Dimension type regularity properties in topological dynamics and C*-algebras

ISem 24

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$$\begin{array}{ccc} \overset{\text{action}}{\alpha} & \xrightarrow[\text{construction}]{\text{crossed product}} & \\ G \curvearrowright C & \rightsquigarrow & C \subset C \rtimes_\alpha G \\ \text{(discrete) group} \quad \text{(unital) } C^*\text{-algebra} & & \text{(unital) } C^*\text{-algebra} \end{array}$$

We now have $3\frac{1}{2}$ interesting objects and may study each of them in terms of the others.

$\leftsquigarrow$ "rigidity" questions

E.g. (i) $G \curvearrowright \ell^\infty(G)$ $\rightsquigarrow$ $\ell^\infty(G) \subset \ell^\infty(G) \rtimes G$ uniform Roe algebra

"$\leftsquigarrow$": $\hookrightarrow$ often encodes "coarse" structure of G

(ii) $G \curvearrowright \partial G$ ≈ Cantor set $\rightsquigarrow$ $C(\partial G) \rtimes G$ purely infinite & classifiable

non-elementary hyperbolic, e.g. F_2 — "boundary of G"

(iii) $G \curvearrowright C(X)$ $\rightsquigarrow$ $C(X) \rtimes G$ simple, finite, and often classifiable

$\mathbb{Z}^d$ — free, minimal — topological space, or even a manifold

(iv) $G \curvearrowright A$ $\rightsquigarrow$ ¿ When is $A \rtimes G$ classifiable ?

cyclic — classifiable

"$\longleftrightarrow$" does not work functorially.

But: A lot of structural properties occur both on the left and on the right.

free and minimal actions $\longleftrightarrow$ simple crossed products

amenable actions $\longleftrightarrow$ nuclear crossed products

finite top. dimension of X resp. α $\longleftrightarrow$ finite nuclear dimension
tower dimension
diagonal dimension

ergodic comparison properties $\longleftrightarrow$ tracial comparison

almost finiteness $\longleftrightarrow$ generalised matrix decompositions
$A \cong A \otimes \mathcal{Z}$

Definition: A C^*-algebra A is nuclear, if the following holds:

$$\forall \varepsilon > 0,\ \mathcal{F} \subset\subset A \text{ finite } \exists\ A \xrightarrow[\text{c.p.c.}]{\psi} \underset{\text{fin. dim.}}{F} \xrightarrow[\text{c.p.c.}]{\varphi} A \ : \ \varphi\psi =_{\mathcal{F},\varepsilon} \mathrm{id}_A$$

A has nuclear dimension at most d, if the following holds:

$$\forall \varepsilon > 0,\ \mathcal{F} \subset\subset A \text{ finite } \exists\ A \xrightarrow[\text{c.p.c.}]{\psi} \underset{\text{fin. dim.}}{F} \xrightarrow[\text{c.p.}]{\varphi} A \ : \ \varphi\psi =_{\mathcal{F},\varepsilon} \mathrm{id}_A$$

and $F = F^{(0)} \oplus \dots \oplus F^{(d)}$

with $\varphi|_{F^{(i)}}$ c.p.c. order zero

$xy = 0 \Rightarrow \varphi(x)\varphi(y) = 0$

$\in F^{(i)}$

$M_{2^\infty} = M_2 \otimes M_2 \otimes M_2 \otimes \dots$

<u>Definition/Theorem:</u> (i) $Z_{2^\infty, 3^\infty} := \{ f : C([0,1], M_{2^\infty} \otimes M_{3^\infty}) \mid f(0) \in M_{2^\infty} \otimes 1_{3^\infty},\ f(1) \in 1_{2^\infty} \otimes M_{3^\infty} \}$

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(ii) $\exists\ 0 \neq J : Z_{2^\infty,3^\infty} \longrightarrow Z_{2^\infty,3^\infty}$: $\forall\, \tau_1, \tau_2 \in T(Z_{2^\infty,3^\infty})$: $\tau_1 \circ J = \tau_2 \circ J$.

(iii) $Z := C^*\text{-}\varinjlim (Z_{2^\infty,3^\infty}, J)$ is simple, monotracial, and does not depend on J.

Definition/Theorem: (i) $\mathcal{Z}_{2^\infty,3^\infty} := \{ f \in C([0,1], M_{2^\infty} \otimes M_{3^\infty}) \mid f(0) \in M_{2^\infty} \otimes 1_{3^\infty},\ f(1) \in 1_{2^\infty} \otimes M_{3^\infty} \}$

(ii) $\exists\ \xi : \mathcal{Z}_{2^\infty,3^\infty} \longrightarrow \mathcal{Z}_{2^\infty,3^\infty}\ :\ \forall\ \tau_1, \tau_2 \in T(\mathcal{Z}_{2^\infty,3^\infty})\ :\ \tau_1 \circ \xi = \tau_2 \circ \xi$.

