are pairwise non-cocycle conjugate.

Warning 2: I started the presentation talking about *cocycle* conjugacy, after which no cocycles were to be seen. It just so happens that cocycles can always be trivialized for Rokhlin actions, which is special. The cocycles are **very** important in general.

Example (Izumi)

Let A and B be two unital Kirchberg algebras that absorb M_{2^∞}. Let $\alpha, \beta : \mathbb{Z}_2 \curvearrowright \mathcal{O}_2$ be two actions as constructed above. Suppose A and B are stably isomorphic, but not isomorphic. (E.g. $A = \mathcal{O}_\infty \otimes M_{2^\infty}$ and $B = \mathcal{O}_\infty^{\mathrm{st}} \otimes M_{2^\infty}$.) Then α and β are cocycle conjugate, but not conjugate.

Warning 3: In general, working only with genuine equivariant maps between C^*-dynamics is too restrictive.

For example, for $G = \mathbb{Z}$, one ends up classifying single automorphisms on classifiable C^*-algebras, but one cannot do it up to conjugacy.

Theorem (Evans–Kishimoto)

Let $\alpha, \beta \in \mathrm{Aut}(A)$ be two single automorphisms on an AF algebra with the Rokhlin property. If $K_0(\alpha) = K_0(\beta)$, then α and β are cocycle conjugate.

The proof involved the invention of what is now called the *Evans–Kishimoto intertwining* method. Roughly speaking, one works very hard for taking care of certain technical obstacles, after which one inductively perturbs α and β with unitary conjugates and/or cocycles in A to push them closer together. If one does this right, then the compositions of the unitary conjugates and the cocycles satisfy a certain Cauchy criterion, and one obtains a cocycle conjugacy via a limit procedure.

(This is a lot more involved than what we saw before!)

My suggested approach is to work in a category where an arrow between C^*-dynamical systems is a pair

$$(\varphi, \mathbb{u}) : (A, \alpha) \to (B, \beta),$$

where $\mathbb{u}$ is a β-cocycle and φ is a $*$-homomorphism which is equivariant with respect to α and $\beta^{\mathbb{u}}$. This can indeed be defined, and this category comes equipped with a flexible notion of (approximate) unitary equivalence

$$(\varphi, \mathbb{u}) \sim_{\mathrm{u}} (\mathrm{Ad}(v) \circ \varphi, v\mathbb{u}_\bullet \beta_\bullet(v)^*), \quad v \in \mathcal{U}(B).$$

This gives one access to an Elliott intertwining machinery with obvious candidates for existence/uniqueness theorems, which are entirely analogous to what we have seen in the first part of the talk.

Theorem (Gabe–S, in progress)

Let G be any countable amenable discrete group. Then outer G-actions on Kirchberg algebras are classified up to cocycle conjugacy via equivariant Kasparov theory (KK-theory).

Warning 4: For actions on general classifiable algebras, we have yet to pass a basic plausibility check w.r.t. their structure!

Open problem: If $A \in \mathfrak{E}$ and $\alpha : G \curvearrowright A$ is an action of an amenable group, is it always cocycle conjugate to $\alpha \otimes \mathrm{id}_{\mathcal{Z}} : G \curvearrowright A \otimes \mathcal{Z}$?

- Amenability of G is known to be necessary.
- If A has no traces or *not too many traces*, then this is known.
- Wide open if (e.g.) the trace space $T(A)$ is the Paulsen simplex.
- The problem becomes somewhat more tractable if G is finite, but remains unsolved even there.

Thank you for your attention!