(iii) $\mathcal{Z} := C^*\text{-}\varinjlim(\mathcal{Z}_{2^\infty,3^\infty}, \xi)$ is simple, monotracial, and does not depend on ξ.

(iv) A C^*-algebra A is said to be $\mathcal{Z}$-stable, if $A \cong A \otimes \mathcal{Z}$.

(v) $\mathcal{Z}$ is $\mathcal{Z}$-stable.

Proposition: A unital and separable C^*-algebra A is $\mathcal{Z}$-stable, if and only if there are c.p.c. order zero maps

$$\Phi, \Psi : M_2 \longrightarrow \ell^\infty(\mathbb{N}, A)/_{c_0(\mathbb{N},A)} \cap A'$$

such that

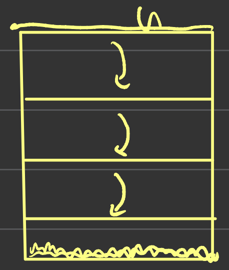
$$\Psi(e_{22}) = 1 - \Phi(1_2)$$

and $\Psi(e_{11})\Phi(e_{11}) = \Psi(e_{11})$.

Theorem: Let A be a separable, simple, unital C^*-algebra, $A \ncong M_r$.
Then, the following are equivalent:
(i) $\dim_{nuc} A < \infty$
(ii) A is nuclear and $\mathcal{Z}$-stable.

How to phrase nuclear dimension and $\mathcal{Z}$-stability for $G \curvearrowright X$?

Rokhlin lemma: $\mathbb{Z} \overset{T}{\curvearrowright} (X, \mu)$ ergodic. $\forall n \in \mathbb{N}, \varepsilon > 0 \; \exists \; U \subset X$ measurable,
$T^k U$ pairwise disjoint, $k = 0, \dots, n$,
$\mu\left(\bigcup_{k=0}^{n} T^k U \right) > 1 - \varepsilon$.



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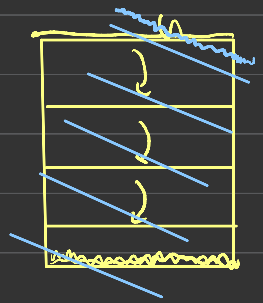

Tower dimension: A coloured topological version, with empty remainder.

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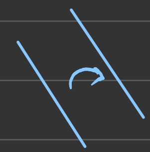
Almost finiteness: An almost invariant dynamic version of
$\Phi, \Psi : \Pi_2 \xrightarrow{\perp} "(G\curvearrowright X)"$



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$\Phi, \Psi : \Pi_2 \xrightarrow{\perp} \text{"}(G \curvearrowright X)\text{"}$

<u>Proposition:</u> $\dim X$, $\dim_{tow}(G \curvearrowright X) < \infty \Rightarrow \dim_{nuc}(C(X) \rtimes G) < \infty$.

$G \curvearrowright X$ almost finite $\Rightarrow C(X) \rtimes G$ $\mathcal{Z}$-stable.

For rigidity questions also keep track of $C(X) \subseteq C(X) \rtimes G$

Cartan pair / C^*-diagonal

<u>Definition:</u> Let $D \subset A$ be a commutative sub-C^*-algebra.
We say the diagonal dimension is at most d, $\dim_{diag}(D \subset A) \leq d$,
if the following holds:
$\forall \mathcal{F} \subset\subset A$ finite, $\varepsilon > 0$ $\exists$ $A \xrightarrow[\text{c.p.c.}]{\psi} F \xrightarrow[\text{c.p.}]{\varphi} A$:
(F fin. dim.)

- $\varphi\psi =_{\mathcal{F},\varepsilon} \mathrm{id}_A$
- $F = F^{(0)} \oplus \ldots \oplus F^{(d)}$, $\varphi|_{F^{(i)}}$ c.p.c. order zero
- F contains a maximal abelian *-subalgebra D_F such that $\psi(D) \subset D_F$ and $\varphi(e) D \varphi(e^*) \subset D$ whenever ee^*, e^*e are rank one projections in D_F.

<u>Theorem:</u> (i) $\dim X$, $\dim_{fdu}(G \curvearrowright X) < \infty \iff \dim_{diag}(C(X) \subset C(X) \rtimes G) < \infty$.

(ii) G finitely generated with word length metric, then
G has finite asymptotic dimension $\iff \dim_{diag}(\ell^\infty(G) \subset \ell^\infty(G) \rtimes G) < \infty$.

Some open problems:

(i) $\dim X$, $\dim_{as} G < \infty \overset{?}{\Rightarrow} \dim_{flow}(G \curvearrowright X) < \infty$

free, minimal, amenable

What about almost finiteness?

[Holds generically!]

(ii) Suppose $C(X) \rtimes G$ is $\mathcal{Z}$-stable. When is $G \curvearrowright X$ almost finite?